Module – 4

ESTIMATION OF DYNAMIC DERIVATIVES

The static stability is a tendency of the aircraft to return to its equilibrium position. In addition to static stability, the aircraft also must be dynamically stable. An airplane can be considered to be dynamically stable if after being disturbed from its equilibrium flight condition the ensuing motion diminishes with time. Of particular interest to the pilot and designer is the degree of dynamic stability. The required degree of dynamic stability usually is specified by the time it takes the motion to damp to half of its initial amplitude or in the case of an unstable motion the time it takes for the initial amplitude or disturbance to double. Also of interest is the frequency or period of the oscillation.

An understanding of the dynamic characteristics of an airplane is important in assessing its handling or flying qualities as well as for designing autopilots. The flying qualities of an airplane are dependent on pilot opinion; that is, the pilot’s likes or dislikes with regard to the various vehicle motions. It is possible to design an airplane that has excellent performance but is considered unsatisfactory by the pilot. Since the early 1960s, considerable research has been directed toward quantifying pilot opinion in terms of aircraft motion characteristics, such as frequency and damping ratio of the aircraft s various modes of motion. Therefore, it is important to understand the dynamic characteristics of an airplane and the relationship of the motion to the vehicle's aerodynamic characteristics and pilot opinion.

Before developing the equations of motion, it is important to review the axis system specified earlier. Figure 3.1 shows the body axis system fixed to the aircraft and the inertial axis system that is fixed to the Earth.

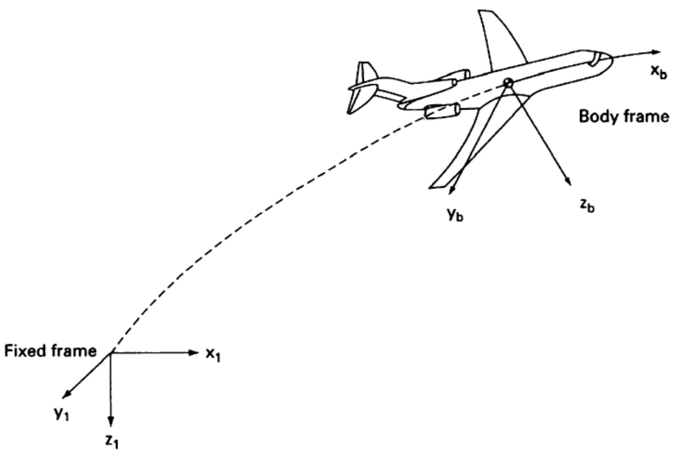


FIGURE 3.1
Body and inertial axis systems.

4.1 Derivation of Rigid Body Equations of Motion

The rigid body equations of motion are obtained from Newton's second law, which states that the summation of all external forces acting on a body is equal to the time rate of change of the momentum of the body; and the summation of the external moments acting on the body is equal to the time rate of change of the moment of momentum (angular momentum). The time rates of change of linear and angular momentum are referred to an absolute or inertial reference frame. For many problems in airplane dynamics, an axis system fixed to the Earth can be used as an inertial reference frame. Newton's second law can be expressed in the following vector equations:

$$\sum \mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) \tag{3.1}$$

$$\sum \mathbf{M} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H} \tag{3.2}$$

The vector equations can be rewritten in scalar form and then consist of three force equations and three moment equations. The force equations can be expressed as follows:

$$F_x = \frac{\mathrm{d}}{\mathrm{d}t}(mu) \qquad F_y = \frac{\mathrm{d}}{\mathrm{d}t}(mv) \qquad F_z = \frac{\mathrm{d}}{\mathrm{d}t}(mw) \tag{3.3}$$

where F_x, F_y, F_z and u, v, w are the components of the force and velocity along the x, y, and z axes, respectively. The force components are composed of contributions due to the aerodynamic, propulsive, and gravitational forces acting on the airplane. The moment equations can be expressed in a similar manner:

$$L = \frac{\mathrm{d}}{\mathrm{d}t}H_x \qquad M = \frac{\mathrm{d}}{\mathrm{d}t}H_y \qquad N = \frac{\mathrm{d}}{\mathrm{d}t}H_z \tag{3.4}$$

where L, M, N and H_x, H_y, H_z are the components of the moment and moment of momentum along the x, y, and z axes, respectively.

Consider the airplane shown in Figure 3.2. If we let δm be an element of mass of the airplane, $\mathbf{v}$ be the velocity of the elemental mass relative to an absolute or inertial frame, and $\delta\mathbf{F}$ be the resulting force acting on the elemental mass, then Newton's second law yields

$$\delta\mathbf{F} = \delta m \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \tag{3.5}$$

and the total external force acting on the airplane is found by summing all the elements of the airplane:

$$\sum \delta\mathbf{F} = \mathbf{F} \tag{3.6}$$

The velocity of the differential mass δm is

$$\mathbf{v} = \mathbf{v}_c + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \tag{3.7}$$

where $\mathbf{v}_c$ is the velocity of the center of mass of the airplane and $\mathrm{d}\mathbf{r}/\mathrm{d}t$ is the velocity of the element relative to the center of mass. Substituting this expression for the

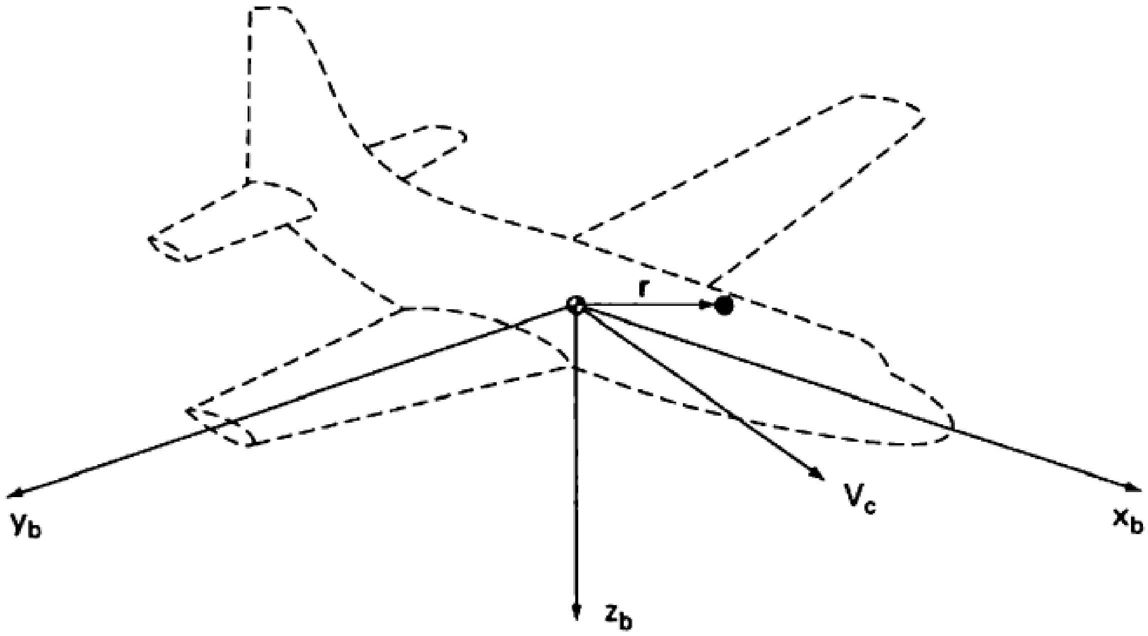


FIGURE 3.2
An element of mass on an airplane.

velocity into Newton's second law yields

$$\sum \delta \mathbf{F} = \mathbf{F} = \frac{d}{dt}\sum\left(\mathbf{v}_c + \frac{d\mathbf{r}}{dt}\right)\delta m \tag{3.8}$$

If we assume that the mass of the vehicle is constant, Equation (3.8) can be rewritten as

$$\mathbf{F} = m\frac{d\mathbf{v}_c}{dt} + \frac{d}{dt}\sum\frac{d\mathbf{r}}{dt}\delta m \tag{3.9}$$

or

$$\mathbf{F} = m\frac{d\mathbf{v}_c}{dt} + \frac{d^2}{dt^2}\sum \mathbf{r}\,\delta m \tag{3.10}$$

Because $\mathbf{r}$ is measured from the center of mass, the summation $\Sigma\,\mathbf{r}\,\delta m$ is equal to 0. The force equation then becomes

$$\mathbf{F} = m\frac{d\mathbf{v}_c}{dt} \tag{3.11}$$

which relates the external force on the airplane to the motion of the vehicle's center of mass.

In a similar manner, we can develop the moment equation referred to a moving center of mass. For the differential element of mass, δm, the moment equation can be written as

$$\delta \mathbf{M} = \frac{d}{dt}\delta \mathbf{H} = \frac{d}{dt}(\mathbf{r}\times\mathbf{v})\,\delta m \tag{3.12}$$

The velocity of the mass element can be expressed in terms of the velocity of the center of mass and the relative velocity of the mass element to the center of mass:

$$\mathbf{v} = \mathbf{v}_c + \frac{d\mathbf{r}}{dt} = \mathbf{v}_c + \boldsymbol{\omega} \times \mathbf{r} \tag{3.13}$$

where $\boldsymbol{\omega}$ is the angular velocity of the vehicle and $\mathbf{r}$ is the position of the mass element measured from the center of mass. The total moment of momentum can be written as

$$\mathbf{H} = \sum \delta\mathbf{H} = \sum (\mathbf{r} \times \mathbf{v}_c)\, \delta m + \sum [\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]\, \delta m \tag{3.14}$$

The velocity $\mathbf{v}_c$ is a constant with respect to the summation and can be taken outside the summation sign:

$$\mathbf{H} = \sum \mathbf{r}\, \delta m \times \mathbf{v}_c + \sum [\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]\, \delta m \tag{3.15}$$

The first term in Equation (3.15) is 0 because the term $\Sigma\, \mathbf{r}\, \delta m = 0$, as explained previously. If we express the angular velocity and position vector as

$$\boldsymbol{\omega} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k} \tag{3.16}$$

and

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3.17}$$

then after expanding Equation (3.15), $\mathbf{H}$ can be written as

$$\begin{aligned} \mathbf{H} = (p\mathbf{i} + q\mathbf{i} + r\mathbf{k}) \sum (x^2 + y^2 + z^2)\, \delta m \\ - \sum (x\mathbf{i} + y\mathbf{i} + z\mathbf{k})(px + qy + rz)\, \delta m \end{aligned} \tag{3.18}$$

