Theory of Vibration - 18AE56

Old VTU Question's Answers

Module – 1

Syllabus:
Introduction: Types of vibrations, S.H.M, principle of super position applied to Simple Harmonic Motions. Beats, Fourier theorem and simple problems.

Part – A & B Questions (Mixing of Questions Expected)

1. Define vibration. List the root causes of vibrations and different methods of eliminating undesirable vibrations

A motion which repeats itself after a certain interval of time may be called as vibration. Vibration is the motion of a particle or a body or a system of connected bodies displaced from the position of equilibrium. Vibration occurs when a system is displaced from a position of stable equilibrium. The system tends to return to this equilibrium position under the action of restoring forces. The system keeps on moving back and forth across its position of equilibrium.

The root causes of vibration are :

i) Unbalanced forces in the machine.
ii) External excitations.
iii) Dry friction between two mating surfaces.
iv) Earthquakes.
v) Winds.

Most vibrations are undesirable as they produce excessive stresses, energy losses, increase bearing loads, induce fatigue, undesirable noise, partial or complete failure of parts etc.

This undesirable vibration can be eliminated or reduced by one or more of the following methods.

1. Using shock absorbers.
2. Dynamic vibration absorbers.
3. Resting the system on proper vibration isolators.
4. Removing the causes of vibrations.

2. Define the terms: (i) Periodic motion (ii) Natural frequency (iii) Resonance (iv) Degree of freedom (v) Simple harmonic motion (vi) Damping (vii) Amplitude

i) Periodic Motion

A motion which repeats itself in equal interval of time is known as periodic motion.

ii) Cycle

It is the motion completed during one time period.

iii) Time Period

It is the time taken to complete one cycle.

iv) Frequency

It is the number of cycles per unit time.

v) Amplitude

The maximum displacement of a vibrating body from the mean position is called amplitude.

vi) Natural frequency

When no external force acts on the body after giving it an initial displacement, then the body is said to be under free or natural vibration. The frequency of free vibration is called natural frequency. It is expressed in rad/sec or Hertz.

vii) Damping

It is the resistance to the motion of the vibrating body.

viii) Resonance

When the frequency of external excitation is equal to the natural frequency of system, a state of resonance is said to have been reached. At resonance the amplitude of vibration is excessively large.

ix) Phase Difference

It is the angle between two rotating vectors representing simple harmonic motions of the same frequency.

Consider two vectors x_1 and x_2 having frequencies ω rad/sec each. The vibrating motions can be expressed as

$$x_1 = A_1 \sin \omega t \qquad \text{----- (1.2.1)}$$

$$x_2 = A_2 \sin (\omega t + \phi) \qquad \text{----- (1.2.2)}$$

Because of the quantity ϕ the two vibrations do not attain their maximum displacements at the same time. The quantity ϕ is known as the phase difference or phase angle.

x) Degrees of freedom

The number of independent co-ordinates required to describe the motion of a system is called degrees of freedom. A system is said to be n-degrees of freedom system, if it needs 'n' independent coordinates to specify completely the configuration of the system at any instant. In single degree of freedom there is only one independent coordinate 'x_1' to specify the configuration as shown in Fig. 1.1 (a). Similarly in two degrees of freedom there are two coordinates x_1, x_2 to specify the configuration as shown in Fig. 1.1 (b) and so on. A cantilever beam shown in Fig. 1.1 (c) has infinite degrees of freedom.

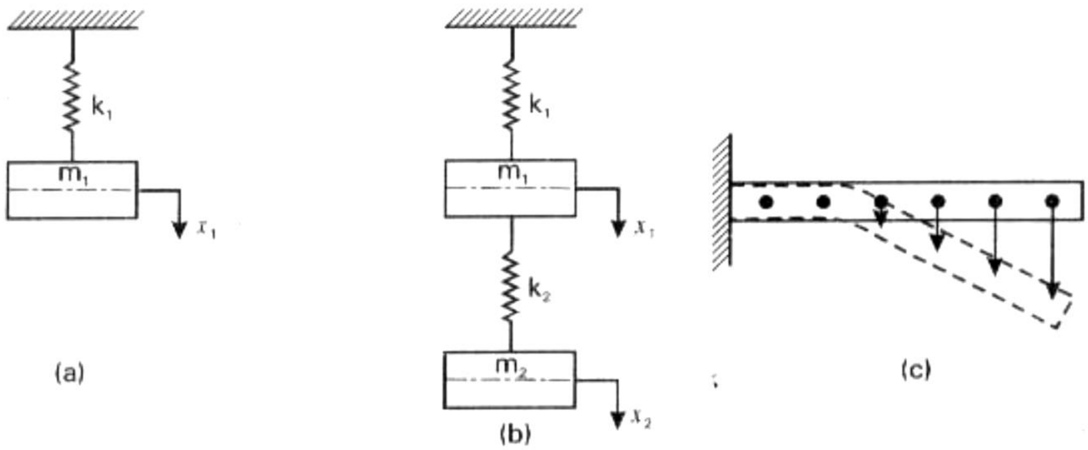


Fig. 1.1

xi) Simple Harmonic Motion

Any motion which repeats itself in equal intervals of time is known as periodic motion. Simple harmonic motion (SHM) is the simplest form of periodic motion. A simple harmonic motion is a reciprocating motion. The motion is periodic and its acceleration is always directed towards the mean position and is proportional to the displacement from the mean position.

If $x(t)$ represents the displacement of a mass in a vibrating system, the motion can be expressed by the equation

$$x = A \sin \omega t \quad \text{----- (1.2.3)}$$

$$\dot{x} = A\,\omega \cos \omega t \quad \text{----- (1.2.4)}$$

$$\ddot{x} = -A\,\omega^2 \sin \omega t = -\omega^2 x \quad \text{----- (1.2.4)}$$

Where x, $\dot{x}$ and $\ddot{x}$ represent the displacement, velocity and acceleration of the body respectively.

3. Briefly explain different types of vibrations.

Vibrations in a system can be classified into three categories; free, forced and self-excited.

Free vibration of a system is the vibration that occurs in the absence of any force, where damping may or may not be present.

An external force that acts on the system causes forced vibrations.

Self-excited vibrations are periodic and deterministic.

1.3.1 Free and Forced Vibrations

When no external force acts on the body after giving it an initial displacement, then the body is said to be under free or natural vibration. The oscillation of a simple pendulum is an example of free vibration.

When the body vibrates under the influence of external force then the body is said to be under forced vibration. Machine tools, electric bells etc., are the suitable examples of forced vibration.

1.3.2 Damped and Undamped Vibrations

If the vibratory system has a damper then there is a reduction in amplitude over every cycle of vibration since the energy of the system will be dissipated due to friction. This type of vibration is called damped vibration.

If the vibratory system has no damper then the vibration is called undamped vibration.

1.3.3 Deterministic and Random Vibrations

If the magnitude of the excitation force or motion acting on a vibrating system is known then the excitation is known as deterministic. The resulting vibration is called the deterministic vibration.

If the magnitude of the excitation force or motion acting on a vibrating system is unknown, but the averages and deviations are known then the excitation is known as non-deterministic. The resulting vibration is called random vibrations.

1.3.4 Longitudinal, Transverse and Torsional Vibrations

When the particles of the shaft or disc moves parallel to the axis of shaft, then the vibrations are known as longitudinal vibrations and is shown in Fig. 1.2(a).

When the particles of the shaft or disc moves approximately perpendicular to the axis of the shaft, then the vibrations are known as transverse vibrations and is shown in Fig. 1.2(b).

When the particles of the shaft or disc moves in a circle about the axis of the shaft i.e., if the shaft gets alternately twisted and untwisted on account of vibratory motion, then the vibrations are known as torsional vibrations and is shown in Fig. 1.2(c).

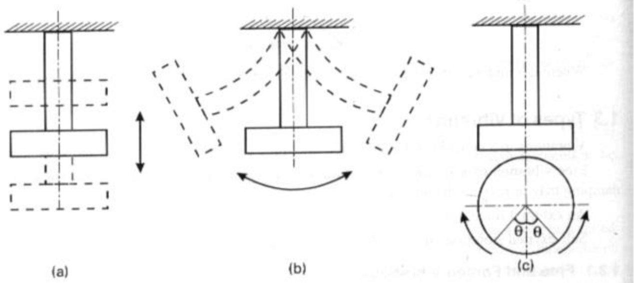


Fig. 1.2

1.3.5 Transient Vibration

The free vibrations continue indefinitely in an ideal system as there is no damping.

There is a reduction in amplitude of vibration continuously because of damping in a real system and vanishes ultimately. The vibrations in a real system is called transient vibration.

4. Explain the phenomenon a beat and obtain in amplitude of resultant motion.

If two harmonic motions passes through a point simultaneously then the resultant displacement at that point is the vector sum of the displacement due to the two motions. This super position of motion is called interference. The phenomenon of beat occurs as a result of interference between two waves of slightly different frequencies moving along the same straight line in the same direction.

