Theory of Vibration - 18AE56

Old VTU Question's Answers

Module – 2

Syllabus:
Undamped Free Vibrations: Single degree of freedom systems. Undamped free vibration, natural frequency of free vibration, Spring and Mass elements, effect of mass of spring, Compound Pendulum.
Damped Free Vibrations: Single degree of freedom systems, different types of damping, concept of critical damping and its importance, study of response of viscous damped systems for cases of under damping, critical and over damping, Logarithmic decrement.

Part – A & B Questions (Mixing of Questions Expected)

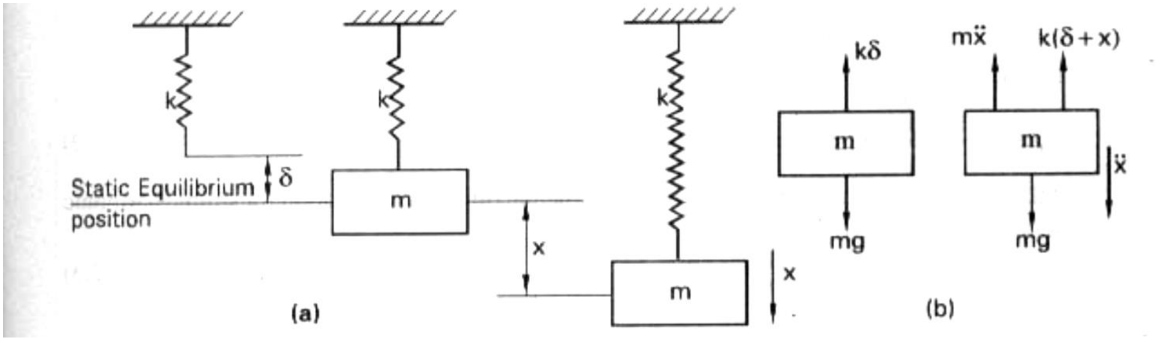
1. Define undamped free vibration.

completely. When no external force acts on the body after giving an initial displacement, then the body is said to be under free or natural vibration. If there is no loss of energy due to friction or resistance throughout the motion of the system, then the free vibration is called undamped free vibration. In this chapter, the free vibration of undamped, single degree of freedom systems are

2. Derive an expression for equation of motion and natural frequency of vibration of a spring mass system in vertical position using Newton's method.

Or

Obtain the differential equation of motion for a single degree of freedom system by: (i) Newton's method (ii) Energy method.



a) Newton's Method

Fig. 2.1 (a) shows a spring mass system in vertical position. The spring is fixed at one end and carries a mass 'm' at its free end. The system is constrained to move in the vertical direction only and the position of the mass can be completely defined by a single variable x which is a function of time 't'. Let the stiffness of the spring be 'k'. In the static equilibrium position, the forces acting on the mass are (i) vertically downward force mg and (ii) vertically upward force $k\delta$ where δ is the static deflection of the spring due to gravitational pull $W = mg$. Therefore in the static equilibrium position $mg = k\delta$.

Let x be positive in the downward direction and negative in the upward direction. Now if the mass is displaced from its static equilibrium position by a distance 'x' and then released, the spring force is $k(\delta + x)$. The free body diagram of the system is shown in Fig. 2.1 (b). Under the forces the body has an acceleration $\ddot{x}$ downwards. The sign for acceleration is positive for downward and negative for upward since these directions are selected same as those for x. If the value of $\ddot{x}$ is negative then it means that it has negative or upward acceleration.

$$\text{Restoring force} = mg - k(\delta + x)$$
$$= k\delta - k\delta - kx = -kx$$
$$\text{Accelerating force} = m\ddot{x}$$

According to Newton's second law of motion

Accelerating force = Restoring force

i.e., $m\ddot{x} = -kx$

i.e., $m\ddot{x} + kx = 0$ ----- (2.2.1)

This is the differential equation of motion for a single degree of freedom spring mass system having free vibrations.

Equation 2.2.1 can also be obtained by using D' Alemberts Principle. It states that a body is in equilibrium, if the resultant force acting on it along with the inertia force is zero. This inertia force is equal to mass times the acceleration of the body and acts through the centre of gravity of the body and the direction of inertia force is opposite to that of accelerating force.

In the displaced spring mass system, the spring force kx acts in the upward direction and accelerating force $m\ddot{x}$ acts in the downward direction. Since the direction of accelerating force is downward, the direction of inertia force is upward. So the body is in static equilibrium under the action of spring force and inertia force.

i.e., $m\ddot{x} + kx = 0$

The equation is same, as obtained by Newton's method.

Equation 2.2.1 can be rewritten as, $\ddot{x} + \frac{k}{m}x = 0$ ----- (2.2.2)

If the vibrating motion is SHM, then the fundamental differential equation for simple harmonic motion is,

$$\ddot{x} + \omega_n^2 x = 0$$ ----- (2.2.3)

Comparing the equation : 2.2.2 and 2.2.3,

$$\omega_n^2 = \frac{k}{m} \quad \therefore \quad \omega_n = \sqrt{\frac{k}{m}} \text{ rad/sec}$$ ----- (2.2.4)

$$\text{Natural frequency } f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hz}$$ ----- (2.2.5)

$$= \frac{1}{2\pi}\sqrt{\frac{mg}{\delta . m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ Hz.}$$ ----- (2.2.6)

b) Energy Method ***[VTU, June/July 2014]***

Consider the spring mass system shown in Fig. 2.1. and according to energy method the total energy of the system at any instant must be constant. Since the mass is displaced from the equilibrium position through a distance x, the velocity at that instant be $\dot{x}$. The total energy of the system consists of (i) Kinetic energy of the mass (ii) Potential energy (gravitational energy) due to the elevation of mass from a reference level (iii) Potential energy (strain energy) of the spring.