The scalar components of $\mathbf{H}$ are

$$\begin{aligned} H_x &= p \sum (y^2 + z^2)\, \delta m - q \sum xy\, \delta m - r \sum xz\, \delta m \\ H_y &= -p \sum xy\, \delta m + q \sum (x^2 + z^2)\, \delta m - r \sum yz\, \delta m \\ H_z &= -p \sum xz\, \delta m - q \sum yz\, \delta m + r \sum (x^2 + y^2)\, \delta m \end{aligned} \tag{3.19}$$

The summations in these equations are the mass moment and products of inertia of the airplane and are defined as follows:

$$\begin{aligned} I_x &= \iiint (y^2 + z^2)\, \delta m \qquad & I_{xy} &= \iiint xy\, \delta m \\ I_y &= \iiint (x^2 + z^2)\, \delta m \qquad & I_{xz} &= \iiint xz\, \delta m \\ I_z &= \iiint (x^2 + y^2)\, \delta m \qquad & I_{yz} &= \iiint yz\, \delta m \end{aligned} \tag{3.20}$$

The terms I_x, I_y, and I_z are the mass moments of inertia of the body about the x, y, and z axes, respectively. The terms with the mixed indexes are called the products of inertia. Both the moments and products of inertia depend on the shape of the body and the manner in which its mass is distributed. The larger the moments of inertia, the greater will be the resistance to rotation. The scaler equations for the moment of momentum follow:

$$H_x = pI_x - qI_{xy} - rI_{xz}$$

$$H_y = -pI_{xy} + qI_y - rI_{yz} \tag{3.21}$$
$$H_z = -pI_{xz} - qI_{yz} + rI_z$$

If the reference frame is not rotating, then as the airplane rotates the moments and products of inertia will vary with time. To avoid this difficulty we will fix the axis system to the aircraft (body axis system). Now we must determine the derivatives of the vectors **v** and **H** referred to the rotating body frame of reference.

It can be shown that the derivative of an arbitrary vector **A** referred to a rotating body frame having an angular velocity **ω** can be represented by the following vector identity:

$$\left.\frac{d\mathbf{A}}{dt}\right|_I = \left.\frac{d\mathbf{A}}{dt}\right|_B + \boldsymbol{\omega} \times \mathbf{A} \tag{3.22}$$

where the subscripts I and B refer to the inertial and body fixed frames of reference. Applying this identity to the equations derived earlier yields

$$\mathbf{F} = m \left.\frac{d\mathbf{v}_c}{dt}\right|_B + m(\boldsymbol{\omega} \times \mathbf{v}_c) \tag{3.23}$$

$$\mathbf{M} = \left.\frac{d\mathbf{H}}{dt}\right|_B + \boldsymbol{\omega} \times \mathbf{H} \tag{3.24}$$

The scalar equations are

$$F_x = m(\dot{u} + qw - rv) \qquad F_y = m(\dot{v} + ru - pw) \qquad F_z = m(\dot{w} + pv - qu)$$
$$L = \dot{H}_x + qH_z - rH_y \qquad M = \dot{H}_y + rH_x - pH_z \qquad N = \dot{H}_z + pH_y - qH_x \tag{3.25}$$

The components of the force and moment acting on the airplane are composed of aerodynamic, gravitational, and propulsive contributions.

By proper positioning of the body axis system, one can make the products of inertia $I_{yz} = I_{xy} = 0$. To do this we are assuming that the xz plane is a plane of symmetry of the airplane. With this assumption, the moment equations can be written as

$$L = I_x\dot{p} - I_{xz}\dot{r} + qr(I_z - I_y) - I_{xz}pq$$
$$M = I_y\dot{q} + rp(I_x - I_z) + I_{xz}(p^2 - r^2) \tag{3.26}$$
$$N = -I_{xz}\dot{p} + I_z\dot{r} + pq(I_y - I_x) + I_{xz}qr$$

4.2 Orientation and Position of The Airplane

The equations of motion have been derived for an axis system fixed to the airplane. Unfortunately, the position and orientation of the airplane cannot be described relative to the moving body axis frame. The orientation and position of the airplane can be defined in terms of a fixed frame of reference as shown in Figure 3.3. At time $t = 0$, the two reference frames coincide.

The orientation of the airplane can be described by three consecutive rotations, whose order is important. The angular rotations are called the Euler angles. The orientation of the body frame with respect to the fixed frame can be determined in the following manner. Imagine the airplane to be positioned so that the body axis system is parallel to the fixed frame and then apply the following rotations:

1. Rotate the x_f, y_f, z_f frame about $0z_f$ through the yaw angle ψ to the frame to x_1, y_1, z_1.
2. Rotate the x_1, y_1, z_1 frame about $0y_1$ through the pitch angle θ bringing the frame to x_2, y_2, z_2.
3. Rotate the x_2, y_2, z_2 frame about $0x_2$ through the roll angle Φ to bring the frame to x_3, y_3, z_3, the actual orientation of the body frame relative to the fixed frame.

Remember that the order of rotation is extremely important.

Having defined the Euler angles, one can determine the flight velocities components relative to the fixed reference frame. To accomplish this, let the velocity components along the x_f, y_f, z_f frame be dx/dt, dy/dt, dz/dt and similarly let the

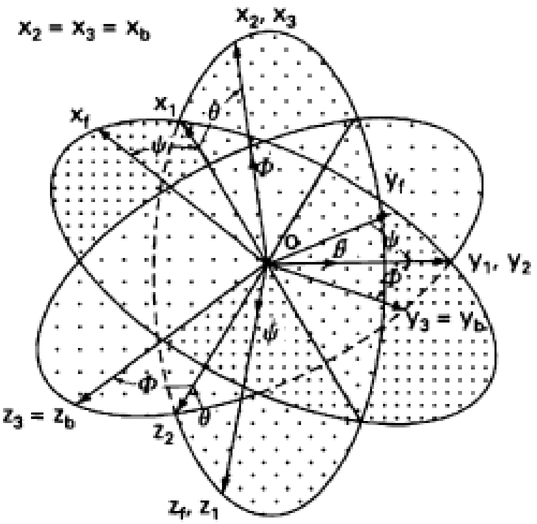
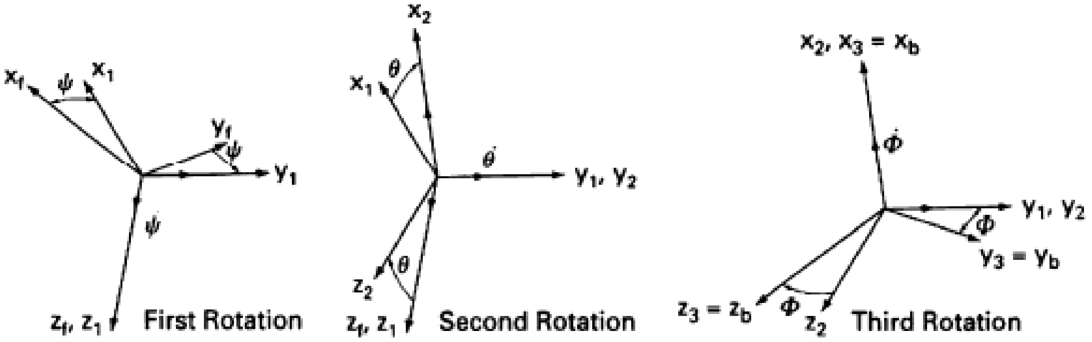


FIGURE 3.3
Relationship between body and inertial axes systems.



subscripts 1 and 2 denote the components along x_1, y_1, z_1 and x_2, y_2, z_2, respectively. Examining Figure 3.3, we can show that

$$\frac{dx}{dt} = u_1 \cos\psi - v_1 \sin\psi \qquad \frac{dy}{dt} = u_1 \sin\psi + v_1 \cos\psi \qquad \frac{dz}{dt} = w_1 \quad (3.27)$$

Before proceeding further, let us use the shorthand notation $S_\psi \equiv \sin\psi$, $C_\psi \equiv \cos\psi$, $S_\theta \equiv \sin\theta$, and so forth. In a manner similar to Equation (3.27), u_1, v_1, and w_1 can be expressed in terms of u_2, v_2, and w_2:

$$u_1 = u_2 C_\theta + w_2 S_\theta \qquad v_1 = v_2 \qquad w_1 = -u_2 S_\theta + w_2 C_\theta \qquad (3.28)$$

and $$u_2 = u \qquad v_2 = vC_\Phi - wS_\Phi \qquad w_2 = vS_\Phi + wC_\Phi \qquad (3.29)$$

where u, v, and w are the velocity components along the body axes x_b, y_b, z_b.

If we back-substitute the preceding equations, we can determine the absolute velocity in terms of the Euler angles and velocity components in the body frame:

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} C_\theta C_\psi & S_\Phi S_\theta C_\psi - C_\Phi S_\psi & C_\Phi S_\theta C_\psi + S_\Phi S_\psi \\ C_\theta S_\psi & S_\Phi S_\theta S_\psi + C_\Phi C_\psi & C_\Phi S_\theta S_\psi - S_\Phi C_\psi \\ -S_\theta & S_\theta C_\theta & C_\Phi C_\theta \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad (3.30)$$

Integration of these equations yields the airplane's position relative to the fixed frame of reference.

The relationship between the angular velocities in the body frame (p, q, and r) and the Euler rates ($\dot{\psi}$, $\dot{\theta}$, and $\dot{\Phi}$) also can be determined from Figure 3.3:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\Phi & C_\theta S_\Phi \\ 0 & -S_\Phi & C_\theta C_\Phi \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad (3.31)$$

Equation (3.31) can be solved for the Euler rates in terms of the body angular velocities:

$$\begin{bmatrix} \dot{\Phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & S_\Phi \tan\theta & C_\Phi \tan\theta \\ 0 & C_\Phi & -S_\Phi \\ 0 & S_\Phi \sec\theta & C_\Phi \sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad (3.32)$$

By integrating these equations, one can determine the Euler angles ψ, θ, and Φ.

4.3 Gravitational and Thrust Forces

The gravitational force acting on the airplane acts through the center of gravity of the airplane. Because the body axis system is fixed to the center of gravity, the gravitational force will not produce any moments. It will contribute to the external force acting on the airplane, however, and have components along the respective body axes. Figure 3.4 shows that the gravitational force components acting along the body axis are a function of the airplane's orientation in space. The gravitational

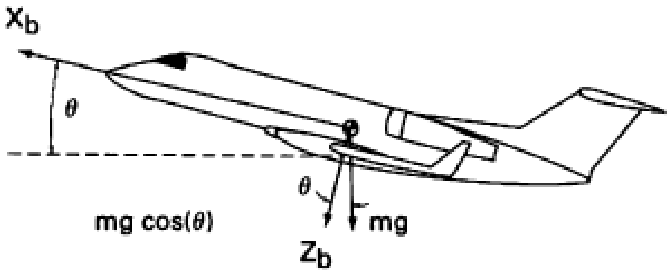


FIGURE 3.4
Components of gravitational force acting along the body axis.