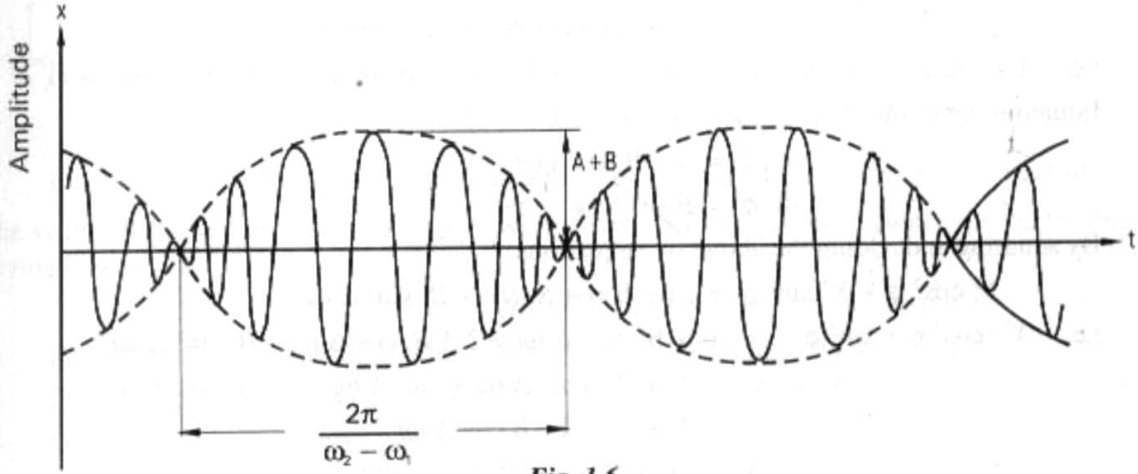


Fig. 1.6

Consider a particle subjected to two different harmonic motions,

$$\left.\begin{aligned} x_1 &= A \sin \omega_1 t \\ x_2 &= B \sin \omega_2 t \end{aligned}\right\} \quad \text{----- (1.7.1)}$$

The resultant motion is given by

$$x = x_1 + x_2 = A \sin \omega_1 t + B \sin \omega_2 t. \quad \text{----- (1.7.2)}$$

If ω_1 and ω_2 are different then the resultant motion is not sinusoidal. But if the two frequencies are slightly different then the phase difference between the rotating vectors keeps on shifting slowly and continuously. The amplitude of the resultant vector is $(A + B)$ when they are in phase and the amplitude of resultant vector is $(A - B)$ when they are in out of phase. Thus the resultant amplitude keeps on changing continuously from a maximum of $(A + B)$ to a minimum of $(A - B)$ with a frequency equal to the difference between the individual component frequencies i.e., $(\omega_2 - \omega_1)$ and this phenomenon is known as beats. The frequency of beats is equal to $(\omega_2 - \omega_1)$ as shown in Fig. 1.6. To get clear and distinct beats the following conditions must be satisfied.

i) The frequency of the beats $(\omega_2 - \omega_1)$ must be small.

ii) The amplitudes A and B should be approximately equal.

The existence of beats can also be shown mathematically.

$$\text{Let} \quad \omega_2 - \omega_1 = \Delta\omega \quad \text{----- (1.7.3)}$$

From equation 1.7.2,

$$\begin{aligned} x &= A \sin \omega_1 t + B \sin \omega_2 t \\ &= A \sin \omega_1 t + B \sin (\omega_1 + \Delta\omega) t \\ &= A \sin \omega_1 t + B [\sin \omega_1 t \cos \Delta\omega.t + \sin \Delta\omega.t \,.\, \cos \omega_1 t] \\ &= (A + B \cos \Delta\omega.t) \sin \omega_1 t + (B \sin \Delta\omega.t) \cos \omega_1 t \end{aligned} \quad \text{----- (1.7.4)}$$

Equation 1.7.4 can be considered as sum of two harmonic motions of frequency ω_1, 90° out of phase and having time dependent amplitudes.

Let $x = X \sin(\omega_1 t + \phi)$

$= X \sin \omega_1 t \cos \phi + X \cos \omega_1 t \sin \phi$

i.e., $X \sin \omega_1 t \cos \phi + X \cos \omega_1 t \sin \phi = (A + B \cos \Delta \omega t) \sin \omega_1 t + (B \sin \Delta \omega t) \cos \omega_1 t$

Equating the terms of $\sin \omega_1 t$ and $\cos \omega_1 t$ on both sides,

$$X \cos \phi = A + B \cos \Delta \omega t$$

$$X \sin \phi = B \sin \Delta \omega t$$

By squaring and adding the above two equations,

$$X^2 \cos^2 \phi + X^2 \sin^2 \phi = (A + B \cos \Delta \omega t)^2 + B^2 \sin^2 \Delta \omega t$$

i.e., $X^2 (\cos^2 \phi + \sin^2 \phi) = A^2 + B^2 \cos^2 \Delta \omega t + 2 A.B. \cos \Delta \omega t + B^2 \sin^2 \Delta \omega t$

i.e., $X^2 = A^2 + B^2 (\cos^2 \Delta \omega t + \sin^2 \Delta \omega t) + 2 AB \cos \Delta \omega t$

$$= A^2 + B^2 + 2 AB \cos \Delta \omega t$$

$$\therefore \quad X = [A^2 + B^2 + 2 AB \cos \Delta \omega t]^{1/2}$$

$=$ Amplitude of resultant motion. ----- (1.7.5)

When the two sinusoidal motions are in phase i.e., phase difference $\Delta \omega t = 0$

Resultant amplitude $X = \sqrt{A^2 + B^2 + 2AB\cos 0} = \sqrt{A^2 + B^2 + 2AB} = A + B.$

When the two sinusoidal motions are out of phase i.e., phase difference $\Delta \omega t = 180°$

Resultant amplitude $X = \sqrt{A^2 + B^2 + 2AB\cos 180} = \sqrt{A^2 + B^2 - 2AB} = A - B.$

Therefore, it is clear from the expression given in equation 1.7.5, the resultant amplitude vary between $(A + B)$ and $(A - B)$ with a frequency $\Delta\omega$.

If the harmonic motion amplitudes are equal i.e., $A = B$, then the

Resultant amplitude $X = [A^2 + A^2 + 2 A.A \cos \Delta \omega t]^{1/2} = [2 A^2 + 2 A^2 \cos \Delta \omega t]^{1/2}$

$= A \{2 (1 + \cos \Delta \omega t)\}^{1/2}$. ----- (1.7.6)

5. Derive an expression for work done by a harmonic force on a harmonic motion.

Let harmonic force $F = F_0 \sin \omega t$ be acting upon a body and the motion of the body being $x = x_0 \sin(\omega t - \phi)$. The workdone by the force F during an interval when the body moves through a small displacement dx is given by $Fdx = F\frac{dx}{dt}.dt$

For a period of one cycle, ωt varies from 0 to 2π and therefore t varies from 0 to $\frac{2\pi}{\omega}$.

$$\therefore \text{ Workdone per cycle } W = \int_0^{2\pi/\omega} F.\frac{dx}{dt}.dt = \int_0^{2\pi/\omega} [F_0 \sin\omega t][\omega x_0 \cos(\omega t - \phi)]dt$$

$$= F_0 . \omega x_0 \int_0^{2\pi/\omega} \{\sin\omega t[\cos\omega t \cos\phi + \sin\omega t \sin\phi]\} dt$$

$$= F_0 . \omega x_0 \left[\cos\phi \int_0^{2\pi/\omega} \sin\omega t . \cos\omega t dt + \sin\phi \int_0^{2\pi/\omega} \sin^2 \omega t \, dt\right]$$

$$= F_0 \, \omega x_0 \left[\cos\phi \int_0^{2\pi/\omega} \left(\frac{\sin 2\omega t}{2}\right) dt + \sin\phi \int_0^{2\pi/\omega} \left(\frac{1 - \cos 2\omega t}{2}\right) dt\right]$$

$$= F_0 . \omega x_0 \left[0 + \frac{\pi}{\omega} \sin\phi\right]$$

$\therefore$ Work done per cycle $W = \pi F_0 x_0 \sin \phi$ ----- (1.10.1)

If $\phi = 0$, then workdone $= 0$. It means if the force is in phase with the displacement then there is no work done per cycle.

If $\phi = 90$, i.e., if the force is in phase with velocity or ahead of displacement by 90°, then work done per cycle $= \pi.F_0 x_0$.

6. The motion of a particle is x = 5 sin wt. Show the relative position and magnitudes of the displacements velocity and acceleration vector at time t = 0 when (i) w = 0.5 rad/sec (ii) w = 2 rad/sec.

Solution :

We have

Displacement Vector $x = 5 \sin \omega t$

Velocity vector $\dot{x} = 5\,\omega \cos \omega t = 5\,\omega \sin\left(\omega t + \frac{\pi}{2}\right)$

Acceleration vector $\ddot{x} = -5\,\omega^2 \sin \omega t = 5\,\omega^2 \sin(\omega t + \pi)$

Therefore, the magnitudes of displacement, velocity and acceleration vectors are 5, 5ω and $5\,\omega^2$ respectively.

The phase difference is such that the velocity vector leads the displacement vector by $\pi/2$ and the acceleration vector leads the velocity vector by $\pi/2$.

i) $\omega = 0.5$ rad/sec

Since the time period is inverse of frequency,

$$\text{Time period } T = \frac{2\pi}{\omega} = \frac{2\pi}{0.5} = 4\,\pi \text{ sec.}$$

The rotating vector diagram is shown in Fig. 1.8(a).

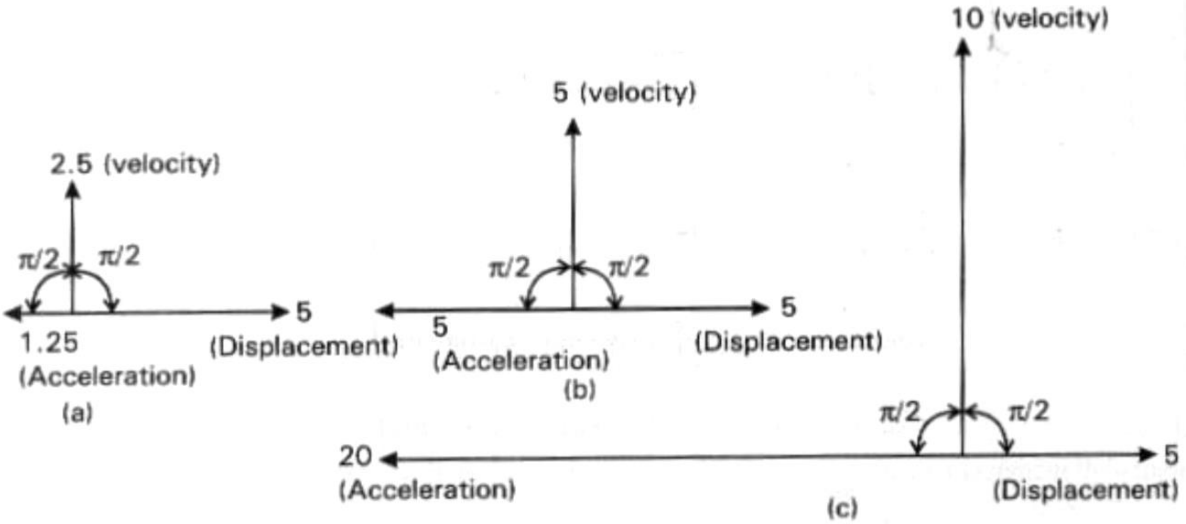


Fig. 1.8

ii) $\omega = 1$ rad/sec

$$\text{Time period } T = \frac{2\pi}{\omega} = \frac{2\pi}{1} = 2\,\pi \text{ secs}$$

The rotating vector diagram is shown in Fig. 1.8(b).

iii) $\omega = 2$ rad/sec

$$\text{Time period } T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi \text{ secs}$$

The rotating vector diagram is shown in Fig. 1.8(c).