Kinetic energy of the mass $(KE) = \frac{1}{2}m\dot{x}^2$ ----- (2.2.7)

Potential energy due to the elevation of mass $= -\,mgx$

Negative sign indicates loss of energy since the level is lowered by a distance x.

The strain energy of the spring is equal to the work done in deforming the spring through a distance x.

In the displaced position, spring force $= k(\delta + x)$

$\therefore$ Average spring force during the deformation $= k\left(\delta + \frac{x}{2}\right)$

Strain energy of the spring $= k\left(\delta + \frac{x}{2}\right).x = k\delta x + \frac{kx^2}{2} = mg\,x + \frac{kx^2}{2}$

$$\therefore \quad \text{Total Potential Energy } (PE) = -mgx + mgx + \frac{kx^2}{2}$$

$$= \frac{kx^2}{2}$$

Now, according to energy method, ----- (2.2.8)

$$KE + PE = \text{Constant}$$

$$\text{i.e.,} \quad \frac{1}{2}m\dot{x}^2 + \frac{kx^2}{2} = \text{Constant} \qquad \text{----- (2.2.9)}$$

Differentiating the above equation,

$$\frac{1}{2}m2\dot{x}.\ddot{x} + \frac{k}{2}\,2x.\dot{x} = 0$$

$$\text{i.e.,} \quad m\ddot{x} + kx = 0 \qquad \text{----- (2.2.10)}$$

Equation 2.2.10 is the same as that obtained by Newton's method (2.2.1).

3. Determine the natural frequency of spring mass system where mass of the spring is also taken into consideration.

Solution :

Fig. 2.19 shows a spring mass system.

If the mass of the spring is taken into account then,

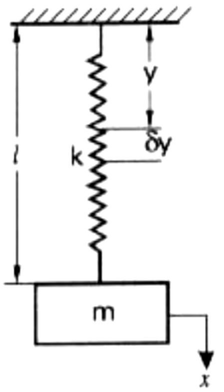


Fig. 2.19

Let x = Displacement of mass

$\dot{x}$ = Velocity of the free end of the spring at the instant under consideration.

m' = Mass of spring wire per unit length

l = Total length of the spring wire.

Consider an elemental length dy at a distance 'y' from the fixed end.

Velocity of the spring wire at the distance y from the fixed end $= \dot{x}\left(\frac{y}{l}\right)$

Kinetic energy of the spring element $dy = \frac{1}{2}(m'dy)\left(\dot{x}\frac{y}{l}\right)^2$

$$\text{Kinetic energy of spring} = \int_0^l \frac{1}{2}m'\dot{x}^2\frac{y^2}{l^2}.dy = \frac{1}{2}m'\frac{\dot{x}^2}{l^2}\int_0^l y^2dy$$

$$= \frac{1}{2}m'\frac{\dot{x}^2}{l^2}\cdot\left(\frac{l^3}{3}\right) = \frac{1}{6}m'l.\dot{x}^2$$

$$= \frac{1}{6} m_s \dot{x}^2 \quad \text{where } m_s = \text{Mass of spring} = m' \, l$$

$\therefore$ Total kinetic energy of the system = KE of mass + KE of spring $= \frac{1}{2} m\dot{x}^2 + \frac{1}{6} m_s \dot{x}^2$

Potential energy of the system $= \frac{1}{2} kx^2$

$\therefore$ Total energy of the system $= \frac{1}{2} m\dot{x}^2 + \frac{1}{6} m_s \dot{x}^2 + \frac{1}{2} kx^2 = \text{Constant}$

Differentiating the above equation with respect to time

$$\frac{1}{2} m 2\dot{x}.\ddot{x} + \frac{1}{6} m_s \, 2\dot{x}.\ddot{x} + \frac{1}{2} k 2x.\dot{x} = 0$$

$$\text{i.e.,} \quad m\ddot{x} + \frac{1}{3} m_s \ddot{x} + kx = 0$$

$$\text{i.e.,} \quad \left(m + \frac{1}{3} m_s\right)\ddot{x} + kx = 0$$

$$\therefore \quad \ddot{x} + \left(\frac{k}{m + \frac{1}{3} m_s}\right) x = 0$$

$$\therefore \text{Circular frequency} \quad \omega_n = \sqrt{\frac{k}{\left(m + \frac{1}{3} m_s\right)}} \text{ rad/sec}$$

$$\text{Natural frequency} \quad f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi}\sqrt{\frac{k}{\left(m + \frac{1}{3} m_s\right)}} \text{ Hz.}$$

4. Show that for finding the natural frequency of a spring mass system, the mass of spring can be taken into account by adding one-third its mass to the main mass.

Similar to Q.No:3 but while finding KE of Mass, "take *m* as *m/3*"

$\therefore$ Total kinetic energy of the system = KE of mass + KE of spring

5. Define and find an expression for undamped natural frequency of a compound pendulum.

Solution :

Fig. 2.20 shows a compound pendulum in the displaced position.