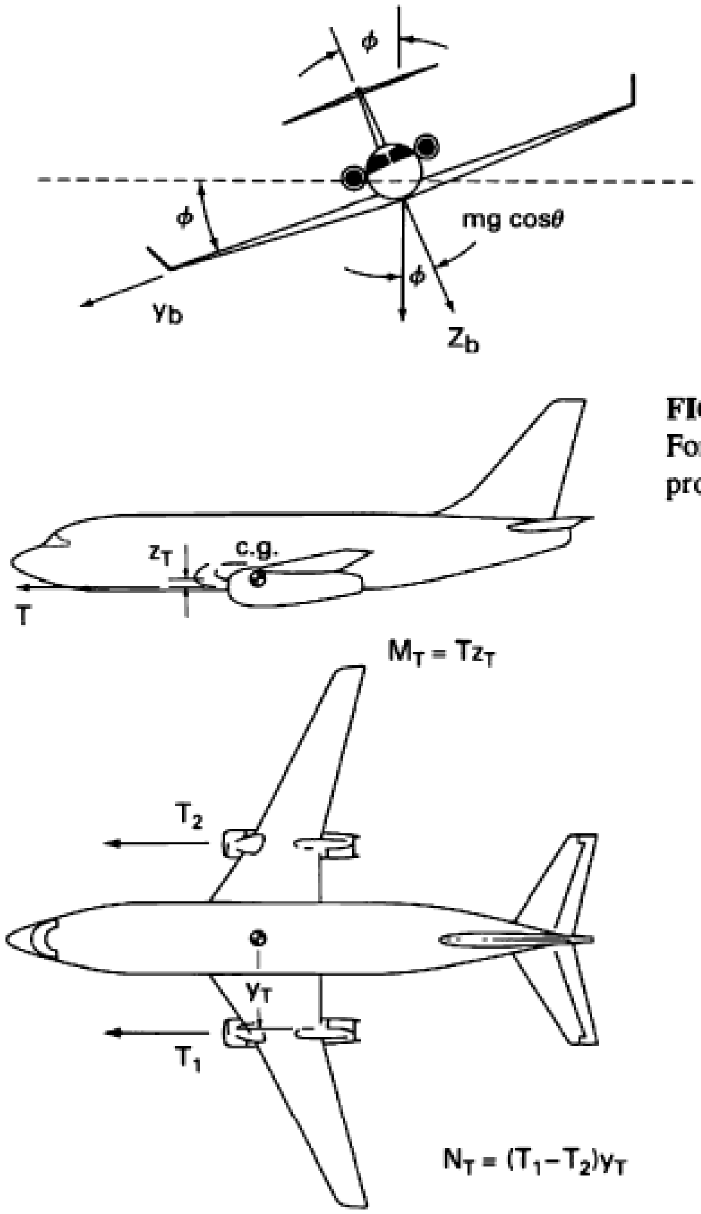


FIGURE 3.5
Force and moments due to propulsion system.

force components along the x, y, and z axes can be easily shown to be

$$
\begin{aligned}
(F_x)_{\text{gravity}} &= -mg\sin\theta \\
(F_y)_{\text{gravity}} &= mg\cos\theta\sin\Phi \\
(F_z)_{\text{gravity}} &= mg\cos\theta\cos\Phi
\end{aligned}
\tag{3.33}
$$

The thrust force due to the propulsion system can have components that act along each of the body axis directions. In addition, the propulsive forces also can create moments if the thrust does not act through the center of gravity. Figure 3.5 shows some examples of moments created by the propulsive system.

The propulsive forces and moments acting along the body axis system are denoted as follows:

$$(F_x)_{\text{propulsive}} = X_T \qquad (F_y)_{\text{propulsive}} = Y_T \qquad (F_z)_{\text{propulsive}} = Z_T \tag{3.34}$$

and

$$(L)_{\text{propulsive}} = L_T \qquad (M)_{\text{propulsive}} = M_T \qquad (N)_{\text{propulsive}} = N_T \tag{3.35}$$

Table 3.1 gives a summary of the rigid body equations of motion.

TABLE 3.1
Summary of kinematic and dynamic equations

$X - mgS_\theta = m(\dot{u} + qw - rv)$ $Y + mgC_\theta S_\Phi = m(\dot{v} + ru - pw)$ $Z + mgC_\theta C_\Phi = m(\dot{w} + pv - qu)$	Force equations
$L = I_x\dot{p} - I_{xz}\dot{r} + qr(I_z - I_y) - I_{xz}pq$ $M = I_y\dot{q} + rp(I_x - I_z) + I_{xz}(p^2 - r^2)$ $N = -I_{xz}\dot{p} + I_z\dot{r} + pq(I_y - I_x) + I_{xz}qr$	Moment equations
$p = \dot{\Phi} - \dot{\psi}S_\theta$ $q = \dot{\theta}C_\Phi + \dot{\psi}C_\theta S_\Phi$ $r = \dot{\psi}C_\theta C_\Phi - \dot{\theta}S_\Phi$	Body angular velocities in terms of Euler angles and Euler rates
$\dot{\theta} = qC_\Phi - rS_\Phi$ $\dot{\Phi} = p + qS_\Phi T_\theta + rC_\Phi T_\theta$ $\dot{\psi} = (qS_\Phi + rC_\Phi)\sec\theta$	Euler rates in terms of Euler angles and body angular velocities

Velocity of aircraft in the fixed frame in terms of Euler angles and body velocity components

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dx}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} C_\theta C_\psi & S_\Phi S_\theta C_\psi - C_\Phi S_\psi & C_\psi S_\theta C_\phi + S_\Phi S_\psi \\ C_\theta S_\psi & S_\Phi S_\theta S_\psi + C_\Phi C_\psi & C_\Phi S_\theta S_\psi - S_\Phi C_\psi \\ -S_\theta & S_\Phi C_\theta & C_\Phi C_\theta \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

4.4 Small Disturbance Theory

The equations developed in the previous section can be linearized using the small-disturbance theory. In applying the small-disturbance theory we assume that the motion of the airplane consists of small deviations about a steady flight condition. Obviously, this theory cannot be applied to problems in which large-amplitude motions are to be expected (e.g., spinning or stalled flight). However, in many cases the small-disturbance theory yields sufficient accuracy for practical engineering purposes.

All the variables in the equations of motion are replaced by a reference value plus a perturbation or disturbance:

$$\begin{array}{lll} u = u_0 + \Delta u & v = v_0 + \Delta v & w = w_0 + \Delta w \\ p = p_0 + \Delta p & q = q_0 + \Delta q & r = r_0 + \Delta r \\ X = X_0 + \Delta X & Y = Y_0 + \Delta Y & Z = Z_0 + \Delta Z \\ M = M_0 + \Delta M & N = N_0 + \Delta N & L = L_0 + \Delta L \\ \delta = \delta_0 + \Delta\delta & & \end{array} \tag{3.36}$$

For convenience, the reference flight condition is assumed to be symmetric and the propulsive forces are assumed to remain constant. This implies that

$$v_0 = p_0 = q_0 = r_0 = \Phi_0 = \psi_0 = 0 \tag{3.37}$$

Furthermore, if we initially align the x axis so that it is along the direction of the airplane's velocity vector, then $w_0 = 0$.

Now, if we introduce the small-disturbance notation into the equations of motion, we can simplify these equations. As an example, consider the X force equation:

$$X - mg \sin \theta = m(\dot{u} + qw - rv) \tag{3.38}$$

Substituting the small-disturbance variables into this equation yields

$$\begin{aligned} X_0 + \Delta X - mg \sin(\theta_0 + \Delta\theta) \\ = m \left[\frac{d}{dt}(u_0 + \Delta u) + (q_0 + \Delta q)(w_0 + \Delta w) - (r_0 + \Delta r)(v_0 + \Delta v) \right] \end{aligned} \tag{3.39}$$

If we neglect products of the disturbance and assume that

$$w_0 = v_0 = p_0 = q_0 = r_0 = \Phi_0 = \psi_0 = 0 \tag{3.40}$$

then the X equation becomes

$$X_0 + \Delta X - mg \sin(\theta_0 + \Delta\theta) = m\,\Delta\dot{u} \tag{3.41}$$

This equation can be reduced further by applying the following trigonometric identity:

$$\sin(\theta_0 + \Delta\theta) = \sin\theta_0 \cos\Delta\theta + \cos\theta_0 \sin\Delta\theta = \sin\theta_0 + \Delta\theta\cos\theta_0$$

Therefore,
$$X_0 + \Delta X - mg(\sin\theta_0 + \Delta\theta\cos\theta_0) = m\,\Delta\dot{u} \tag{3.42}$$

If all the disturbance quantities are set equal to 0 in these equation, we have the reference flight condition

$$X_0 - mg \sin\theta_0 = 0 \tag{3.43}$$

This reduces the X-force equation to

$$\Delta X - mg\,\Delta\theta\cos\theta_0 = m\,\Delta\dot{u} \tag{3.44}$$

The force ΔX is the change in aerodynamic and propulsive force in the x direction and can be expressed by means of a Taylor series in terms of the perturbation variables. If we assume that ΔX is a function only of u, w, δ_e, and δ_T, then ΔX can be expressed as

$$\Delta X = \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w + \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{\partial X}{\partial \delta_T}\Delta\delta_T \tag{3.45}$$

where $\partial X/\partial u$, $\partial X/\partial w$, $\partial X/\partial\delta_e$, and $\partial X/\partial\delta_T$, called stability derivatives, that are evaluated at the reference flight condition. The variables δ_e and δ_T are the change in elevator angle and throttle setting, respectively. If a canard or all-moveable stabilator is used for longitudinal control, then the control term would be replaced by

$$\frac{\partial X}{\partial \delta_H}\Delta\delta_H \qquad \text{or} \qquad \frac{\partial X}{\partial \delta_c}\Delta\delta_c$$

Substituting the expression for ΔX into the force equation yields:

$$\frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w + \frac{\partial X}{\partial \delta_e}\Delta \delta_e + \frac{\partial X}{\partial \delta_T}\Delta \delta_T - mg\,\Delta\theta\cos\theta_0 = m\,\Delta\dot{u} \quad (3.46)$$

or on rearranging

$$\left(m\frac{\mathrm{d}}{\mathrm{d}t} - \frac{\partial X}{\partial u}\right)\Delta u - \left(\frac{\partial X}{\partial w}\right)\Delta w + (mg\cos\theta_0)\,\Delta\theta = \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{\partial X}{\partial \delta_T}\Delta\delta_T$$

The equation can be rewritten in a more convenient form by dividing through by the mass m:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - X_u\right)\Delta u - X_w\,\Delta w + (g\cos\theta_0)\,\Delta\theta = X_{\delta_e}\,\Delta\delta_e + X_{\delta_T}\,\Delta\delta_T \quad (3.47)$$

where $X_u = \partial X/\partial u/m$, $X_w = \partial X/\partial w/m$, and so on are aerodynamic derivatives divided by the airplane's mass.