7. Add the following harmonic motion and check the solution graphically.

$x_1 = 2\ Cos(\omega t + 0.5)$

$x_2 = 5\ Sin(\omega t + 1.0)$

Analytical Method

The resultant motion is given by

$$x = x_1 + x_2 = 2\cos(\omega t + 0.5) + 5\sin(\omega t + 1.0)$$

As the frequency is same for both x_1 and x_2 the resultant motion can also be written as, $x = A\sin(\omega t + \theta)$

$$\therefore\ A\sin(\omega t + \theta) = 2\cos(\omega t + 0.5) + 5\sin(\omega t + 1.0)$$

i.e., $A\sin\omega t\cos\theta + A\cos\omega t\sin\theta = 2(\cos\omega t)(\cos 0.5) - 2(\sin\omega t)(\sin 0.5)$
$+ 5(\sin\omega t)(\cos 1.0) + 5\cos\omega t(\sin 1.0)$

i.e., $\sin\omega t(A\cos\theta) + \cos\omega t(A\sin\theta) = 5.9625\cos\omega t + 1.7427\sin\omega t$

Equating the corresponding coefficients of $\cos\omega t$ and $\sin\omega t$ on both sides,

$$A\cos\theta = 1.7427$$

$$A\sin\theta = 5.9625$$

Squaring and adding

$$A^2\cos^2\theta + A^2\sin^2\theta = 1.7427^2 + 5.9625^2$$

$$\text{i.e., } A^2(\cos^2\theta + \sin^2\theta) = 38.5883$$

$$\therefore\ A = 6.212$$

$$\tan\theta = \frac{5.9625}{1.7427}$$

$$\therefore\ \theta = 73.708° = 1.2864 \text{ radian}$$

$$\therefore \text{ Resultant motion } x = 6.212\sin(\omega t + 1.2864) = 6.212\sin(\omega t + 73.708°)$$

Graphical Method

$$x_1 = 2\cos(\omega t + 0.5) = 2\sin\left(\omega t + 0.5 + \frac{\pi}{2}\right) = 2\sin(\omega t + 2.0708) = 2\sin(\omega t + 118.648°)$$

$$x_2 = 5\sin(\omega t + 1.0) = 5\sin(\omega t + 57.296°)$$

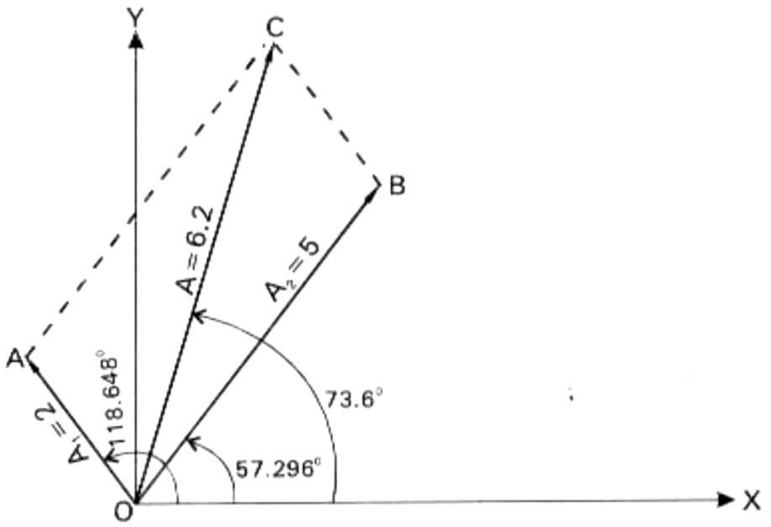


Fig. 1.9

The vector diagram can be drawn as shown in Fig. 1.9.

Steps :

i) Draw *OX* and *OY*.

ii) Draw *OA* making 118.648° with *OX* in the ccw direction and equal to 2cm.
ie., It representes $x_1 = 2\sin(\omega t + 118.648°)$.

iii) Draw *OB* making 57.296° with *OX* in the ccw direction and equal to 5cm.
ie., It representes $x_2 = 5\sin(\omega t + 57.296°)$.

iv) From B draw a line parallel to OA and from A draw a line parallel to OB. Intersection of these two lines gives C.

v) On measurement, the sum of the two vectors is 6.2cm (i.e., $OC = 6.2$cm) at an angle of 73.6° in the ccw direction with OX.

$$\therefore \; x = 6.2 \sin(\omega t + 73.6^\circ) = 6.2 \sin(\omega t + 1.285)$$

8. Add the following harmonic motions analytically and check the solution graphically

$x_1 = 3\sin(\omega t + 60)$, $x_2 = 4\cos(\omega t + 10)$

Solution :

Analytical Method

The resultant motion is given by

$$x = x_1 + x_2 = 3\sin(\omega t + 30^\circ) + 4\cos(\omega t + 10^\circ)$$

As the frequency is same for both x_1 and x_2, the resultant motion can also be written as, $x = A\sin(\omega t + \theta)$

$$\therefore \; A\sin(\omega t + \theta) = 3\sin(\omega t + 30^\circ) + 4\cos(\omega t + 10^\circ)$$

i.e., $A \sin \omega t \cos\theta + A\cos\omega t \sin\theta = 3\sin\omega t \cos 30^\circ + 3\cos\omega t \sin 30^\circ$
$+ 4\cos\omega t \cos 10^\circ - 4\sin\omega t \sin 10^\circ$

i.e., $\sin\omega t\,(A\cos\theta) + \cos\omega t\,(A\sin\theta) = 1.9035\sin\omega t + 5.4392\cos\omega t$

Equating the corresponding coefficients of $\cos\omega t$ and $\sin\omega t$ on both sides,

$$A\cos\theta = 1.9035$$

$$A\sin\theta = 5.4392$$

Squaring and adding

$$A^2\cos^2\theta + A^2\sin^2\theta = 1.9035^2 + 5.4392^2$$

$$\text{i.e., } A^2(\cos^2\theta + \sin^2\theta) = 33.20815$$

$$\therefore \; A = 5.76265$$

$$\tan\theta = \frac{5.4392}{1.9035}$$

$$\therefore \; \theta = 70.712^\circ$$

$\therefore$ Resultant motion $x = \mathbf{5.76265\sin(\omega t + 70.712^\circ)}$

Graphical Method

$$x_1 = 3\sin(\omega t + 30^\circ)$$

$$x_2 = 4\cos(\omega t + 10^\circ) = 4\sin(\omega t + 10^\circ + 90^\circ) = 4\sin(\omega t + 100^\circ)$$

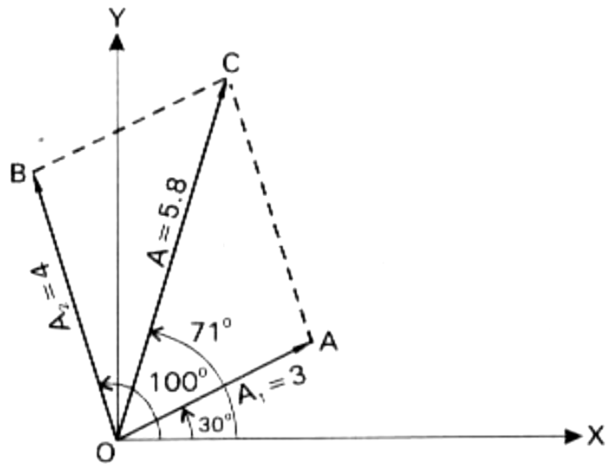


Fig. 1.10

The vector diagram can be drawn as shown in Fig. 1.10.

Steps :

i) Draw OX and OY.

ii) Draw OA making 30° with OX in the ccw direction and equal to 3cm. ie., It representes $x_1 = 3(\sin\omega t + 30^\circ)$.

iii) Draw OB making 100° with ox in the ccw directions and equal to 4cm.

ie., It representes $x_2 = 4 \sin(\omega t + 100)$.

iv) From B draw a line parallel to OA and from A draw a line parallel to OB. Intersection of these two lines gives C.

v) On measurement, the sum of the two vectors is 5.8cm (ie., OC = 5.8cm) at an angle of 71° in the ccw direction with OX.

$$\therefore \quad x = 5.8 \sin(\omega t + 71°)$$

9. Split the harmonic motion $x = 5\sin\left(\omega t + \frac{\pi}{4}\right)$ into two harmonic motions one having phase of zero and the other of 60°.