Let m = Mass of the rigid body = w/g

l = Distance of point of suspension from G

O = Point of suspension

G = Centre of gravity

I = Moment of inertia of the body about O

$= mk^2 + ml^2 = m(k^2 + l^2)$

k = Radius of gyration of the body

If OG is displaced by an angle θ,

Restoring torque = $-(mg \sin\theta)\, l = -mgl\theta$ since θ is small $\sin\theta \approx \theta$

According to Newton's second law

Accelerating torque = Restoring torque

$$\text{i.e.,} \quad I\ddot{\theta} = -mgl\,\theta$$

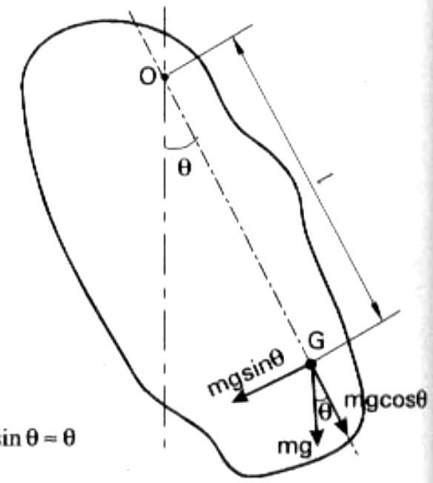


Fig. 2.20

$$\text{i.e.,}\quad \ddot{\theta}+\frac{mgl}{I}\theta = 0$$

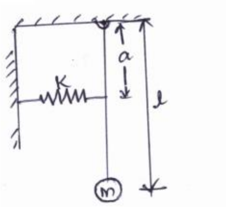
$$\text{i.e.,}\quad \ddot{\theta}+\frac{mgl}{m\left(k^2+l^2\right)}\theta = 0$$

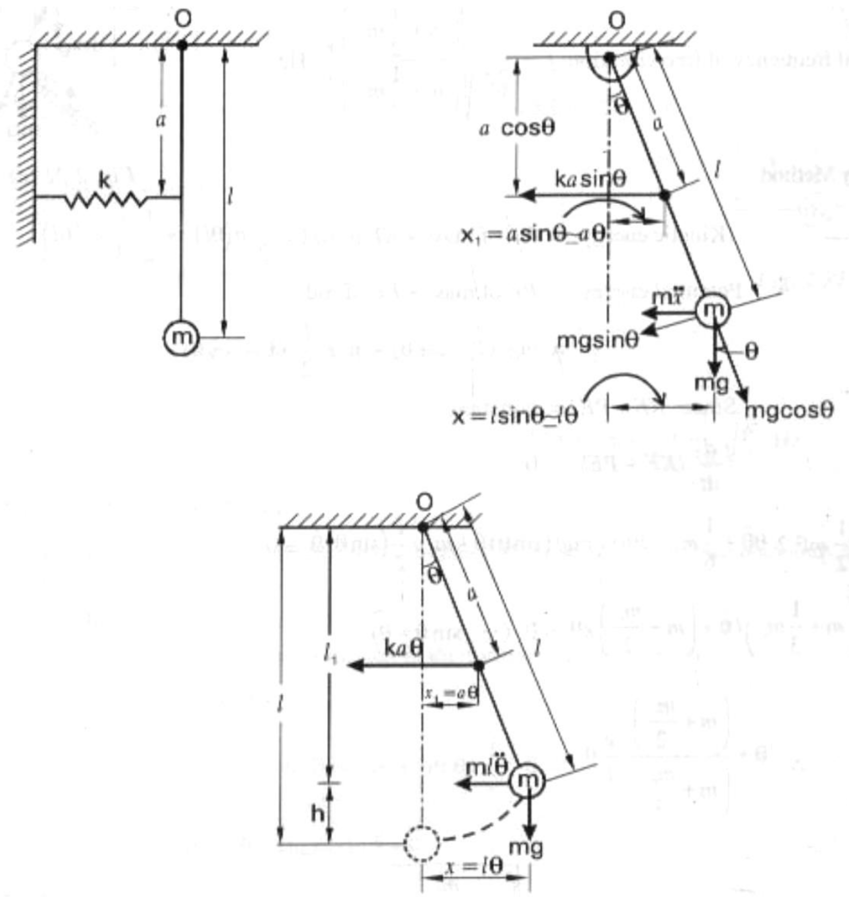
$$\therefore\quad \ddot{\theta}+\frac{gl}{\left(k^2+l^2\right)}\theta = 0$$

$$\therefore \text{ Circular frequency}\quad \omega_n = \sqrt{\frac{gl}{k^2+l^2}}\ \text{rad/sec}$$

$$\text{Natural frequency}\quad f_n = \frac{1}{2\pi}\,\omega_n = \frac{1}{2\pi}\sqrt{\frac{gl}{k^2+l^2}}\ \text{Hz.}$$

6. Determine natural frequency of the system shown in Fig by: (i) Newton method (ii) Energy method





Fig(c).22

Solution :

(a) Newtons method

MI of mass about O, $I_o = ml^2$

For a small angular displacement θ, the free body diagram of the system is shown in Fig. 2.22 (b).
According to Newton's second law of motion in the form of torques,

$I_0\,\ddot{\theta} = \Sigma M$ *where* ΣM = Sum of restoring couples about O

$$\text{i.e., } ml^2\,\ddot{\theta} = -(mg\sin\theta)\,l - (ka\sin\theta)(a\cos\theta)$$

$$\text{i.e., } ml^2\,\ddot{\theta} + mgl\,\theta + ka^2\,\theta = 0 \quad (\because\ \sin\theta \approx \theta \text{ and } \cos\theta \approx 1 \text{ for small angle})$$

$$\text{i.e., } \ddot{\theta} + \left(\frac{mgl + ka^2}{ml^2}\right)\theta = 0$$

$$\therefore \text{ Circular frequency } \omega_n = \sqrt{\frac{mgl + ka^2}{ml^2}} \text{ rad/sec.} = \sqrt{\left(\frac{g}{l} + \frac{ka^2}{ml^2}\right)} \text{ rad/sec}$$

$$\text{Natural frequency } f_n = \frac{1}{2\pi}\omega_n \text{ Hz} = \frac{1}{2\pi}\sqrt{\frac{mgl + ka^2}{ml^2}} \text{ Hz.} = \frac{1}{2\pi}\sqrt{\left(\frac{g}{l} + \frac{ka^2}{ml^2}\right)} \text{ Hz}$$