The change in aerodynamic forces and moments are functions of the motion variables Δu, Δw, and so forth. The aerodynamic derivatives usually the most important for conventional airplane motion analysis follow:

$$\left.\begin{aligned}
\Delta X &= \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w + \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \frac{\partial X}{\partial \delta_T}\Delta\delta_T \\
\Delta Y &= \frac{\partial Y}{\partial v}\Delta v + \frac{\partial Y}{\partial p}\Delta p + \frac{\partial Y}{\partial r}\Delta r + \frac{\partial Y}{\partial \delta_r}\Delta\delta_r \\
\Delta Z &= \frac{\partial Z}{\partial u}\Delta u + \frac{\partial Z}{\partial w}\Delta w + \frac{\partial Z}{\partial \dot{w}}\Delta\dot{w} + \frac{\partial Z}{\partial q}\Delta q \\
&\quad + \frac{\partial Z}{\partial \delta_e}\Delta\delta_e + \frac{\partial Z}{\partial \delta_T}\Delta\delta_T
\end{aligned}\right\} \quad (3.48)$$

$$\left.\begin{aligned}
\Delta L &= \frac{\partial L}{\partial v}\Delta v + \frac{\partial L}{\partial p}\Delta p + \frac{\partial L}{\partial r}\Delta r + \frac{\partial L}{\partial \delta_r}\Delta\delta_r + \frac{\partial L}{\partial \delta_a}\Delta\delta_a \\
\Delta M &= \frac{\partial M}{\partial u}\Delta u + \frac{\partial M}{\partial w}\Delta w + \frac{\partial M}{\partial \dot{w}}\Delta\dot{w} + \frac{\partial M}{\partial q}\Delta q \\
&\quad + \frac{\partial M}{\partial \delta_e}\Delta\delta_e + \frac{\partial M}{\partial \delta_T}\Delta\delta_T \\
\Delta N &= \frac{\partial N}{\partial v}\Delta v + \frac{\partial N}{\partial p}\Delta p + \frac{\partial N}{\partial r}\Delta r + \frac{\partial N}{\partial \delta_r}\Delta\delta_r + \frac{\partial N}{\partial \delta_a}\Delta\delta_a
\end{aligned}\right\} \quad (3.49)$$

The aerodynamic forces and moments can be expressed as a function of all the motion variables; however, in these equations only the terms that are usually significant have been retained. Note also that the longitudinal aerodynamic control surface was assumed to be an elevator. For aircraft that use either a canard or combination of longitudinal controls, the elevator terms in the preceding equations can be replaced by the appropriate control derivatives and angular deflections.

The complete set of linearized equations of motion is presented in Table 3.2.

TABLE 3.2
The linearized small-disturbance longitudinal and lateral rigid body equation of motion

Longitudinal equations

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w\,\Delta w + (g\cos\theta_0)\,\Delta\theta = X_{\delta_e}\,\Delta\delta_e + X_{\delta_T}\,\Delta\delta_T$$

$$-Z_u\,\Delta u + \left[(1 - Z_{\dot{w}})\frac{d}{dt} - Z_w\right]\Delta w - \left[(u_0 + Z_q)\frac{d}{dt} - g\sin\theta_0\right]\Delta\theta = Z_{\delta_e}\,\Delta\delta_e + Z_{\delta_T}\,\Delta\delta_T$$

$$-M_u\,\Delta u - \left(M_{\dot{w}}\frac{d}{dt} + M_w\right)\Delta w + \left(\frac{d^2}{dt^2} - M_q\frac{d}{dt}\right)\Delta\theta = M_{\delta_e}\,\Delta\delta_e + M_{\delta_T}\,\Delta\delta_T$$

Lateral equations

$$\left(\frac{d}{dt} - Y_v\right)\Delta v - Y_p\,\Delta p + (u_0 - Y_r)\,\Delta r - (g\cos\theta_0)\,\Delta\phi = Y_{\delta_r}\,\Delta\delta_r$$

$$-L_v\,\Delta v + \left(\frac{d}{dt} - L_p\right)\Delta p - \left(\frac{I_{xz}}{I_x}\frac{d}{dt} + L_r\right)\Delta r = L_{\delta_a}\Delta\delta_a + L_{\delta_r}\Delta\delta_r$$

$$-N_v\,\Delta v - \left(\frac{I_{xz}}{I_z}\frac{d}{dt} + N_p\right)\Delta p + \left(\frac{d}{dt} - N_r\right)\Delta r = N_{\delta_a}\,\Delta\delta_a + N_{\delta_r}\,\Delta\delta_r$$

4.5 Aerodynamic Force and Moment Representation

In previous sections we represented the aerodynamic force and moment contributions by means of the aerodynamic stability coefficients. We did this without explaining the rationale behind the approach.

The method of representing the aerodynamic forces and moments by stability coefficients was first introduced by Bryan over three-quarters of a century ago [3.1, 3.3]. The technique proposed by Bryan assumes that the aerodynamic forces and moments can be expressed as a function of the instantaneous values of the perturbation variables. The perturbation variables are the instantaneous changes from the reference conditions of the translational velocities, angular velocities, control deflection, and their derivatives. With this assumption, we can express the aerodynamic forces and moments by means of a Taylor series expansion of the perturbation variables about the reference equilibrium condition. For example, the change in the force in the x direction can be expressed as follows:

$$\Delta X(u, \dot{u}, w, \dot{w}, \ldots, \delta_e, \dot{\delta}_e) = \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial \dot{u}}\Delta\dot{u} + \cdots + \frac{\partial X}{\partial \delta_e}\Delta\delta_e + \text{H.O.T. (higher order terms)} \tag{3.50}$$

The term $\partial X/\partial u$, called the stability derivative, is evaluated at the reference flight condition.

The contribution of the change in the velocity u to the change ΔX in the X force is just $[\partial X/\partial u]\,\Delta u$. We can also express $\partial X/\partial u$ in terms of the stability coefficient C_{x_u} as follows:

$$\frac{\partial X}{\partial u} = C_{x_u}\frac{1}{u_0}QS \tag{3.51}$$

where

$$C_{x_u} = \frac{\partial C_x}{\partial(u/u_0)} \tag{3.52}$$

Note that the stability derivative has dimensions, whereas the stability coefficient is defined so that it is nondimensional.

The preceding discussion may seem as though we are making the aerodynamic force and moment representation extremely complicated. However, by assuming that the perturbations are small we need to retain only the linear terms in Equation (3.50). Even though we have retained only the linear terms, the expressions still may include numerous first-order terms. Fortunately, many of these terms also can be neglected because their contribution to a particular force or moment is negligible. For example, we have examined the pitching moment in detail in Chapter 2. If we express the pitching moment in terms of the perturbation variables, as indicated next,

$$M(u, v, w, \dot{u}, \dot{v}, \dot{w}, p, q, r, \delta_a, \delta_e, \delta_r) = \frac{\partial M}{\partial u}\Delta u + \frac{\partial M}{\partial u}\Delta v + \frac{\partial M}{\partial w}\delta w + \cdots + \frac{\partial M}{\partial p}\Delta p + \cdots \tag{3.53}$$

it should be quite obvious that terms such as $(\partial M/\partial v)\,\Delta v$ and $(\partial M/\partial p)\,\Delta p$ are not going to be significant for an airplane. Therefore, we can neglect these terms in our analysis.

In the following sections, we shall use the stability derivative approach to represent the aerodynamic forces and moments acting on the airplane. The expressions developed for each of the forces and moments will include only the terms usually important in studying the airplane's motion. The remaining portion of this chapter is devoted to presentation of methods for predicting the longitudinal and lateral stability coefficients. We will confine our discussion to methods that are applicable to subsonic flight speeds. Note that many of the stability coefficients vary significantly with the Mach number. This can be seen by examining the data on the A-4D airplane in Appendix B or by examining Figure 3.6.

We have developed a number of relationships for estimating the various stability coefficients; for example, expressions for some of the static stability coefficients such as C_{m_α}, C_{n_β} and C_{l_β} were formulated in Chapter 2. Developing prediction methods for all of the stability derivatives necessary for performing vehicle motion analysis would be beyond the scope of this book. Therefore, we shall confine our attention to the development of several important dynamic derivatives and simply refer the reader to the *US Air Force Stability and Control DATCOM* [3.4]. This report is a comprehensive collection of aerodynamic stability and control prediction techniques, which is widely used through the aviation industry.

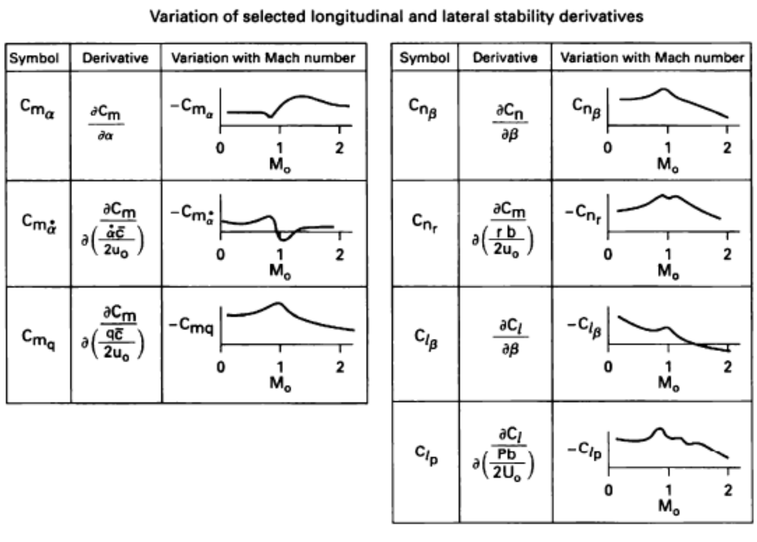


FIGURE 3.6
Variation of selected longitudinal and lateral derivatives with the Mach number.