Solution :

$$x = 5\sin\left(\omega t + \pi/4\right) = 5\sin(\omega t + 45°)$$

Analytical Method

Let the equations are,

$$x_1 = A_1 \sin(\omega t + 0) = A_1 \sin \omega t$$

$$x_2 = A_2 \sin(\omega t + 60°)$$

$$\therefore \quad x = x_1 + x_2 = A_1 \sin \omega t + A_2 (\sin \omega t + 60°)$$

$$\text{i.e.,} \quad 5\sin(\omega t + 45°) = A_1 \sin \omega t + A_2 (\sin \omega t + 60°)$$

i.e., $5 \sin \omega t \cos 45° + 5 \cos \omega t \sin 45° = A_1 \sin \omega t + A_2 \sin \omega t \cos 60° + A_2 \cos \omega t \sin 60°$

i.e., $\sin \omega t (5 \cos 45°) + \cos \omega t (5 \sin 45°) = \sin \omega t (A_1 + A_2 \cos 60°) + \cos \omega t (A_2 \sin 60°)$

By equating the corresponding coefficients of $\cos \omega t$ and $\sin \omega t$ on both sides,

$$A_2 \sin 60° = 5 \sin 45°; \quad \therefore \quad A_2 = 4.08$$

$$A_1 + A_2 \cos 60° = 5 \cos 45°$$

$$\text{i.e.,} \quad A_1 + 4.08 \times \cos 60° = 5\cos 45°; \quad \therefore \quad A_1 = 1.49$$

$\therefore$ The equations of harmonic motions are

$$x_1 = 1.49 \sin \omega t$$

$$x_2 = 4.08 \sin(\omega t + 60°)$$

Graphical Method

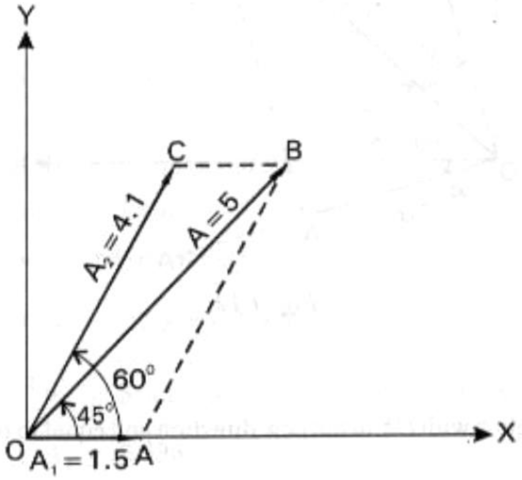


Fig. 1.11

The vector diagram can be drawn as shown in Fig. 1.11.

Steps :

i) Draw OX and OY.

ii) Draw OB making 45° with OX in the ccw direction and equal to 5 cm i.e., it represents $x = 5\sin(\omega t + 45°)$.

iii) From O draw a line making 60° with OX in the ccw direction and from B draw a line parallel to OX. Intersection of these two lines gives C.

iv) From B draw a line parallel to CO till it intersects OX at A.

v) Measurement of $OA = A_1 = 1.5$ and $OC = A_2 = 4.1$

$\therefore x_1 = 1.5 \sin \omega t$ and $x_2 = 4.1 \sin(\omega t + 60^\circ)$

10. A harmonic motion is given by the equation $x = 6\sin(2t + \phi)$. Determine its two components such that one leads it by 30^0 and other lags it by 60^0 using analytical method. (Similar to Q.No.9, *Sol is in Graphical Method)

Solution :

Graphical Method

Let the two components are,

$$x_1 = A_1 \sin(2t + \phi - 60^\circ)$$
$$x_2 = A_2 \sin(2t + \phi + 30^\circ)$$

The vector diagram can be drawn as shown in Fig. 1.12.

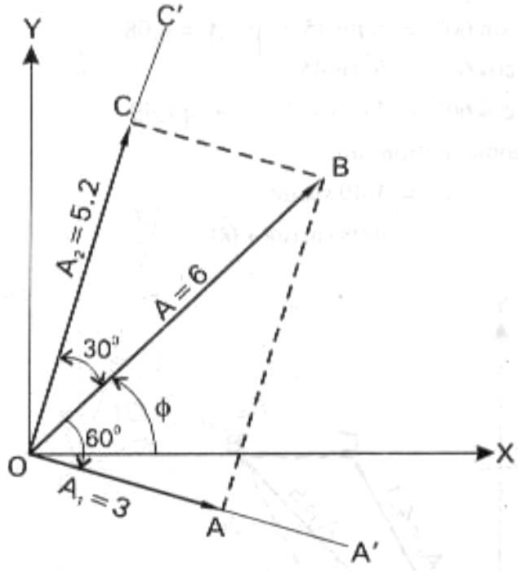


Fig. 1.12

Steps :

i) Draw OX and OY.

ii) Draw OB making any angle ϕ with OX in the ccw direction and equal to 6 cm. It represents $x = 6\sin(2t + \phi)$.

iii) From O draw a line OA' making 60° *cw* with OB (i.e., lagging by 60°).

iv) From O draw a line OC' making 30° *ccw* with OB (i.e., leading by 30°).

v) From B draw a line parallel to OC' till it intersects OA' at A. Similarly from B draw a line parallel to OA' till it intersects OC' at C.

vi) Measurement of $OA = A_1 = 3$ and $OC = A_2 = 5.2$ $\therefore x_1 = 3\sin(2t + \phi - 60^\circ)$ and $x_2 = 5.2\sin(2t + \phi + 30^\circ)$

11. Add the following harmonic motion analytically and check the solution graphically: $x_1 = 4\cos(wt + 10^\circ)$ and $x_2 = 6\sin(wt + 60^\circ)$

Analytical method

The resultant motion is given by,

$$x = x_1 + x_2 = 4\cos(\omega t + 10^\circ) + 6\sin(\omega t + 60^\circ)$$

As the frequency is same for both x_1 and x_2, the resultant motion can also be written as, $x = A\sin(\omega t + \theta)$

$$\therefore \quad A\sin(\omega t + \theta) = 4\cos(\omega t + 10^\circ) + 6\sin(\omega t + 60^\circ)$$

i.e., $A\sin\omega t\cos\theta + A\cos\omega t\sin\theta = 4\cos\omega t\cos 10^\circ - 4\sin\omega t\sin 10^\circ + 6\sin\omega t\cos 60^\circ + 6\cos\omega t\sin 60^\circ$

i.e., $\sin\omega t\,(A\cos\theta) + \cos\omega t\,(A\sin\theta) = 9.1354\cos\omega t + 2.3054\sin\omega t$

Equating the coefficients of $\sin\omega t$ and $\cos\omega t$

$$A\cos\theta = 2.3054 \qquad \text{--- (i)}$$

$$A\sin\theta = 9.1354 \qquad \text{--- (ii)}$$

Squaring and adding equation (i) and (ii), we get

$$A^2\cos^2\theta + A^2\sin^2\theta = 2.3054^2 + 9.1354^2$$

i.e., $A^2\left(\cos^2\theta + \sin^2\theta\right) = 88.77$; i.e., $A^2 = 88.77$ [$\because \cos^2\theta + \sin^2\theta = 1$]

$$\therefore \quad A = \mathbf{9.4218}$$

Equation (ii) divided by equation (i) gives

$$\frac{A\sin\theta}{A\cos\theta} = \tan\theta = \frac{9.1354}{2.3054}\ ; \ \therefore \theta = 75.8366^\circ = 1.3236 \text{ radians}$$

$\therefore$ Resultant motion $x = 9.4218\sin\left(\omega t + 75.8366^\circ\right)$

Graphical method

$$x_1 = 4\cos\left(\omega t + 10^\circ\right) = 4\sin\left(\omega t + 10^\circ + 90^\circ\right) = 4\sin\left(\omega t + 100^\circ\right)$$

$$x_2 = 6\sin\left(\omega t + 60^\circ\right)$$

The vector diagram can be drawn as shown in Fig. 1.14

Steps :

(i) Draw OX and OY

(ii) Draw OA making 100° with OX in the *ccw* direction as the given angle is + ve (i.e., + 100°) and equal to 4 cm. It represents $x_1 = 4\sin\left(\omega t + 100^\circ\right)$

(iii) Draw OB making 60° with OX in the *ccw* direction as the given angle is + ve (i.e., + 60°) and equal to 6 cm. It represents $x_2 = 6\sin\left(\omega t + 60^\circ\right)$

(iv) From B draw a line parallel to OA and from A draw a line parallel to OB. Intersection of these two lines gives C.

(v) On measurement, the sum of the two vectors is 9.4 cm (i.e., OC = 9.4 cm) at an angle of 75.8° in the ccw direction with OX.

$\therefore$ Resultant motion $x = 9.4\sin\left(\omega t + 75.8^\circ\right)$

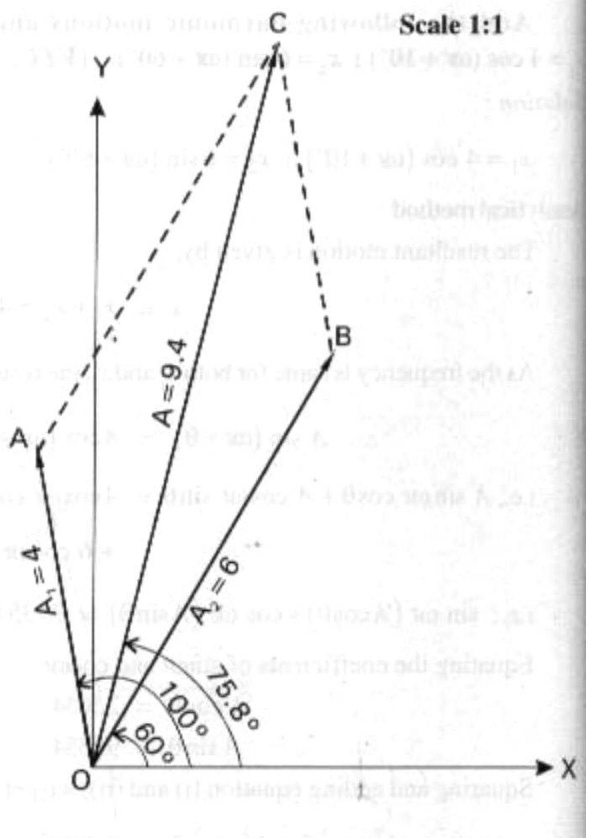


Fig. 1.14

12. A harmonic motion is given by the equation n(t) = 5 sin(15t – π/4) cm where phase angle is in radians and t in seconds. Find (i) Period of motion (ii) Frequency (iii) Maximum displacement, velocity and acceleration.