(b) Energy method

From Fig. 2.22 (c), $h = l - l_1 = l - l\cos\theta = l\,(1-\cos\theta)$ $\left[\because\ \cos\theta = \frac{l_1}{l}\right]$

$$x = l\sin\theta \cong l\theta \text{ and } x_1 = a\sin\theta \cong a\theta \quad [\because\ \sin\theta \cong \theta]$$

$$KE = \frac{1}{2}m\,\dot{x}^2 = \frac{1}{2}m\left(l\dot{\theta}\right)^2 = \frac{1}{2}ml^2\,\dot{\theta}^2 \quad \left[\because\ \dot{x} = l\dot{\theta}\right]$$

$$PE = PE \text{ of mass} + PE \text{ of spring} = mgh + \frac{1}{2}k\,x_1^2 = mgl\,(1-\cos\theta) + \frac{1}{2}ka^2\theta^2 \quad (\because\ x_1 = a\theta)$$

According to energy method, $KE + PE = \text{Constant}$

$$\therefore \frac{d}{dt}(KE + PE) = 0 \text{ ; i.e., } \frac{d}{dt}\left(\frac{1}{2}ml^2\dot{\theta}^2 + mgl\,(1-\cos\theta) + \frac{1}{2}ka^2\theta^2\right) = 0$$

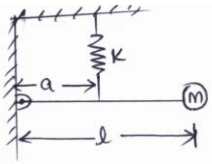
$$\text{i.e., } \frac{1}{2}ml^2\,2\dot{\theta}\,\ddot{\theta} + mgl\,(\sin\theta)\,\dot{\theta} + \frac{1}{2}ka^2\,2\theta\cdot\dot{\theta} = 0$$

$$\text{i.e., } ml^2\,\ddot{\theta} + mgl\,\theta + ka^2\theta = 0 \text{ ; i.e., } \ddot{\theta} + \left(\frac{mgl + ka^2}{ml^2}\right)\theta = 0 \; [\because \text{ For small angular displaced } \sin\theta \cong \theta\,]$$

$$\therefore \text{ Circular frequency } \omega_n = \sqrt{\frac{mgl + ka^2}{ml^2}} \text{ rad/sec}$$

$$\text{Natural frequency } f_n = \frac{1}{2\pi}\omega_n\ Hz = \frac{1}{2\pi}\sqrt{\frac{mgl + ka^2}{ml^2}}\ Hz = \frac{1}{2\pi}\sqrt{\frac{g}{l} + \frac{ka^2}{ml^2}}\ Hz$$

7. Determine natural frequency of the system shown in Fig by: (i) Newton method (ii) Energy method



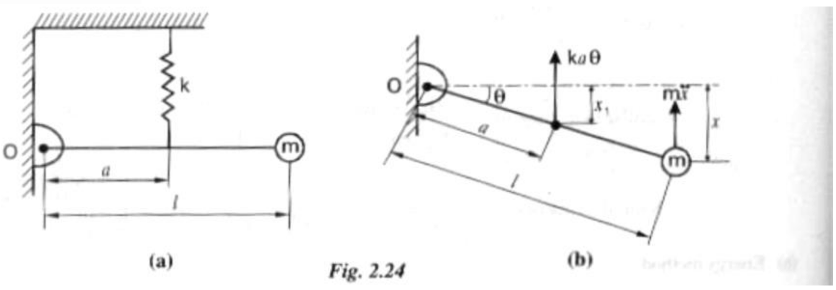


Fig. 2.24

Solution :

(a) Newton's method

Moment of inertia of mass about 0, $I_0 = ml^2$

For a small angular displacement θ, the free body diagram of the system is shown in Fig. 2.24 (b). According to Newton's second law of motion in the form of torque equation,

$$I_0\,\ddot{\theta} = \Sigma M \quad \text{where } \Sigma M = \text{Sum of restoring couples about } O$$

$$\text{i.e., } ml^2\,\ddot{\theta} = -ka\theta.a \quad (\because \text{ For small angular displacement, } \sin\theta \cong \theta\,)$$

$$\text{i.e., } ml^2\ddot{\theta} + ka^2\theta = 0\ ;\ \therefore\ \ddot{\theta} + \frac{ka^2}{ml^2}\theta = 0$$

$$\therefore \text{ Circular frequency of vibration } \omega_n = \sqrt{\frac{k}{m}\cdot\left(\frac{a}{l}\right)^2}\ \text{rad/sec} = \frac{a}{l}\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

$$\text{Natural frequency of vibration of the system } f_n = \frac{1}{2\pi}\omega_n Hz = \frac{1}{2\pi}\sqrt{\frac{k}{m}\left(\frac{a}{l}\right)^2}\ \text{Hz} = \frac{1}{2\pi}\left(\frac{a}{l}\right)\sqrt{\frac{k}{m}}\ \text{Hz.}$$

(b) Energy method

From Fig. 2.24 (b), $x = l\sin\theta \cong l\theta$ and $x_1 = a\sin\theta \cong a\theta$ [∵ For small angular displacement, $\sin\theta \cong \theta$]

$$KE = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\left(l\dot{\theta}\right)^2 = \frac{1}{2}ml^2\dot{\theta}^2 \quad (\because\ \dot{x} = l\dot{\theta})$$

$$PE = PE \text{ of spring} = \frac{1}{2}k\,x_1^2 = \frac{1}{2}k\,(a\theta)^2 = \frac{1}{2}ka^2\theta^2 \quad (\because\ x_1 = a\theta)$$

As the direction of gravitational pull acts parallel to the direction of spring force, for static equilibrium, the gravitational pull (mg) must be equal and opposite to that of the spring force due to static deformation δ. i.e., $mg = k\delta$. Hence the *PE* due to the mass is not required to be considered.