4.6 Derivatives Due To Change in Forward Speed

The drag, lift, and pitching moments vary with changes in the airplane's forward speed. In addition the thrust of the airplane is also a function of the forward speed. The aerodynamic and propulsive forces acting on the airplane along the X body axes are the drag force and the thrust. The change in the X force, that is, ΔX due to a change in forward speed, can be expressed as

$$\Delta X = \frac{\partial X}{\partial u}\Delta u = -\frac{\partial D}{\partial u}\Delta u + \frac{\partial T}{\partial u}\Delta u \tag{3.54}$$

or

$$\frac{\partial X}{\partial u} = -\frac{\partial D}{\partial u} + \frac{\partial T}{\partial u} \tag{3.55}$$

The derivative $\partial X/\partial u$ is called the speed damping derivative. Equation (3.55) can be rewritten as

$$\frac{\partial X}{\partial u} = -\frac{\rho S}{2}\left(u_0^2 \frac{\partial C_D}{\partial u} + 2u_0 C_{D_0}\right) + \frac{\partial T}{\partial u} \tag{3.56}$$

where the subscript 0 indicates the reference condition. Expressing $\partial X/\partial u$ in coefficient form yields

$$C_{X_u} = -(C_{D_u} + 2C_{D_0}) + C_{T_u} \tag{3.57}$$

where

$$C_{D_u} = \frac{\partial C_D}{\partial(u/u_0)} \quad \text{and} \quad C_{T_u} = \frac{\partial C_T}{\partial(u/u_0)} \tag{3.58}$$

are the changes in the drag and thrust coefficients with forward speed. These coefficients have been made nondimensional by differentiating with respect to (u/u_0). The coefficient C_{D_u} can be estimated from a plot of the drag coefficient versus the Mach number:

$$C_{D_u} = \mathbf{M}\frac{\partial C_D}{\partial \mathbf{M}} \tag{3.59}$$

where $\mathbf{M}$ is the Mach number of interest. The thrust term C_{T_u} is 0 for gliding flight; it also is a good approximation for jet powered aircraft. For a variable pitch propeller and piston engine power plant, C_{T_u} can be approximated by assuming it to be equal to the negative of the reference drag coefficient (i.e., $C_{T_u} = -C_{D_0}$).

The change in the Z force with respect to forward speed can be shown to be

$$\frac{\partial Z}{\partial u} = -\frac{1}{2}\rho S u_0[C_{L_u} + 2C_{L_0}] \tag{3.60}$$

or in coefficient form as

$$C_{Z_u} = -[C_{L_u} + 2C_{L_0}] \tag{3.61}$$

The coefficient C_{L_u} arises form the change in lift coefficient with the Mach number. C_{L_u} can be estimated from the Prandtl-Glauent formula, which corrects the incompressible lift coefficient for the Mach number effects:

$$C_L = \frac{C_L|_{M=0}}{\sqrt{1-\mathbf{M}^2}} \tag{3.62}$$

Differentiating the list coefficient with respect to the Mach number yields

$$\frac{\partial C_L}{\partial \mathbf{M}} = \frac{\mathbf{M}}{1 - \mathbf{M}^2} C_L \tag{3.63}$$

but
$$C_{L_u} = \frac{\partial C_L}{\partial (u/u_0)} = \frac{u_o}{a} \frac{\partial C_L}{\partial \left(\frac{u}{a}\right)} \tag{3.64}$$

$$= \mathbf{M} \frac{\partial C_L}{\partial \mathbf{M}} \tag{3.65}$$

where a is the speed of sound.

C_{L_u} therefore can be expressed as

$$C_{L_u} = \frac{\mathbf{M}^2}{1 - \mathbf{M}^2} C_{L_0} \tag{3.66}$$

This coefficient can be neglected at low flight speeds but can become quite large near the critical Mach number for the airplane.

The change in the pitching moment due to variations in the forward speed can be expressed as

$$\Delta M = \frac{\partial M}{\partial u} \Delta u \tag{3.67}$$

or
$$\frac{\partial M}{\partial u} = C_{m_u} \rho S \bar{c} u_0 \tag{3.68}$$

The coefficient C_{m_u} can be estimated as follows:

$$C_{m_u} = \frac{\partial C_m}{\partial \mathbf{M}} \mathbf{M} \tag{3.69}$$

The coefficient C_{m_u} depends on the Mach number but also is affected by the elastic properties of the airframe. At high speeds aeroelastic bending of the airplane can cause large changes in the magnitude of C_{m_u}.

4.7 Derivatives Due To The Pitching Velocity, q

The stability coefficients C_{z_q} and C_{m_q} represent the change in the Z force and pitching moment coefficients with respect to the pitching velocity q . The aerodynamic characteristics of both the wing and the horizontal tail are affected by the pitching motion of the airplane. The wing contribution usually is quite small in comparison to that produced by the tail. A common practice is to compute the tail contribution and then increase it by 10 percent to account for the wing. Figure 3.7 shows an airplane undergoing a pitching motion.

As illustrated in Figure 3.7, the pitching rate q causes a change in the angle of attack at the tail, which results in a change in the lift force acting on the tail:

$$\Delta L_t = C_{L_{\alpha_t}} \Delta\alpha_t Q_t S_t \tag{3.70}$$

or

$$\Delta Z = -\Delta L_t = -C_{L_{\alpha_t}} \frac{ql_t}{u_0} Q_t S_t \tag{3.71}$$

$$C_Z = \frac{Z}{QS} \tag{3.72}$$

$$\Delta C_Z = -C_{L_{\alpha_t}} \frac{ql_t}{u_0} \eta \frac{S_t}{S} \tag{3.73}$$

$$C_{Z_q} \equiv \frac{\partial C_z}{\partial(q\bar{c}/2u_0)} = \frac{2u_0}{\bar{c}} \frac{\partial C_z}{\partial q} \tag{3.74}$$

$$C_{z_q} = -2C_{L_{\alpha_t}} \eta V_H \tag{3.75}$$

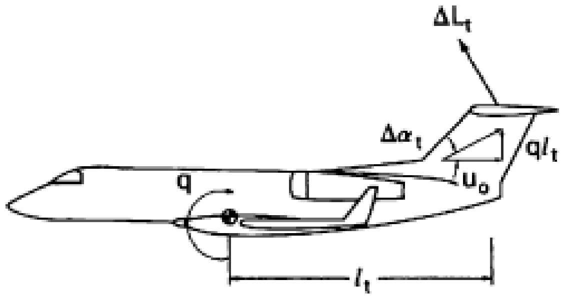


FIGURE 3.7
Mechanism for aerodynamic force due to pitch rate.

The pitching moment due to the change in lift on the tail can be calculated as follows:

$$\Delta M_{cg} = -l_t \Delta L_t \tag{3.76}$$

$$\Delta C_{m_{cg}} = -V_H \eta C_{L_{\alpha_t}} \frac{ql_t}{u_0} \tag{3.77}$$

$$C_{m_q} \equiv \frac{\partial C_m}{\partial(q\bar{c}/2u_0)} = \frac{2u_0}{\bar{c}} \frac{\partial C_m}{\partial q} \tag{3.78}$$

$$C_{m_q} = -2C_{L_{\alpha_t}} \eta V_H \frac{l_t}{\bar{c}} \tag{3.79}$$

Equations (3.75) and (3.79) represent the tail contribution to C_{z_q} and C_{m_q}, respectively. The coefficients for the complete airplane are obtained by increasing the tail values by 10 percent to account for the wing and fuselage contributions.

4.8 Derivatives Due To The Time Rate of Change of Angle of Attack

The stability coefficients $C_{z_{\dot{\alpha}}}$ and $C_{m_{\dot{\alpha}}}$ arise because of the lag in the wing downwash getting to the tail. As the wing angle of attack changes, the circulation around the wing will be altered. The change in circulation alters the downwash at the tail; however, it takes a finite time for the alteration to occur. Figure 3.8 illustrates the lag in flow field development. If the airplane is traveling with a forward velocity u_0, then a change in circulation imparted to the trailing vortex wake will take the increment in time $\Delta t = l_t/u_0$ to reach the tail surface.

The lag in angle of attack at the tail can be expressed as

$$\Delta\alpha_t = \frac{d\varepsilon}{dt}\Delta t \tag{3.80}$$

where $$\Delta t = l_t/u_0 \tag{3.81}$$

or $$\Delta\alpha_t = \frac{d\varepsilon}{dt}\frac{l_t}{u_0} = \frac{d\varepsilon}{d\alpha}\frac{d\alpha}{dt}\frac{l_t}{u_0} \tag{3.82}$$

$$= \frac{d\varepsilon}{d\alpha}\dot{\alpha}\frac{l_t}{u_0} \tag{3.83}$$

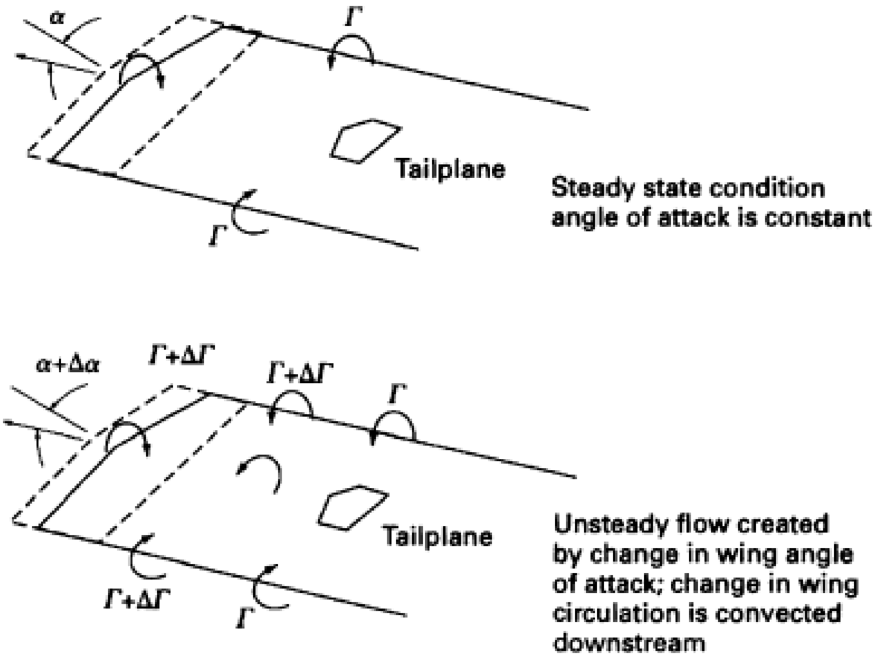


FIGURE 3.8
Mechanism for aerodynamic force due to the lag in flow field development.