Solution :

$$x(t) = 5\sin\left(15t - \frac{\pi}{4}\right)$$

Let the harmonic motion be in the form $x = A\sin(\omega t - \phi)$ where A = Amplitude, ϕ = phase angle and ω = Frequency in radians. Comparing the two equations,

Amplitude $A = 5\text{cm}$; Phase angle $\phi = \frac{\pi}{4} = 45°$, Circular frequency $\omega = 15$ rad/sec

i) Period of motion $T = \frac{2\pi}{\omega} = \frac{2\pi}{15} = 0.4189$ seconds.

ii) Frequency $f = \frac{\omega}{2\pi} = \frac{15}{2\pi} = 2.387$ Hz.

iii) Maximum displacement $x_{max} = A = 5$ cm.

iv) Maximum velocity $\dot{x}_{max} = A\omega = 5 \times 15 = 75$ cm/sec.

v) Maximum Acceleration $\ddot{x}_{max} = A\omega^2 = 5 \times 15^2 = 1125$ cm/sec^2.

13. A force $P_0\sin wt$ acts on a displacement $x_0 \sin(wt - \pi/6)$. If P_0 = 25N, x_0 = 0.05m $w = 2\pi$ rad/sec. Find the work done during i) First second ii) First 1/4 of second.

Similar Problem:

A force $F_0 \sin\omega t$ acts on a displacement $x_0 \sin\left(\omega t - \frac{\pi}{4}\right)$ where $F_0 = 20$ N, $x_0 = 0.025$ m and $\omega = 10\pi$ rad/sec. Determine the workdone during (i) First second (ii) First $\frac{1}{20}$th second (iii) First $\frac{1}{40}$th second.

Solution :

$$\text{Work done } W = \int_0^{t_1} F\frac{dx}{dt}dt = \int_0^{t_1} F_0(\sin\omega t)x_0\omega\cos\left(\omega t - \frac{\pi}{4}\right)dt$$

$$= F_0 . x_0 . \omega \int_0^{t_1}(\sin\omega t)\cos\left(\omega t - \frac{\pi}{4}\right)dt$$

$$= F_0 . x_0 . \omega \int_0^{t_1}\frac{1}{2}\left[\sin\left(2\omega t - \frac{\pi}{4}\right) + \sin\frac{\pi}{4}\right]dt$$

$$= \frac{F_0 . x_0 . \omega}{2}\left[-\frac{\cos\left(2\omega t - \frac{\pi}{4}\right)}{2\omega} + 0.707t\right]_0^{t_1}$$

$$= \frac{20 \times 0.025 \times 10\pi}{2}\left[-\frac{\cos\left(20\pi t_1 - \frac{\pi}{4}\right)}{20\pi} + \frac{\cos\left(\frac{\pi}{4}\right)}{20\pi} + 0.707t_1\right]$$

$$= 7.854\left[\frac{-\cos\left(20\pi t_1 - \frac{\pi}{4}\right)}{20\pi} + \frac{\cos\left(\frac{\pi}{4}\right)}{20\pi} + 0.707 t_1\right]$$

i) First second i.e., $t_1 = 1$ sec

$$\text{Workdone } W = 7.854\left[-\frac{\cos\left(20\pi\times 1 - \frac{\pi}{4}\right)}{20\pi} + \frac{\cos\frac{\pi}{4}}{20\pi} + 0.707\times 1\right] = 5.553\ \text{Nm}$$

ii) When $t_1 = \frac{1}{20}$ sec

$$\text{Work done } W = 7.854\left[-\frac{\cos\left(20\pi\times\frac{1}{20} - \frac{\pi}{4}\right)}{20\pi} + \frac{\cos\frac{\pi}{4}}{20\pi} + 0.707\times\frac{1}{20}\right] = 0.4544\ \text{Nm}$$

iii) When $t_1 = \frac{1}{40}$ sec

$$\text{Work done } W = 7.854\left[-\frac{\cos\left(20\pi\times\frac{1}{40} - \frac{\pi}{4}\right)}{20\pi} + \frac{\cos\frac{\pi}{4}}{20\pi} + 0.707\times\frac{1}{40}\right]$$

$$= 0.13882\ \text{Nm}$$

14. Split the harmonic motion $x = 10\sin\left(\omega t + \frac{\pi}{6}\right)$ into two harmonic motions having phase angle of zero and the other of 45^0. Verify answers by graphical method.

Solution is Similar to Q.No.9

15. Add the following harmonic motions analytically,

$$x_1 = 4\cos(\omega t + 20^\circ)$$
$$x_2 = 7\sin(\omega t + 45^\circ)$$

Solution is Similar to Q.No.8

16. A body is subjected to two harmonic motions as given below; $x_1 = 15\sin\left(\omega t + \frac{\pi}{6}\right)$ And $x_2 = 8\cos\left(\omega t + \frac{\pi}{3}\right)$. What harmonic motion should be given to the body to bring it to equilibrium?

Similar Problem Solution:

Add the following two simple harmonic motions and check it graphically.

$x_1 = 10\cos\left(\omega t + \frac{\pi}{3}\right)$; $x_2 = 8\sin\left(\omega t + \frac{\pi}{6}\right)$ **[*VTU, Dec. 06 / Jan 07*]**

Solution :

$$x_1 = 10\cos\left(\omega t + \frac{\pi}{3}\right) = 10\cos\left(\omega t + \frac{180^\circ}{3}\right) = 10\cos(\omega t + 60^\circ)$$

Analytical method

The resultant motion is given by,

$$x = x_1 + x_2 = 4\sin(\omega t + 60°) - 6\cos(\omega t + 120°)$$

As the frequency is same for both x_1 and x_2, the resultant motion can also be written as

$$x = A\sin(\omega t + \theta)$$

$$\therefore \quad A\sin(\omega t + \theta) = 4\sin(\omega t + 60°) - 6\cos(\omega t + 120°)$$

i.e., $A\sin\omega t\cos\theta + A\cos\omega t\sin\theta = 4\sin\omega t\cos 60° + 4\cos\omega t\sin 60° - 6\cos\omega t\cos 120° + 6\sin\omega t\sin 120°$

i.e., $\sin\omega t\,(A\cos\theta) + \cos\omega t\,(A\sin\theta) = \sin\omega t\,[4\cos 60° + 6\sin 120°] + \cos\omega t\,[4\sin 60° - 6\cos 120°]$

i.e., $\sin\omega t\,(A\cos\theta) + \cos\omega t\,(A\sin\theta) = \sin\omega t\,(7.196) + \cos\omega t\,(6.464)$

Equating the coefficients of $\sin\omega t$ and $\cos\omega t$

$$A\cos\theta = 7.196 \quad \text{—(i)}$$

$$A\sin\theta = 6.464 \quad \text{—(ii)}$$

Squaring and adding equation (i) and (ii), we get

$$A^2\cos^2\theta + A^2\sin^2\theta = 7.196^2 + 6.464^2$$

i.e., $A^2\left[\cos^2\theta + \sin^2\theta\right] = 93.566$; i.e., $A^2 = 93.56 \left[\because \cos^2\theta + \sin^2\theta = 1\right]$

$$\therefore \quad \mathbf{A = 9.673}$$

Equation (ii) divided by equation (i) gives,

$$\frac{A\sin\theta}{A\cos\theta} = \tan\theta = \frac{6.464}{7.196} \; ; \; \therefore \boldsymbol{\theta = 41.93°}$$

$$\therefore \quad \text{Resultant motion } x = 9.673\sin(\omega t + 41.93°)$$

Graphical method

$$x_1 = 4\sin\left(\omega t + \frac{\pi}{3}\right) = 4\sin(\omega t + 60°)$$

$$x_2 = -6\cos\left(\omega t + \frac{2\pi}{3}\right) = -6\cos(\omega t + 120°) = -6\cos(\omega t + 30° + 90°) = -6\left[-\sin(\omega t + 30°)\right]$$

$$= +6\sin(\omega t + 30°)$$

The vector diagram can be drawn as shown in Fig. 1.19

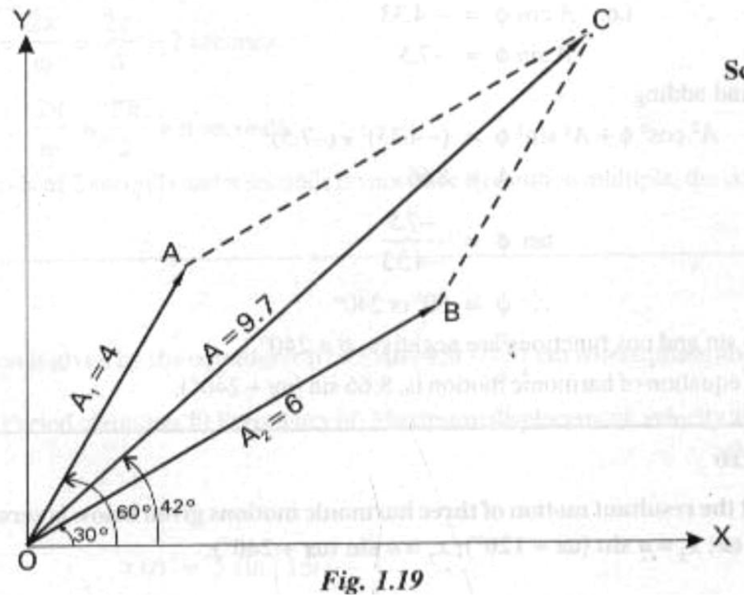


Fig. 1.19

Steps :

(i) Draw OX and OY

(ii) Draw OA making 60° with OX in the ccw direction as the given angle is + ve (i.e., +60°) and equal to 4cm. It represents $x_1 = 4\sin(\omega t + 60°)$

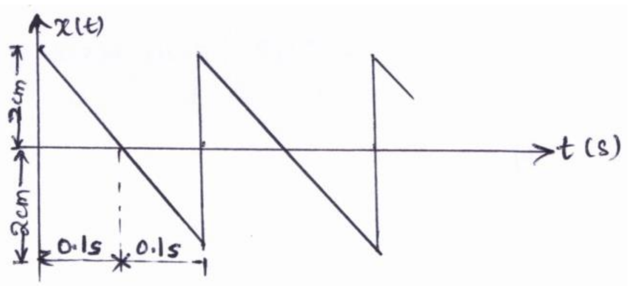
(iii) Draw OB making 30° with OX in the ccw direction as the given angle is +ve (i.e., + 30°) and equal to 6 cm. It represents $x_2 = 6\sin(\omega t + 30°)$

(iv) From B draw a line parallel to OA and from A draw a line parallel to OB. Intersection of these two lines gives C.