According to energy method, $KE + PE = \text{Constant}$. $\therefore\ \frac{d}{dt}(KE + PE) = 0$

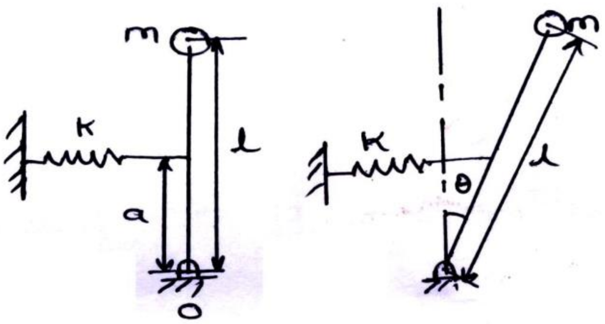
$$\text{i.e., } \frac{d}{dt}\left[\frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ka^2\theta^2\right] = 0\ ;\ \text{i.e., } \frac{1}{2}ml^2 2\dot{\theta}\ddot{\theta} + \frac{1}{2}ka^2 2\theta\cdot\dot{\theta} = 0$$

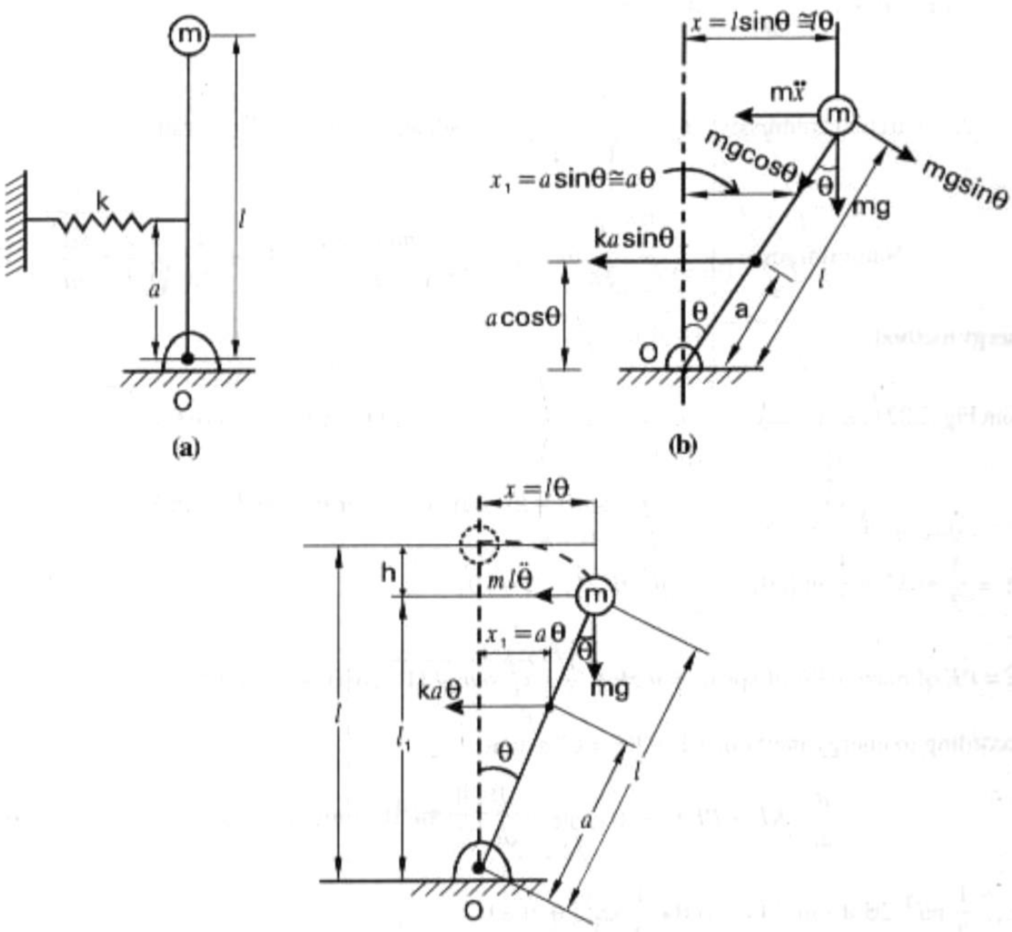
$$\text{i.e., } ml^2\,\ddot{\theta} + ka^2\theta = 0\ ;\ \text{i.e., } \ddot{\theta} + \frac{ka^2}{ml^2}\theta = 0$$

$$\therefore \text{ Circular frequency of vibration } \omega_n = \sqrt{\frac{ka^2}{ml^2}} = \frac{a}{l}\sqrt{\frac{k}{m}}\ \text{rad/sec}$$

$$\text{Natural frequency of vibration } f_n = \frac{1}{2\pi}\omega_n\ Hz = \frac{1}{2\pi}\cdot\frac{a}{l}\sqrt{\frac{k}{m}}\ Hz$$

8. Determine the natural frequency of the system shown in Fig, by i) Newton's method ii) Energy method.





(a) (b)

(c)

Fig. 2.23

Solution :

(a) Newton's method

MI of mass about O, $I_0 = ml^2$.

For a small angular displacement θ, the free body diagram of the system is as shown in Fig. 2.23 (b). Here the moment due to inertia force and the spring force are in the same direction and the moment due to gravitational force is in the opposite direction.