The change in the lift force can be expressed as

$$\Delta L_t = C_{L_{\alpha_t}}\Delta\alpha_t Q_t S_t \tag{3.84}$$

or in terms of the z force coefficient

$$\Delta C_z = -\frac{\Delta L_t}{QS} = -C_{L_{\alpha_t}}\Delta\alpha_t\,\eta\frac{S_t}{S} \tag{3.85}$$

$$= -C_{L_{\alpha_t}}\frac{d\varepsilon}{d\alpha}\dot{\alpha}\frac{l_t}{u_0}\eta\frac{S_t}{S} \tag{3.86}$$

$$C_{Z_{\dot{\alpha}}} \equiv \frac{\partial C_z}{\partial(\dot{\alpha}\bar{c}/2u_0)} = \frac{2u_0}{\bar{c}} \frac{\partial C_z}{\partial \dot{\alpha}} \tag{3.87}$$

$$= -2V_H \eta C_{L_{\alpha_t}} \frac{d\varepsilon}{d\alpha} \tag{3.88}$$

The pitching moment due to the lag in the downwash field can be calculated as follows:

$$\Delta M_{cg} = -l_t \, \Delta L_t = -l_t C_{L_{\alpha_t}} \Delta\alpha_t \, Q_t S_t \tag{3.89}$$

$$\Delta C_{m_{cg}} = -V_H \eta C_{L_{\alpha_t}} \frac{d\varepsilon}{d\alpha} \dot{\alpha} \frac{l_t}{u_0} \tag{3.90}$$

$$C_{m_{\dot{\alpha}}} = \frac{\partial C_m}{\partial(\dot{\alpha}\bar{c}/2u_0)} = \frac{2u_0}{\bar{c}} \frac{\partial C_m}{\partial \dot{\alpha}} \tag{3.91}$$

$$= -2C_{L_{\alpha_t}} \eta V_H \frac{l_t}{\bar{c}} \frac{d\varepsilon}{d\alpha} \tag{3.92}$$

Equations (3.89) and (3.92) yield only the tail contribution to these stability coefficients. To obtain an estimate for the complete airplane these coefficients are increased by 10 percent. A summary of the equations for estimating the longitudinal stability coefficients is included in Table 3.3.

TABLE 3.3
Equations for estimating the longitudinal stability coefficients

	X-force derivatives	Z-force derivatives	Pitching moment derivatives
u	$C_{X_u} = -[C_{D_u} + 2C_{D_0}] + C_{T_u}$	$C_{z_u} = -\frac{M^2}{1-M^2} C_{L_0} - 2C_{L_0}$	$C_{m_u} = \frac{\partial C_m}{\partial M} M_0$
α	$C_{X_\alpha} = C_{L_0} - \frac{2C_{L_0}}{\pi e} \frac{C_{L_\alpha}}{AR}$	$C_{Z_\alpha} = -(C_{L_\alpha} + C_{D_0})$	$C_{m_\alpha} = C_{L_{\alpha w}}\left(\frac{x_{cg}}{\bar{c}} - \frac{X_{ac}}{\bar{c}}\right) + C_{m_{\alpha fus}} - \eta V_H C_{L_{\alpha t}}\left(1 - \frac{d\varepsilon}{d\alpha}\right)$
$\dot{\alpha}$	0	$C_{Z_{\dot{\alpha}}} = -2\eta C_{L_{\alpha t}} V_H \frac{d\varepsilon}{d\alpha}$	$C_{m_{\dot{\alpha}}} = -2\eta C_{L_{\alpha t}} V_H \frac{l_t}{c} \frac{d\varepsilon}{d\alpha}$
q	0	$C_{z_q} = -2\eta C_{L_{\alpha t}} V_H$	$C_{m_q} = -2\eta C_{L_{\alpha t}} V_H \frac{l_t}{c}$
α_e	0	$C_{Z_{\delta_e}} = -C_{L_{\delta_e}} = -\frac{S_t}{S}\eta \frac{dC_{L_t}}{d\delta_e}$	$C_{m_{\delta_e}} = -\eta V_H \frac{dC_{L_t}}{d\delta_e}$

Subscript 0 indicates reference values and M is the Mach number.

AR	Aspect ratio	V_H	Horizontal tail volume ratio
C_{D_0}	Reference drag coefficient	M	Flight mach number
C_{L_0}	Reference lift coefficient	S	Wing area
C_{L_α}	Airplane lift curve slope	S_t	Horizontal tail area
$C_{L_{\alpha w}}$	Wing lift curve slope	$\frac{d\varepsilon}{d\alpha}$	Change in downwash due to a change in angle of attack
$C_{L_{\alpha t}}$	Tail lift curve slope	η	Efficiency factor of the horizontal tail
$\bar{c}$	Mean aerodynamic chord		
e	Oswald's span efficiency factor		
l_t	Distance from center of gravity to tail quarter chord		

4.9 Derivative Due To The Rolling Rate, p

The stability coefficients C_{y_p}, C_{n_p}, and C_{l_p} arise due to the rolling angular velocity, p. When an airplane rolls about its longitudinal axis, the roll rate creates a linear velocity distribution over the vertical, horizontal, and wing surfaces. The velocity distribution causes a local change in angle of attack over each of these surfaces that results in a change in the lift distribution and, consequently, the moment about the center of gravity. In this section we will examine how the roll rate creates a rolling moment. Figure 3.9 shows a wing planform rolling with a positive rolling velocity. On the portion of the wing rolling down, an increase in angle of attack is created by the rolling motion. This results in an increase in the lift distribution over the downward-moving wing. If we examine the upward-moving part of the wing we observe that the rolling velocity causes a decrease in the local angle of attack and

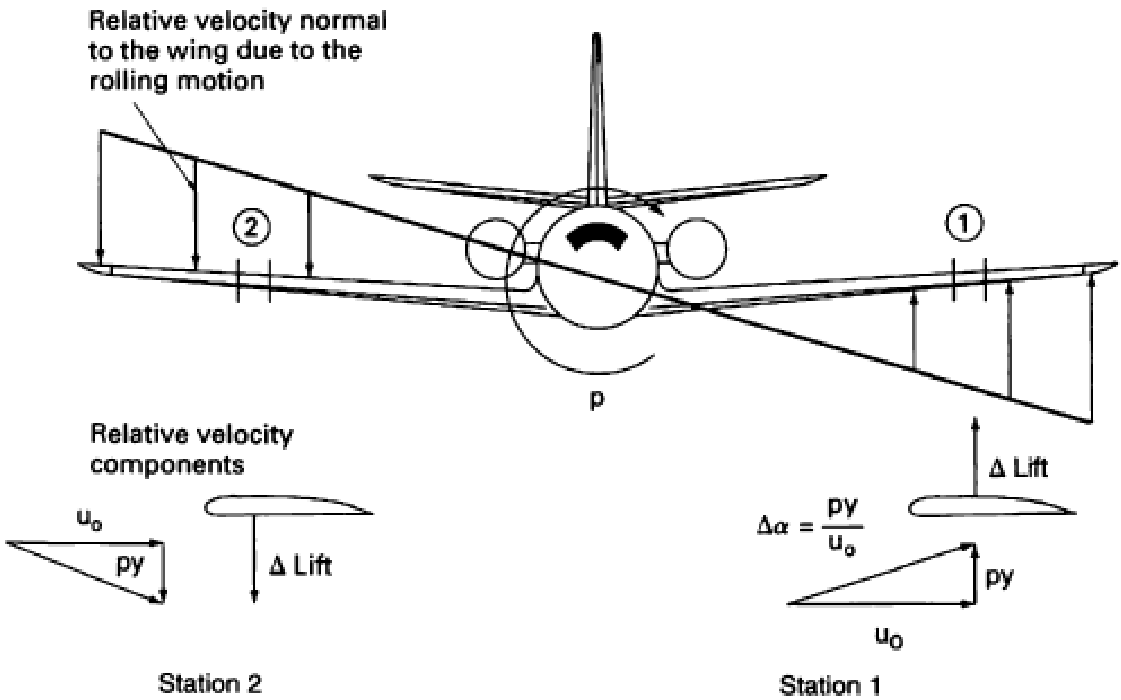


FIGURE 3.9
Wing planform undergoing a rolling motion.

the lift distribution decreases. The change in the lift distribution across the wing produces a rolling moment that opposes the rolling motion and is proportional to the roll rate, p. In Figure 3.9 the negative rolling velocity induces a positive rolling moment.

An estimate of the rolling damping derivative, C_{l_p}, due to the wing surface can be developed in the following manner. The incremental lift force created by rolling motion can be expressed as

$$d(\text{Lift}) = C_{l_\alpha} \Delta\alpha Q c \, dy \tag{3.93}$$

where $\Delta\alpha = py/u_0$.

The incremental roll moment can be estimated by multiplying the incremental lift by the moment arm y:

$$dL = -C_{l_\alpha}\left(\frac{py}{u_0}\right)Qcy\ dy \tag{3.94}$$

The total roll moment now can be calculated by integrating the moment contribution across the wing:

$$L = -2\int_0^{b/2} C_{l_\alpha}\left(\frac{py}{u_0}\right)Qcy\ dy \tag{3.95}$$

or in coefficient form

$$C_l = -\frac{2p}{Sbu_0}\int_0^{b/2} C_{l_\alpha} cy^2\ dy \tag{3.96}$$

To simplify this integral, the sectional lift curve slope is approximated by the wing lift curve slope as follows:

$$C_l = -\frac{2C_{L_{\alpha_w}}}{Sb}\left(\frac{p}{u_0}\right)\int_0^{b/2} cy^2\ dy \tag{3.97}$$

The roll damping coefficient C_{l_p} is defined in terms of a nondimensional roll rate:

$$C_{l_p} \equiv \frac{\partial C_l}{\partial\left(\frac{pb}{2u_0}\right)} \tag{3.98}$$

Differentiating Equation (3.98) yields

$$C_{l_p} = -\frac{4C_{L_{\alpha_w}}}{Sb^2}\int_0^{b/2} cy^2\ dy \tag{3.99}$$

which provides an estimate to C_{l_p}, the roll damping coefficient due to wing surface. From this simple analysis we readily can see that C_{l_p} depends on the wing span. Wings of large span or high aspect ratio will have larger roll damping than low aspect ratio wings of small wing span.