(v) On measurement $OC = 9.7$ cm at an angle of 42° in the ccw direction with OX.

$$\therefore \quad \text{Resultant motion } x = 9.7\sin(\omega t + 42°)$$

17. Represent the periodic motion given in the following Fig. by harmonic series.



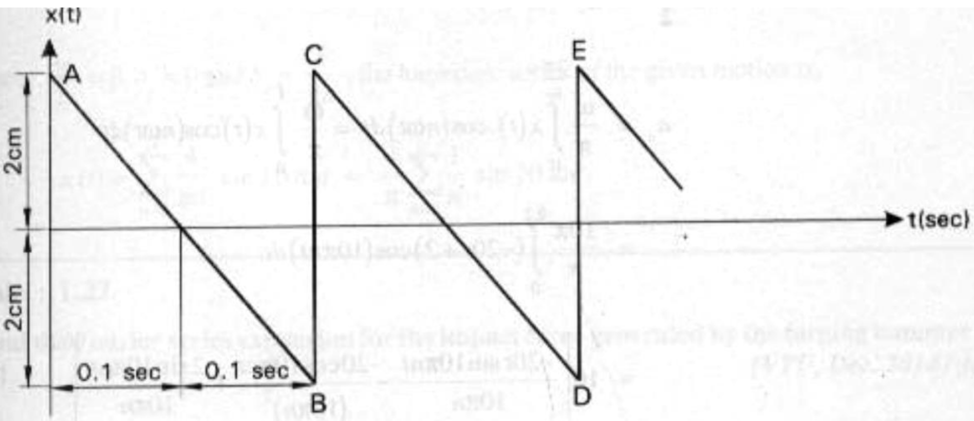


Fig. 1.20

Solution :

Fourier series of the above periodic motion is represented by,

$$x(t) = \frac{a_0}{2} + a_1\cos\omega t + a_2\cos 2\omega t + \text{-----} + b_1\sin\omega t + b_2\sin 2\omega t + \text{----}$$

Equation for straight line is, $y = mx + c$

$$\text{i.e., } x(t) = mt + c \quad \text{--- (i)}$$

For AB, when $t = 0$; $x(t) = 2$

Substituting is equation (i) $2 = 0 + c$; $\therefore c = 2$

when $t = 0.2$ sec, $x(t) = -2$

Substituting these values in equation (i)

$$-2 = 0.2m + 2 \quad (\because c = 2)$$

$$\therefore m = -20. \quad \therefore \text{Equation for } AB \text{ is, } x(t) = -20t + 2$$

The equation of the curve for one cycle is

$$x(t) = -20t + 2, \quad 0 \le t \le 0.2$$

$$\text{Time period } T = \frac{2\pi}{\omega} = 0.2 \text{ sec}$$

$$\therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\pi \text{ rad/sec}$$

$$\frac{a_0}{2} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} x(t)\,dt = \frac{\omega}{2\pi}\int_0^{T} x(t)\,dt = \frac{10\pi}{2\pi}\int_0^{0.2}(-20t+2)\,dt$$

$$= 5\left[-\frac{20t^2}{2}+2t\right]_0^{0.2} = 5\left[-\frac{20\times0.2^2}{2}+2\times0.2-0\right]=0$$

$$\therefore \frac{a_0}{2} = 0$$

$$a_n = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} x(t).\cos(n\omega t)\,dt = \frac{\omega}{\pi}\int_0^{T} x(t)\cos(n\omega t)\,dt$$

$$= \frac{10\pi}{\pi}\int_0^{0.2}(-20t+2)\cos(10\pi nt)\,dt$$

$$= 10\left[\frac{-20t\sin 10\pi nt}{10\pi n} - \frac{20\cos 10\pi nt}{(10\pi n)^2} + \frac{2\sin 10\pi nt}{10\pi n}\right]_0^{0.2}$$

$$= 10\left[\frac{-20\times0.2\sin 10\pi n\times0.2}{10\pi n} - \frac{20\cos 10\pi n\times0.2}{(10\pi n)^2} + \frac{2\sin 10\pi n\times0.2}{10\pi n} + \frac{20\cos 10\pi n\times 0}{(10\pi n)^2}\right]$$

$$= 10\left[0 - \frac{20\cos 2\pi n}{(10\pi n)^2} + 0 + \frac{20\cos 0}{(10\pi n)^2}\right] = 0$$

$\therefore a_n = 0.$ For all values of n

$$b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\sin(n\omega t)\,dt = \frac{\omega}{\pi}\int_0^{T} x(t)\sin(n\omega t)\,dt$$

$$= \frac{10\pi}{\pi}\int_0^{0.2}(-20t+2)\sin(10\pi nt)\,dt$$

$$= 10\left[\frac{-20t\{-\cos(10\pi nt)\}}{10\pi n} - \frac{20\sin 10\pi nt}{(10\pi n)^2} + \frac{2\{-\cos(10\pi nt)\}}{10\pi n}\right]_0^{0.2}$$

$$= 10\left[\frac{+20t\cos 10\pi nt}{10\pi n} - \frac{20\sin 10\pi nt}{(10\pi n)^2} - \frac{2\cos(10\pi nt)}{10\pi n}\right]_0^{0.2}$$

$$= 10\left[\frac{20\times0.2\cos 10\pi n\times0.2}{10\pi n} - \frac{2\times\cos 10\pi n\times0.2}{10\pi n} + \frac{2\cos 0}{10\pi n}\right]$$

$$= 10\left[\frac{4-2+2}{10\pi n}\right] = \frac{4}{\pi n}$$

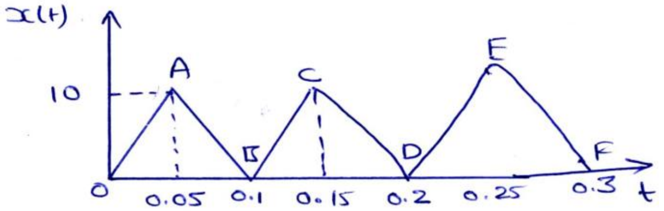
$\therefore b_n = \frac{4}{\pi n}$. For all the values of n

Fourier Series of harmonic function is, $x(t) = \frac{a_0}{2} + \sum_{n=1}^{\alpha}(a_n \cos n\,\omega t + b_n \sin n\,\omega t)$.

Since $\frac{a_0}{2} = 0, a_n = 0$ and $b_n = \frac{4}{\pi n}$, the harmonic series of the given motion is,

$$x(t) = \sum_{n=1}^{\alpha}\frac{4}{\pi n}\sin 10\,\pi nt = \frac{4}{\pi}\sum_{n=1}^{\alpha}\frac{1}{n}\sin 10\,\pi nt.$$

18. Determine the Fourier series for the curve shown in Fig.



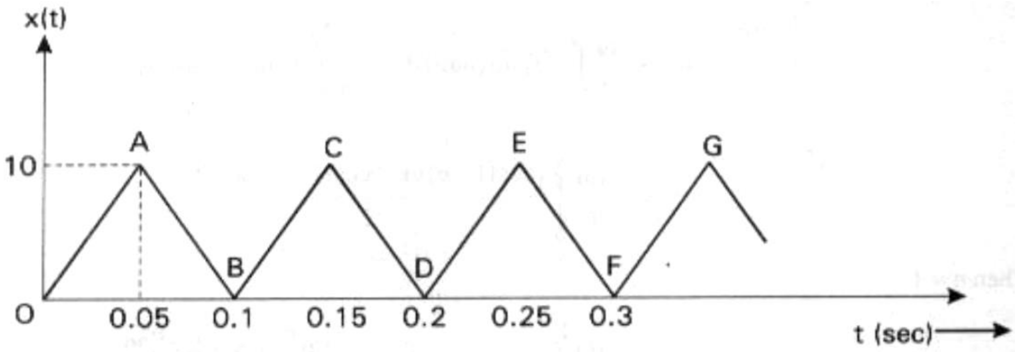


Fig. 1.22

Solution :

Fourier series of the above periodic motion is represented by

$$x(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + \text{-----} + b_1 \sin \omega t + b_2 \sin 2\omega t + \text{----}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos n\omega t + b_n \sin n\omega t$$

Equation for straight line is, $y = mx + c$, i.e., $x(t) = mt + c$ —(i)

For OA, when $t = 0$; $x(t) = 0$, Substituting in equation (i) $0 = 0 + c$; $\therefore c = 0$

when $t = 0.05$; $x(t) = 10$, Substituting in equation (i) $10 = 0.05m + 0$ $(\because c = 0)$; $\therefore m = 200$

$\therefore$ Equation for OA is, $x(t) = 200t$. $0 \le t \le 0.05$

For AB, when $t = 0.05$; $x(t) = 10$, Substituting in equation (i) $10 = 0.05\,m + c$ —(ii)

when $t = 0.1$; $x(t) = 0$, Substituting in equation (i) $0 = 0.1\,m + c$ —(iii)