Therefore according to Newton's second law,

$$I_0\,\ddot{\theta} = \Sigma M \quad \text{where } \Sigma M = \text{Sum of restoring couples about } O$$

i.e., $$I_0\,\ddot{\theta} = -(ka\sin\theta)(a\cos\theta) + (mg\sin\theta)\,l$$

i.e., $I_0\,\ddot{\theta} = -(ka\theta)\,a + mgl\theta$ ($\because$ For small angular displacement θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$)

i.e., $$ml^2\,\ddot{\theta} + ka^2\,\theta - mgl\theta = 0$$

i.e., $$\ddot{\theta} + \left(\frac{ka^2 - mgl}{ml^2}\right)\theta = 0$$

$\therefore$ Circular frequency $$\omega_n = \sqrt{\frac{ka^2 - mgl}{ml^2}} \text{ rad/sec.} = \sqrt{\left(\frac{ka^2}{ml^2} - \frac{g}{l}\right)} \text{ rad / sec}$$

Natural frequency $$f_n = \frac{1}{2\pi}\,\omega_n,\ Hz = \frac{1}{2\pi}\sqrt{\frac{ka^2 - mgl}{ml^2}} \text{ Hz.} = \frac{1}{2\pi}\sqrt{\left(\frac{ka^2}{ml^2} - \frac{g}{l}\right)}\ Hz$$

(b) Energy method

From Fig. 2.23 (c), $$h = l - l_1 = l - l\cos\theta = l\,(1 - \cos\theta) \quad \left[\because\ \cos\theta = \frac{l_1}{l}\right]$$

$$x = l\sin\theta \cong l\theta \text{ and } x_1 = a\sin\theta \cong a\theta \quad (\because \sin\theta \cong \theta)$$

$$KE = \frac{1}{2} m\dot{x}^2 = \frac{1}{2} m\left(l\dot{\theta}\right)^2 = \frac{1}{2} ml^2\dot{\theta}^2 \quad (\because \dot{x} = l\dot{\theta})$$

$$PE = PE \text{ of mass} + PE \text{ of spring} = -mgh + \frac{1}{2} k x_1^2 = -mgl(1-\cos\theta) + \frac{1}{2} ka^2\theta^2 \quad (\because x_1 = a\theta)$$

[Loss of energy due to mass, since the level is lowered by a distance h. $\therefore -mgh$]

According energy method, $KE + PE = \text{Constant}$. $\therefore \frac{d}{dt}(KE + PE) = 0$

i.e., $$\frac{d}{dt}\left[\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta) + \frac{1}{2}ka^2\theta^2\right] = 0$$

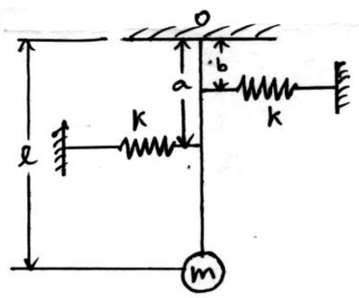
i.e., $$\frac{1}{2}ml^2 2\dot{\theta}\ddot{\theta} - mgl(\sin\theta)\dot{\theta} + \frac{1}{2}ka^2 2\theta\dot{\theta} = 0$$

i.e., $ml^2\ddot{\theta} - mgl\theta + ka^2\theta = 0$; i.e., $\ddot{\theta} + \left(\frac{ka^2 - mgl}{ml^2}\right)\theta = 0 \quad (\because \sin\theta \cong \theta)$

$$\therefore \text{ Circular frequency } \omega_n = \sqrt{\frac{ka^2 - mgl}{ml^2}} \text{ rad/sec.} = \sqrt{\left(\frac{ka^2}{ml^2} - \frac{g}{l}\right)} \text{ rad / sec}$$

$$\text{Natural frequency } f_n = \frac{1}{2\pi}\omega_n, Hz = \frac{1}{2\pi}\sqrt{\frac{ka^2 - mgl}{ml^2}} \text{ Hz.} = \frac{1}{2\pi}\sqrt{\left(\frac{ka^2}{ml^2} - \frac{g}{l}\right)} Hz$$

9. Determine the natural frequency of the system shown in Fig, using Newton's method.



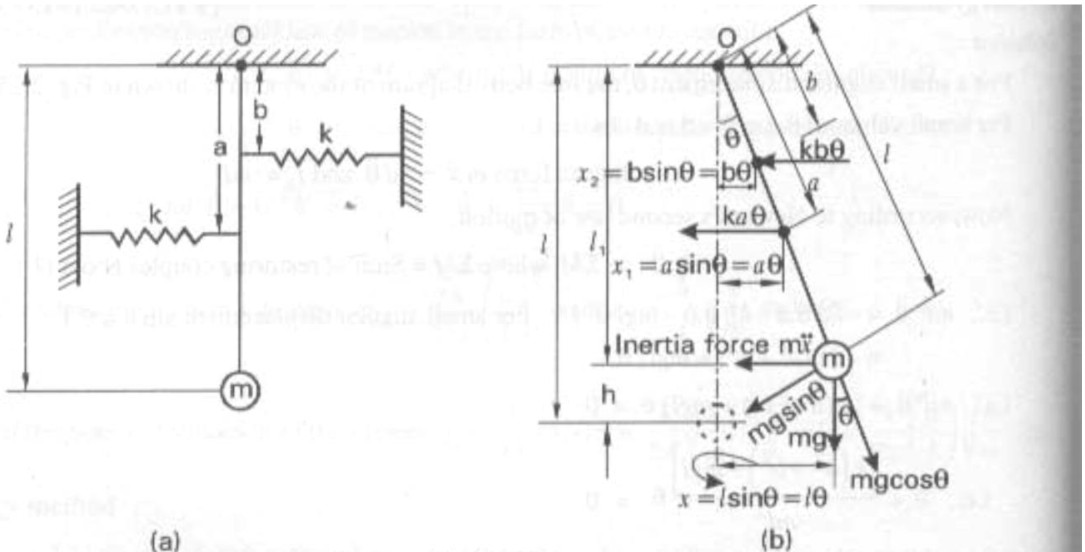


Fig. 2.25

Solution :

For a small angular displacement θ, the free body diagram of the system is shown in Fig. 2.25 (b).