The roll damping of the airplane is made up of contributions from the wing, horizontal, and vertical tail surfaces. The wing, typically being the largest aerodynamic surface, provides most of the roll damping. This is not necessarily the case for aircraft having low aspect ratio wings or missile configurations; for these configurations, the other components may contribute as much to the roll damping coefficients as the wing.

4.10 Derivative Due To The Yawing Rate, r

The stability coefficient C_{y_r}, C_{n_r}, and C_{l_r} are caused by the yawing angular velocity, r. A yawing rate causes a change in the side force acting on the vertical tail surface as illustrated in Figure 3.10. As in the case of the other angular rate coefficients the angular motion creates a local change in the angle of attack or in this case a change in sideslip angle of the vertical tail.

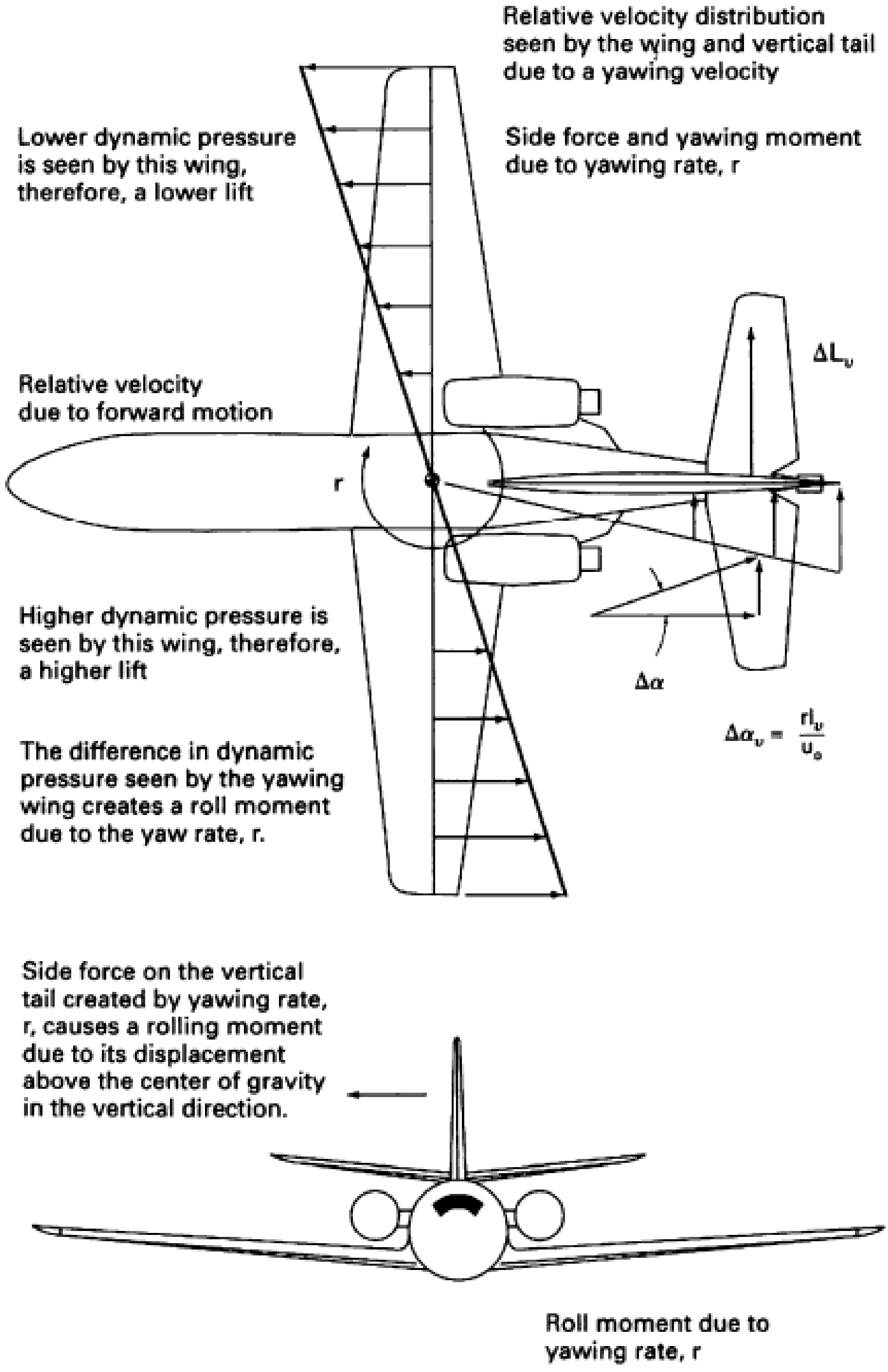


FIGURE 3.10
Influence of the yawing rate on the wing and vertical tail.

A positive yaw rate produces a negative sideslip angle on the vertical tail. The side force created by the negative sideslip angle is in the positive direction:

$$Y = -C_{L_{\alpha_v}} \Delta\beta\, Q_v S_v \tag{3.100}$$

where $\Delta\beta = -rl_v/u_0$ for a positive yawing rate. Rewritting Equation (3.100) in coefficient form yields

$$C_y = \frac{C_{L_{\alpha_v}}\left(\frac{rl_v}{u_0}\right)Q_v S_v}{QS} \tag{3.101}$$

$$= C_{L_{\alpha_v}}\left(\frac{rl_v}{u_0}\right)\eta_v \frac{S_v}{S} \tag{3.102}$$

The stability coefficient C_{y_r} is defined in terms of the nondimensional yaw rate as follows:

$$C_{y_r} \equiv \frac{\partial C_y}{\partial\left(\frac{rb}{2u_0}\right)} \tag{3.103}$$

Taking the derivative of C_y with respect to $rb/2u_0$ yields

$$C_{y_r} = 2C_{L_{\alpha_v}}\eta_v \frac{S_v}{S}\frac{l_v}{b} \tag{3.104}$$

The term $C_{L_{\alpha_v}}\eta_v \frac{S_v}{S}$ is approximately $-C_{y\beta_{\text{tail}}}$; therefore,

$$C_{y_r} = -2C_{y_{\beta_{\text{tail}}}}\frac{l_v}{b} \tag{3.105}$$

The stability coefficients, C_{n_r}, which is the change in yaw moment coefficient with respect to a nondimensional yaw rate $rb/(2u_0)$, is made up of contributions from the wing and the vertical tail. The vertical tail contribution is derived next. The yaw moment produced by the yawing rate is a result of the sideslip angle induced on the vertical tail. A positive yaw rate produces a negative sideslip at the vertical tail or a positive side force on the tail. A positive side force causes a negative yawing moment; therefore,

$$N = C_{L_{\alpha_v}} \Delta\beta Q_v S_v l_v \tag{3.106}$$

But $\Delta\beta = -rl_v/u_0$ for a positive yawing rate:

$$N = -C_{L_{\alpha_v}} \left(\frac{rl_v}{u_0}\right) Q_v S_v l_v \tag{3.107}$$

Or in coefficient form

$$C_n = -C_{L_{\alpha_v}} \left(\frac{rl_v}{u_0}\right) \eta_v V_v \tag{3.108}$$

where $\eta_v = Q_v/Q$ and $V_v = S_v l_v/Sb$,

The stability coefficient C_{n_r} is defined as

$$C_{n_r} \equiv \frac{\partial C_n}{\partial\left(\dfrac{rb}{2u_0}\right)} \tag{3.109}$$

$$= -2C_{L_{\alpha_v}} \eta_v V_v \frac{l_v}{b} \tag{3.110}$$

The vertical tail contribution to C_{n_r} also can expressed in terms of the side force coefficient with respect to sideslip:

$$C_{n_r} \simeq 2C_{y_{\beta_{\text{tail}}}} \left(\frac{l_v}{b}\right)^2 \tag{3.111}$$

The yaw rate, r, also produces a roll moment. The stability coefficient C_{l_r} is due to both the wing and the vertical tail. An expression for estimating C_{l_r} is given in Table 3.4. As shown earlier the yawing rate creates a side force on the vertical tail that is proportional to the yaw rate, r. Because this force acts above the center of gravity a rolling moment is created. The contribution of the wing to C_{l_r} is due to the change in velocity across the wing in the plane of the motion. Development of an expression for C_{l_r} due to the wing and the vertical tail is left as an exercise problem at the end of this chapter.

In this section we have attempted to provide a physical explanation of some of the stability coefficients. This was accomplished by simple models of the flow physics responsible for the creation of the force and moments due to the motion variables such as p, q, and r. Most of the simple expressions developed for estimating a particular stability coefficient were limited to only the contribution due to the primary aircraft component; that is, either the wing, horizontal, or vertical tail surface. To provide a more complete analysis of the aerodynamic stability coefficients a more detailed analysis is required than has been presented in this chapter. References [3.4] and [3.5] provide a more complete set of stability and control prediction methods.

The stability coefficients C_{l_p}, C_{n_r}, C_{z_q}, C_{m_q}, $C_{z_{\dot\alpha}}$, and $C_{m_{\dot\alpha}}$ all oppose the motion of the vehicle and thus can be considered as damping terms. This will become more apparent as we analyze the motion of an airplane in Chapters 4 and 5.