(ii) – (iii) gives, $10 = -0.05\text{m}$; $\therefore m = 200$

Substituting the value of m in equation (ii), $10 = (0.05)(-200) + c$; $\therefore c = 20$

$\therefore$ Equation for AB is, $x(t) = -200t + 20 \quad 0.05 \le t \le 0.1$

Hence the equation for the given motion is,

$$x(t) = \begin{vmatrix} 200t & 0 \le t \le 0.05 \\ -200t + 20 & 0.05 \le 0.1 \end{vmatrix}$$

$$\text{Time period } T = 0.1 \text{ sec}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.1} = 20\,\pi \text{ rad / sec}$$

$$\frac{a_0}{2} = \frac{\omega}{2\pi}\int_0^T x(t)\,dt = \frac{\omega}{2\pi}\left[\int_0^{0.05} 200t\,dt + \int_{0.05}^{0.1}(-200t + 20)\,dt\right]$$

$$= \frac{20\pi}{2\pi}\left[\left(\frac{200t^2}{2}\right)_0^{0.05} + \left(-\frac{200t^2}{2} + 20t\right)_{0.05}^{0.1}\right]$$

$$= 10\left[(100 \times 0.05^2 - 0) + \{(-100 \times 0.1^2 + 20 \times 0.1) - (-100 \times 0.05^2 + 20 \times 0.05)\}\right]$$

$$= 10\,[0.25 - 1 + 2 + 0.25 - 1] = 5$$

$$\therefore \frac{a_0}{2} = 5$$

$$a_n = \frac{\omega}{\pi}\int_0^T x(t)\cos(n\omega t)\,dt$$

$$= \frac{20\pi}{\pi}\left[\int_0^{0.05}(200t)\cos(n\omega t)\,dt + \int_{0.05}^{0.1}(-200t+20)\cos(n\omega t)\,dt\right]$$

$$= 20\left[\int_0^{0.05}(200t)\cos(20\pi nt)\,dt + \int_{0.05}^{0.1}(-200t+20)\cos(20\pi nt)\,dt\right]$$

$$= 20\left[\left\{\frac{200t}{20\pi n}\sin 20\pi nt + \frac{200\cos(20\pi nt)}{(20\pi n)^2}\right\}_0^{0.05} + \left\{\frac{-200t}{20\pi n}\sin 20\pi nt - \frac{200\cos(20\pi nt)}{(20\pi n)^2} + \frac{20\sin(20\pi nt)}{20\pi n}\right\}_{0.05}^{0.1}\right]$$

$$= 20\left[\left\{\frac{200\times 0.05}{20\pi n}\sin(20\pi n\times 0.05) + \frac{200\cos(20\pi n\times 0.05)}{(20\pi n)^2}\right\} - \left\{0 + \frac{200\cos 0}{(20\pi n)^2}\right\}\right.$$

$$+\left\{\frac{-200\times 0.1}{20\pi n}\sin(20\pi n\times 0.1) - \frac{200\cos(20\pi n\times 0.1)}{(20\pi n)^2} + \frac{20\sin(20\pi n\times 0.1)}{20\pi n}\right\}$$

$$\left.-\left\{\frac{-200\times 0.05}{20\pi n}\sin(20\pi n\times 0.05) - \frac{200\cos(20\pi n\times 0.05)}{(20\pi n)^2} + \frac{20\sin(20\pi n\times 0.05)}{20\pi n}\right\}\right]$$

$$= 20\left[\frac{\sin n\pi}{2\pi n} + \frac{\cos n\pi}{2n^2\pi^2} - \frac{200}{400n^2\pi^2} - \frac{\sin 2\pi n}{\pi n} - \frac{\cos 2\pi n}{2\pi^2 n^2} + \frac{\sin 2\pi n}{\pi n} + \frac{\sin \pi n}{2\pi n} + \frac{\cos \pi n}{2\pi^2 n^2} - \frac{\sin \pi n}{\pi n}\right]$$

$$= 20\left[\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{2n^2\pi^2} - \frac{\cos 2\pi n}{2n^2\pi^2}\right]$$

$$= \frac{20}{n^2\pi^2}\left[\cos n\pi - \frac{1}{2} - \frac{1}{2}\cos 2n\pi\right]$$

$$\therefore a_n = 0. \quad \text{If } n \text{ is even}$$

$$a_n = -\frac{40}{n^2\pi^2}. \quad \text{If } n \text{ is odd}$$

$$b_n = \frac{\omega}{\pi}\int_0^T x(t)\sin(n\omega t)\,dt$$

$$= \frac{\omega}{\pi}\left[\int_0^{0.05}(200t)\sin(n\omega t)\,dt + \int_{0.05}^{0.1}(-200t+20)\sin(n\omega t)\,dt\right]$$

$$= \frac{20\pi}{\pi}\left[\int_0^{0.05}(200t)\sin(20\pi nt)\,dt + \int_{0.05}^{0.1}(-200t+20)\sin(20\pi nt)\,dt\right]$$

$$= 20\left[\left\{\frac{-200t}{20\pi n}\cos(20\pi nt) + \frac{200\sin 20\pi nt}{(20\pi n)^2}\right\}_0^{0.05} + \left\{\frac{200t}{20\pi n}\cos 20\pi nt - \frac{200\sin 20\pi nt}{(20\pi n)^2} - \frac{20\cos(20\pi nt)}{20\pi n}\right\}_{0.05}^{0.1}\right]$$

$$= 20\left[\left\{\frac{-200\times 0.05}{20\pi n}\cos(20\pi n\times 0.05)+\frac{200\sin(20\pi n\times 0.05)}{(20\pi n)^2}\right\}-\{-0+0\}\right.$$

$$+\left\{\frac{200\times 0.1}{20\pi n}\cos(20\pi n\times 0.1)-\frac{200\sin(20\pi n\times 0.1)}{(20\pi n)^2}-\frac{20\cos(20\pi n\times 0.1)}{20\pi n}\right\}$$

$$\left.-\left\{\frac{200\times 0.05}{20\pi n}\cos(20\pi n\times 0.05)-\frac{200\sin(20\pi n\times 0.05)}{(20\pi n)^2}-\frac{20\cos(20\pi n\times 0.05)}{20\pi n}\right\}\right]$$

$$= 20\left[-\frac{\cos\pi n}{2\pi n}+\frac{\sin\pi n}{2\pi^2 n^2}+\frac{\cos 2\pi n}{\pi n}-\frac{\sin 2\pi n}{2\pi^2 n^2}-\frac{\cos 2\pi n}{\pi n}-\frac{\cos\pi n}{2\pi n}+\frac{\sin\pi n}{2\pi^2 n^2}+\frac{\cos\pi n}{\pi n}\right]$$

$$= \frac{20}{n^2\pi^2}\left[\sin\pi n-\frac{1}{2}\sin 2\pi n\right]$$

$$\text{i.e., } b_n = \frac{20}{n^2\pi^2}\left[\sin n\pi-\frac{1}{2}\sin 2\pi n\right]$$

$$\therefore b_n = \mathbf{0}. \quad \text{For all values of } n$$

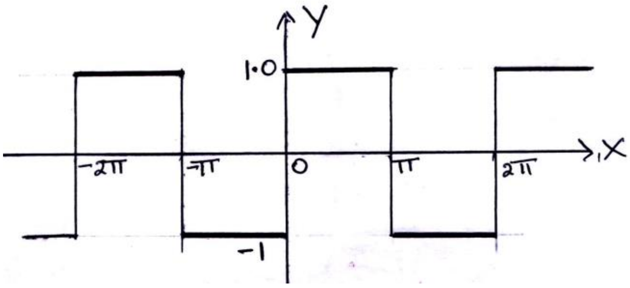
Harmonic series of the given motion is, $x(t) = \frac{a_0}{2}+\sum_{n=1}^{\alpha}(a_n\cos n\omega t+b_n\sin n\omega t)$

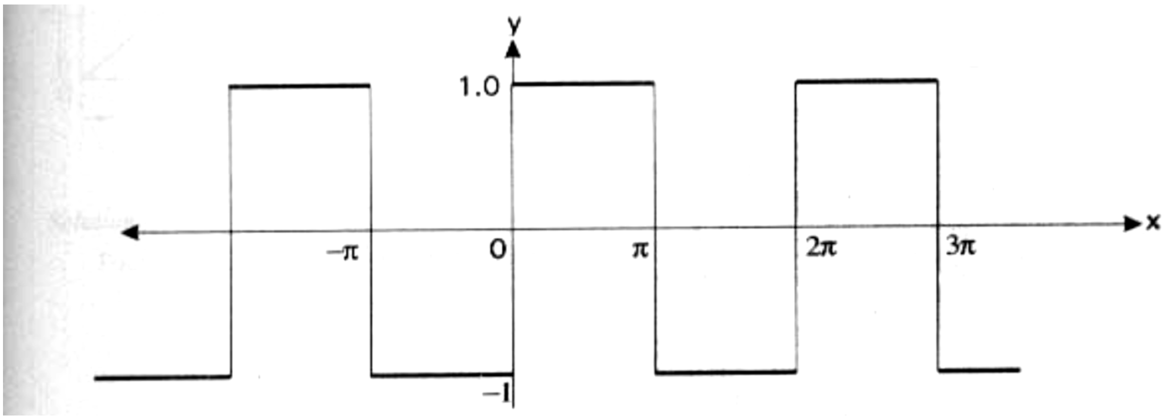
$$x(t) = 5+\sum_{n=1,3,5}^{\alpha}\left\{-\frac{40}{n^2\pi^2}\cos 20\pi nt\right\} \quad (\because b_n = 0)$$

$$= 5-\frac{40}{1^2\times\pi^2}\cos 20\pi t-\frac{40}{3^2\times\pi^2}\cos 60\pi t-\frac{40}{5^2\times\pi^2}\cos 100\pi t+-----$$

$$= 5-\frac{40}{\pi^2}\left[\frac{\cos 20\pi t}{1^2}+\frac{\cos 60\pi t}{3^2}+\frac{\cos 100\pi t}{5^2}+-----\right]$$