For small values of θ, $\sin\theta \approx \theta$ and $\cos\theta = 1$.

$$\text{Inertia force } m\ddot{x} = ml\ddot{\theta} \text{ and } I_0 = ml^2$$

Now, according to Newton's second law of motion,

$$I_0\,\ddot{\theta} = \Sigma M \text{ where } \Sigma M = \text{Sum of restoring couples about } O$$

i.e., $ml^2\,\ddot{\theta} = -ka\,\theta.a - kb\,\theta.b - mgl\,\theta$ [∵ For small angular displacement $\sin\theta \cong \theta$]

$$= -[k(a^2+b^2)+mgl]\,\theta$$

i.e., $ml^2\ddot{\theta} + [k(a^2+b^2)+mgl]\,\theta = 0$

i.e., $\ddot{\theta} + \dfrac{\left[k\left(a^2+b^2\right)+mgl\right]}{ml^2}\theta = 0$

∴ Circular frequency of vibration $\omega_n = \sqrt{\dfrac{\left(ka^2+kb^2+mgl\right)}{ml^2}}$ rad/sec

$$\text{Natural frequency } f_n = \frac{1}{2\pi}\,\omega_n\text{ Hz} = \frac{1}{2\pi}\sqrt{\frac{\left(ka^2+kb^2+mgl\right)}{ml^2}}\text{ Hz.}$$

(b) Energy method

From Fig. 2.25 (b), $h = l - l_1 = l - l\cos\theta = l\,(1-\cos\theta)$ $\left(\because \cos\theta = \dfrac{l_1}{l}\right)$

$$x = l\sin\theta \cong l\theta,\; x_1 = a\sin\theta \cong a\theta \text{ and } x_2 = b\sin\theta \cong b\theta \quad [\because \sin\theta \cong \theta]$$

$$KE = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\left(l\dot{\theta}\right)^2 = \frac{1}{2}ml^2\,\dot{\theta}^2 \quad (\because \dot{x} = l\dot{\theta})$$

$$PE = PE \text{ of mass} + PE \text{ of springs} = mgh + \frac{1}{2}k\,x_1^2 + \frac{1}{2}k\,x_2^2$$

$$= mgl\,(1-\cos\theta) + \frac{1}{2}ka^2\theta^2 + \frac{1}{2}k\,b^2\theta^2 \quad [\because x_1 = a\theta \text{ and } x_2 = b\theta]$$

According to energy method, $KE + PE = $ Constant.

$$\therefore \frac{d}{dt}(KE+PE) = 0\;;\text{ i.e., } \frac{d}{dt}\left[\frac{1}{2}ml^2\,\dot{\theta}^2 + mgl\,(1-\cos\theta) + \frac{1}{2}ka^2\theta^2 + \frac{1}{2}kb^2\right]$$

i.e., $\dfrac{1}{2}ml^2\,2\dot{\theta}\,\ddot{\theta} + mgl\,(\sin\theta)\dot{\theta} + \dfrac{1}{2}ka^2\,2\theta\,\dot{\theta} + \dfrac{1}{2}kb^2\,2\theta\dot{\theta} = 0$

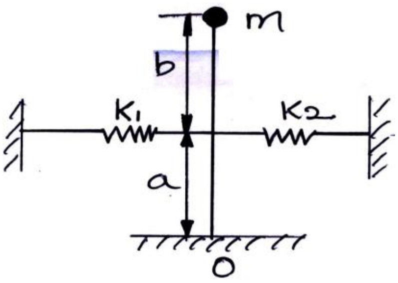
i.e., $ml^2\ddot{\theta} + mgl\theta + ka^2\theta + kb^2\theta = 0$ $[\because \sin\theta \cong \theta]$

i.e., $\ddot{\theta} + \dfrac{\left(mgl+ka^2+kb^2\right)}{ml^2}\theta = 0$

i.e., Circular frequency of vibration $\omega_n = \sqrt{\dfrac{\left(mgl+ka^2+kb^2\right)}{ml^2}}$ rad / sec

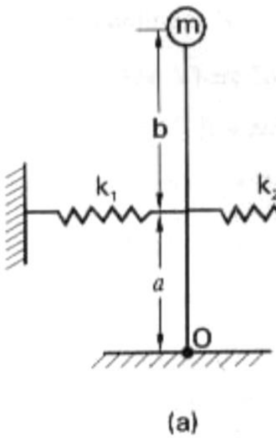
Natural frequency of vibration $f_n = \dfrac{1}{2\pi}\,\omega_n Hz = \dfrac{1}{2\pi}\sqrt{\dfrac{\left(mgl+ka^2+kb^2\right)}{ml^2}}\,Hz$

10. Find the natural frequency of vibration of the system for small amplitudes. If K_1, K_2, a and b or (a+b) are fixed, determine the value of "b" for which the system will not vibrate. Find maximum acceleration of the mass.