TABLE 3.4
Equations for estimating the lateral stability coefficients

	Y-force derivatives	Yawing moment derivatives	Rolling moment derivatives
β	$C_{y_\beta} = -\eta \frac{S_v}{S} C_{L_{\alpha_v}}\left(1+\frac{d\sigma}{d\beta}\right)$	$C_{n_\beta} = C_{n_{\beta_{wf}}} + \eta_v V_v C_{L_{\alpha_v}}\left(1+\frac{d\sigma}{d\beta}\right)$	$C_{l_\beta} = \left(\frac{C_{l_\beta}}{\Gamma}\right)\Gamma + \Delta C_{l_\beta}$ (see Figure 3.11)
p	$C_{y_p} = C_L \frac{AR+\cos\Lambda}{AR+4\cos\Lambda}\tan\Lambda$	$C_{n_p} = -\frac{C_L}{8}$	$C_{l_p} = -\frac{C_{L_\alpha}}{12}\frac{1+3\lambda}{1+\lambda}$
r	$C_{y_r} = -2\left(\frac{l_v}{b}\right)(C_{y_\beta})_{tail}$	$C_{n_r} = -2\eta_v V_v\left(\frac{l_v}{b}\right)C_{L_{\alpha_v}}$	$C_{l_r} = \frac{C_L}{4} - 2\frac{l_v}{b}\frac{z_v}{b}C_{y_{\beta_{tail}}}$
δ_a	0	$C_{n_{\delta_a}} = 2KC_{L_0}C_{l_{\delta_a}}$ (see Figure 3.12)	$C_{l_{\delta_a}} = \frac{2C_{L_{\alpha_w}}\tau}{Sb}\int_{y_1}^{y_2} cy\,dy$
δ_r	$C_{y_{\delta_r}} = \frac{S_v}{S}\tau C_{L_{\alpha_v}}$	$C_{n_{\delta_r}} = -V_v\eta_v\tau C_{L_{\alpha_v}}$	$C_{l_{\delta_r}} = \frac{S_v}{S}\left(\frac{z_v}{b}\right)\tau C_{L_{\alpha_v}}$

- AR Aspect ratio
- b Wingspan
- C_{L_0} Reference lift coefficient
- C_{L_α} Airplane lift curve slope
- $C_{L_{\alpha_w}}$ Wing lift curve slope
- $C_{L_{\alpha_t}}$ Tail lift curve slope
- $\bar{c}$ Mean aerodynamic chord
- K empirical factor
- l_v Distance from center of gravity to vertical tail aerodynamic center
- V_v Vertical tail volume ratio
- S Wing area
- S_v Vertical tail area
- z_v Distance from center of pressure of vertical tail to fuselage centerline
- Γ Wing dihedral angle
- Λ Wing sweep angle
- η_v Efficiency factor of the vertical tail
- λ Taper ratio (tip chord/root chord)
- $\frac{d\sigma}{d\beta}$ Change in sidewash angle with a change in sideslip angle

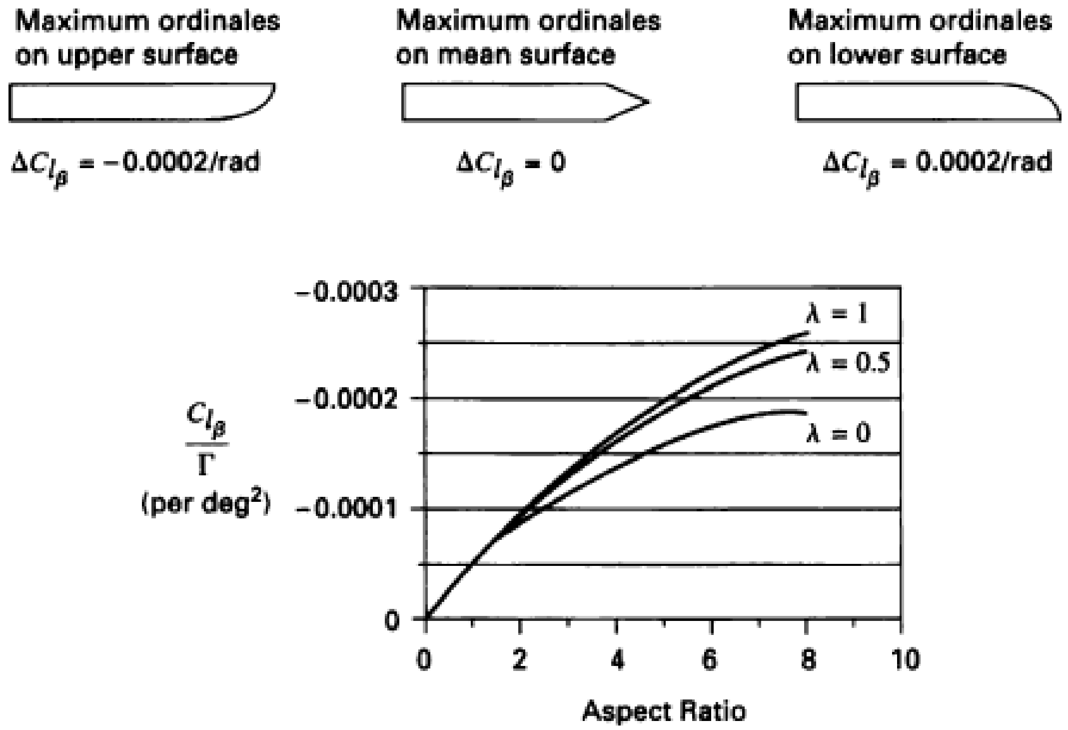


FIGURE 3.11
Tip shape and aspect ratio effect on C_{l_β}.

$$\eta = \frac{Y_1}{b_w/2} = \frac{\text{Spanwise distance from centerline to the inboard edge of the aileron control}}{\text{Semispan}}$$

FIGURE 3.12
Empirical factor for $C_{n_{\delta_a}}$ estimate.

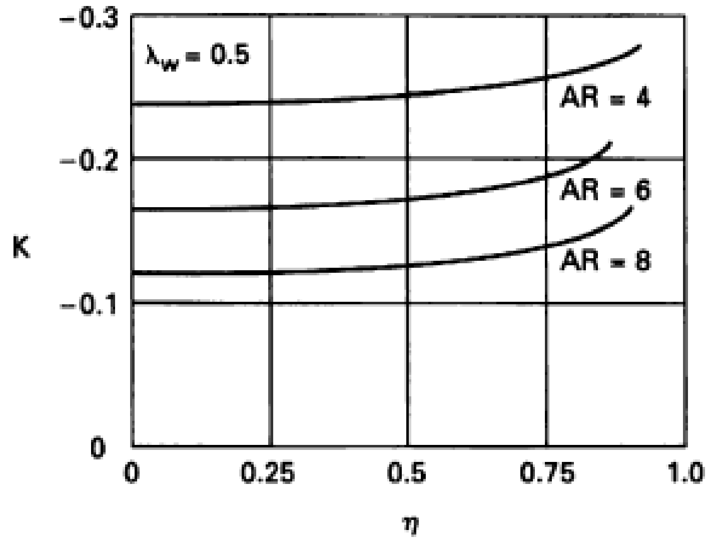


TABLE 3.5
Summary of longitudinal derivatives

$X_u = \dfrac{-(C_{D_u} + 2C_{D_0})QS}{mu_0}\,(\mathrm{s}^{-1})$	$X_w = \dfrac{-(C_{D_\alpha} - C_{L_0})QS}{mu_0}\,(\mathrm{s}^{-1})$
$Z_u = \dfrac{-(C_{L_u} + 2C_{L_0})QS}{mu_0}\,(\mathrm{s}^{-1})$	
$Z_w = \dfrac{-(C_{L_\alpha} + C_{D_0})QS}{mu_0}\,(\mathrm{s}^{-1})$	$Z_{\dot{w}} = -C_{z_{\dot{\alpha}}}\dfrac{c}{2u_0}QS/(u_0 m)$
$Z_\alpha = u_0 Z_w\,(\mathrm{ft/s^2})$ or $(\mathrm{m/s^2})$	$Z_{\dot{\alpha}} = u_0 Z_{\dot{w}}\,(\mathrm{ft/s})$ or $(\mathrm{m/s})$
$Z_q = -C_{Z_q}\dfrac{c}{2u_0}QS/m\,(\mathrm{ft/s})$ or $(\mathrm{m/s})$	$Z_{\delta_e} = -C_{Z_{\delta_e}}QS/m\,(\mathrm{ft/s^2})$
$M_u = C_{m_u}\dfrac{(QSc)}{u_0 I_y}\left(\dfrac{1}{\mathrm{ft\cdot s}}\right)$ or $\left(\dfrac{1}{\mathrm{m\cdot s}}\right)$	
$M_w = C_{m_\alpha}\dfrac{(QS\bar{c})}{u_0 I_y}\left(\dfrac{1}{\mathrm{ft\cdot s}}\right)$ or $\left(\dfrac{1}{\mathrm{m\cdot s}}\right)$	$M_{\dot{w}} = C_{m_{\dot{\alpha}}}\dfrac{\bar{c}}{2u_0}\dfrac{QS\bar{c}}{u_0 I_y}\,(\mathrm{ft}^{-1})$
$M_\alpha = u_0 M_w\,(\mathrm{s}^{-2})$	$M_{\dot{\alpha}} = u_0 M_{\dot{w}}\,(\mathrm{s}^{-1})$
$M_q = C_{m_q}\dfrac{\bar{c}}{2u_0}(QS\bar{c})/I_y\,(\mathrm{s}^{-1})$	$M_{\delta_e} = C_{m_{\delta_e}}(QS\bar{c})/I_y\,(\mathrm{s}^{-2})$

TABLE 3.6
Summary of lateral directional derivatives

$Y_\beta = \dfrac{QSC_{y\beta}}{m}$ (ft/s²) or (m/s²)	$N_\beta = \dfrac{QSbC_{n\beta}}{I_z}$ (s^{-2})	$L_\beta = \dfrac{QSbC_{l\beta}}{I_x}$ (s^{-2})
$Y_p = \dfrac{QSbC_{y_p}}{2mu_0}$ (ft/s) (m/s)	$N_p = \dfrac{QSb^2C_{n_p}}{2I_z u_0}$ (s^{-1})	$L_p = \dfrac{QSb^2C_{l_p}}{2I_x u_0}$ (s^{-1})
$Y_r = \dfrac{QSbC_{y_r}}{2mu_0}$ (ft/s) or (m/s)	$N_r = \dfrac{QSb^2C_{n_r}}{2I_z u_0}$ (s^{-1})	$L_r = \dfrac{QSb^2C_{l_r}}{2I_x u_0}$ (s^{-1})
$Y_{\delta a} = \dfrac{QSC_{y\delta_a}}{m}$ (ft/s²) or (m/s²)	$Y_{\delta_r} = \dfrac{QSC_{y\delta_r}}{m}$ (ft/s²) or (m/s²)	
$N_{\delta_a} = \dfrac{QSbC_{n\delta_a}}{I_z}$ (s^{-2})	$N_{\delta_r} = \dfrac{QSbC_{n\delta_r}}{I_z}$ (s^{-2})	
$L_{\delta_a} = \dfrac{QSbC_{l\delta_a}}{I_x}$ (s^{-2})	$L_{\delta_r} = \dfrac{QSbC_{l\delta_r}}{I_x}$ (s^{-2})	

As noted earlier, there are many more derivatives for which we could develop prediction methods. The few simple examples presented here should give the reader an appreciation of how one would go about determining estimates of the aerodynamic stability coefficients. A summary of some of the theoretical prediction methods for some of the more important lateral and longitudinal stability coefficients is presented in Tables 3.3 and 3.4. Tables 3.5 and 3.6 summarize the longitidinal and lateral derivatives.