19. Develop the Fourier series for the curve shown in Fig





Solution :

Fourier Series of the above curve is,

$$y = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \text{-----} + b_1 \sin x + b_2 \sin 2x + \text{-----}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos nx + b_n \sin nx$$

when $x = 0$ to $-\pi$, $y = -1$ and when $x = 0$ to $+\pi$, $y = +1$

The equation of the curve for the cycle is,

$$y = \begin{vmatrix} -1.0 & -\pi \le x \le 0 \\ 1.0 & 0 \le x \le \pi \end{vmatrix}$$

$$\text{Time period } T = 2\pi. \therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1 \frac{\text{rad}}{\text{sec}}$$

$$\therefore \frac{a_0}{2} = \frac{\omega}{2\pi}\int_0^T y\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} y dx + \int_0^{\pi} y dx\right] = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)dx + \int_0^{\pi}(1)dx\right]$$

$$= \frac{1}{2\pi}\left[-(x)_{-\pi}^{0} + (x)_0^{\pi}\right] = \frac{1}{2\pi}\left[-(0+\pi) + (\pi - 0)\right] = 0$$

$$\therefore \frac{\boldsymbol{a_0}}{\mathbf{2}} = \mathbf{0}$$

$$a_n = \frac{\omega}{\pi}\int_0^T y \cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos nx\,dx + \int_0^{\pi}(1)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{\sin nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{\sin(-n\pi)}{n} + \frac{\sin n\pi}{n} - 0\right]$$

$$= \frac{1}{n\pi}\left[-\sin \pi n + \sin \pi n\right] = 0$$

$\therefore \boldsymbol{a_n = 0.}$ For all values of n

$$b_n = \frac{\omega}{\pi}\int_0^T y \sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nx\,dx + \int_0^{\pi}(1)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(+\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-\frac{\cos nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n} - \frac{\cos(-n\pi)}{n} - \frac{\cos n\pi}{n} + \frac{1}{n}\right]$$

$$= \frac{1}{n\pi}\left[1 - \cos n\pi - \cos n\pi + 1\right]$$

$$= \frac{1}{n\pi}\left[2 - 2\cos n\pi\right] = \frac{2}{n\pi}\left[1 - \cos n\pi\right]$$

$\therefore \boldsymbol{b_n} = \dfrac{\mathbf{4}}{\boldsymbol{n\pi}}$. For odd values of n

$\boldsymbol{b_n} = \mathbf{0.}$ For even values n

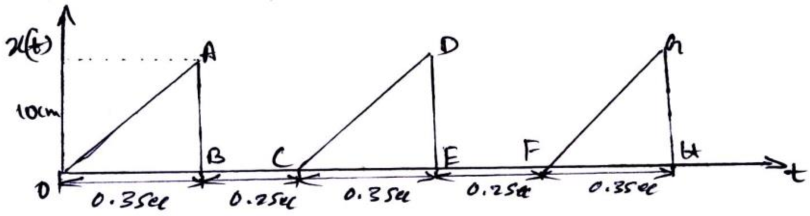
$\therefore$ Fourier Series of the given curve is,

$$y = \frac{a_0}{2} + \sum_{n=1}^{\alpha}(a_n \cos nx + b_n \sin nx)$$

$$\therefore y = \sum_{n=1,3,5,--}^{\alpha}\left(\frac{4}{n\pi}\right)\sin nx \quad \left[\because \frac{a_0}{2} = 0 \text{ and } a_n = 0\right]$$

$$= \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \text{-----}\right]$$

20. Represent the periodic motion given in Fig. by harmonic series.



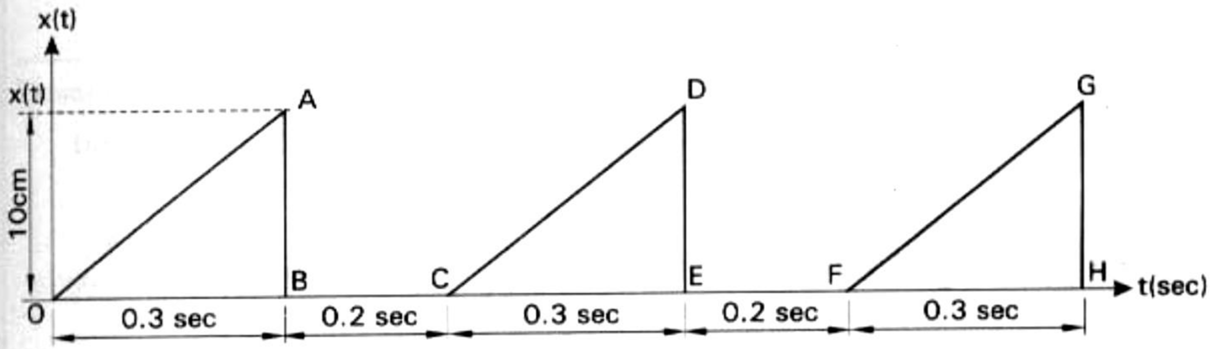


Fig. 1.26

Solution :

Fourier services of the above motion is,

$$x(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + \text{-----} + b_1 \sin \omega t + b_2 \sin 2\omega t + \text{-----}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\alpha} (a_n \cos n\omega t + b_n \sin n\omega t)$$

Equation for straight line is, $y = mx + c$

i.e., $x(t) = mt + c$ —(

For OA, when $t = 0, x(t) = 0$

Substituting these values in equation (i), $0 = 0 + c$; $\therefore c = 0$

when $t = 0.3$; $x(t) = 10$

Substituting these values in equation (i)

$$10 = 0.3\,t + 0 \quad (\because c = 0)$$

$$\text{i.e., } t = \frac{10}{0.3} = \frac{100}{3}$$

$\therefore$ Equation for OA is, $x(t) = \frac{100}{3} t \quad 0 \le t \le 0.3$

B to C i.e., when $t = 0.3$ sec to 0.5 sec, $x(t) = 0$

Hence the equation of the curve for the cycle is,

$$x(t) = \begin{vmatrix} \frac{100t}{3} & 0 \le t \le 0.3 \\ 0 & 0.3 \le t \le 0.5 \end{vmatrix}$$

Time period T = 0.5 sec.

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.5} = 4\pi \text{ rad/sec}$$

$$\frac{a_0}{2} = \frac{\omega}{2\pi} \int_0^T x(t)dt = \frac{\omega}{2\pi} \int_0^{0.3} x(t)dt = \frac{\omega}{2\pi} \int_0^{0.3} \left(\frac{100t}{3}\right) dt$$

$$= \frac{4\pi}{2\pi}\left(\frac{100}{3}\right)\left(\frac{t^2}{2}\right)_0^{0.3} = (2)\left(\frac{100}{3}\right)\left(\frac{0.3^2}{2}\right) = 3$$

$$\therefore \frac{a_0}{2} = 3$$

$$a_n = \frac{\omega}{\pi}\int_0^T x(t)\cos(n\omega t)\,dt = \frac{\omega}{\pi}\int_0^{0.3} x(t)\cos(n\omega t)\,dt = \frac{4\pi}{\pi}\int_0^{0.3}\left(\frac{100t}{3}\right)\cos(4\pi nt)\,dt$$

$$= \frac{400}{3}\left[\frac{t\sin 4\pi nt}{4\pi n} + \frac{\cos 4\pi nt}{(4\pi n)^2}\right]_0^{0.3}$$

$$= \frac{400}{3}\left[\frac{0.3\sin(4\pi n \times 0.3)}{4\pi n} + \frac{\cos(4\pi n \times 0.3)}{(4\pi n)^2} - 0 - \frac{1}{(4\pi n)^2}\right]$$

$$= \frac{10\sin(1.2\pi n)}{\pi n} + \frac{25}{3}\frac{\cos(1.2\pi n)}{\pi^2 n^2} - \frac{25}{3}\cdot\frac{1}{\pi^2 n^2}$$

$$\therefore a_n = \frac{10}{\pi n}\sin(1.2\pi n) + \frac{8.33}{\pi^2 n^2}\{\cos(1.2\pi n) - 1\}$$

$$b_n = \frac{\omega}{\pi}\int_0^T x(t)\sin(n\omega t)\,dt = \frac{\omega}{\pi}\int_0^{0.3} x(t)\sin(n\omega t)\,dt = \frac{4\pi}{\pi}\int_0^{0.3}\left(\frac{100t}{3}\right)\sin(4\pi nt)\,dt$$

$$= \frac{400}{3}\left[\frac{-t\cos 4\pi nt}{4\pi n} + \frac{\sin 4\pi nt}{(4\pi n)^2}\right]_0^{0.3}$$

$$= \frac{400}{3}\left[\frac{-0.3\cos(4\pi n \times 0.3)}{4\pi n} + \frac{\sin(4\pi n \times 0.3)}{(4\pi n)^2}\right]$$

$$\therefore b_n = -\frac{10}{\pi n}\cos(1.2\pi n) + \frac{8.33}{\pi^2 n^2}\sin(1.2\pi n)$$

Harmonic series of the given motion is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\alpha}(a_n \cos n\omega t + b_n \sin n\omega t)$$

$$x(t) = 3 + \sum_{n=1}^{\alpha}\left\{\frac{10}{\pi n}\sin 1.2\pi n + \frac{8.33}{\pi^2 n^2}(\cos 1.2\pi n - 1)\right\}\cos 4\pi nt$$

$$+ \sum_{n=1}^{\alpha}\left\{-\frac{10}{\pi n}\cos 1.2\pi n + \frac{8.33}{\pi^2 n^2}\sin 1.2\pi n\right\}\sin 4\pi nt$$