[VTU, June/July 2015]

Solution :



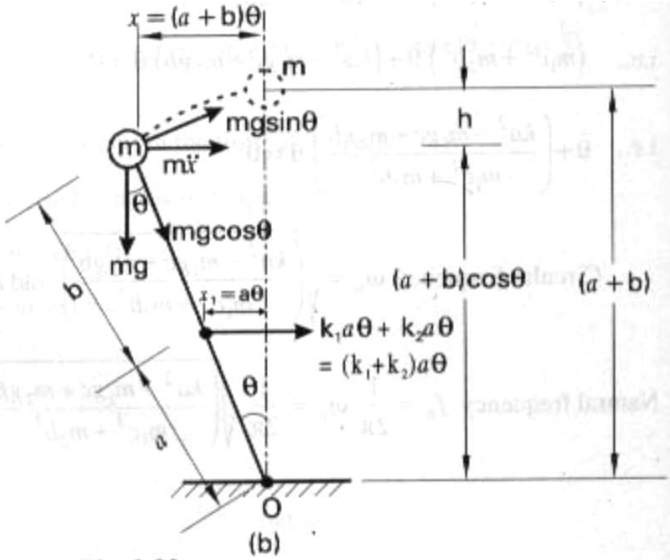


Fig. 2.30

Fig. 2.30 (b) shows the free body diagram of the given system.

For small values of θ, $\sin\theta \approx \theta$ and $\cos\theta = 1$. $I_0 = m(a+b)^2$

(i) Natural frequency of oscillation

Newton's method

According to Newton's second law of motion in the form of torques,

$I_0\,\ddot{\theta} = \Sigma M$ where ΣM = Sum of restoring couples about 0

$$\text{i.e.,}\quad m(a+b)^2\,\ddot{\theta} = -(k_1+k_2)\,a\,\theta.a + mg\,(a+b)\,\theta$$

$$\text{i.e.,}\quad m(a+b)^2\,\ddot{\theta} + (k_1+k_2)\,a^2\,\theta - mg\,(a+b)\,\theta = 0$$

$$\text{i.e.,}\quad m(a+b)^2\,\ddot{\theta} + \{(k_1+k_2)\,a^2 - mg\,(a+b)\}\,\theta = 0$$

$$\text{i.e.,}\quad \ddot{\theta} + \frac{\{(k_1+k_2)a^2 - mg(a+b)\}}{m(a+b)^2}\cdot\theta = 0$$

$$\text{i.e.,}\quad \ddot{\theta} + \left\{\frac{(k_1+k_2)a^2}{m(a+b)^2} - \frac{g}{(a+b)}\right\}\theta = 0$$

$$\therefore \text{ Circular frequency } \omega_n = \sqrt{\left\{\frac{(k_1+k_2)a^2}{m(a+b)^2} - \frac{g}{(a+b)}\right\}} \text{ rad/sec.}$$

$$\text{Natural frequency of oscillation } f_n = \frac{1}{2\pi}\,\omega_n = \frac{1}{2\pi}\sqrt{\left\{\frac{(k_1+k_2)a^2}{m(a+b)^2} - \frac{g}{(a+b)}\right\}} \text{ Hz.}$$

Energy method

From Fig. 2.30 (b)

$x = (a+b)\,\theta\ ;\quad x_1 = a\theta\ ;\quad h = (a+b) - (a+b)\cos\theta = (a+b)\,(1-\cos\theta)$

KE of the system $= \dfrac{1}{2}\, m_1\, \dot{x}^2 = \dfrac{1}{2}\, m\left[(a+b)\,\dot{\theta}\right]^2 = \dfrac{1}{2}\, m\,(a+b)^2\,\dot{\theta}^2$

PE of the system = PE of mass + PE of springs

$$= -mgh + \frac{1}{2}\, k_1\, x_1^2 + \frac{1}{2} k_2\, x_1^2 = -mg\,(a+b)(1-\cos\theta) + \frac{1}{2}\, k_1 a^2\theta^2 + \frac{1}{2}\, k_2 a^2\theta^2$$

$$= -mg\,(a+b)\,(1-\cos\theta) + \frac{1}{2}\,(k_1+k_2)\,a^2\theta^2$$

[Loss of energy due to mass, since the level is lowered $\therefore -mgh$]

According to energy method, $KE + PE$ = Constant

$$\text{i.e.,}\quad \frac{d}{dt}\,(KE+PE) = 0$$

$$\text{i.e.,}\quad \frac{d}{dt}\left[\frac{1}{2}\, m\,(a+b)^2\,\dot{\theta}^2 - mg\,(a+b)\,(1-\cos\theta) + \frac{1}{2}\,(k_1+k_2)a^2\,\theta^2\right] = 0$$

$$\text{i.e.,}\quad \frac{1}{2}\, m\,(a+b)^2\, 2\dot{\theta}\,\ddot{\theta} - mg\,(a+b)\,(\sin\theta)\,\dot{\theta} + \frac{1}{2}\,(k_1+k_2)\,a^2\, 2\theta\,\dot{\theta} = 0$$

$$\text{i.e.,}\quad m\,(a+b)^2\,\ddot{\theta} - mg\,(a+b)\,\theta + (k_1+k_2)\,a^2\theta = 0 \quad [\because \sin\theta \cong \theta]$$

$$\text{i.e.,}\quad \ddot{\theta} + \left[\frac{(k_1+k_2)\,a^2 - mg\,(a+b)}{m\,(a+b)^2}\right]\theta = 0$$

$$\therefore\quad \text{Circular frequency } \omega_n = \sqrt{\left[\frac{(k_1+k_2)\,a^2 - mg\,(a+b)}{m\,(a+b)^2}\right]}\ \text{rad / sec}$$

$$\text{Natural frequency } f_n = \frac{1}{2\pi}\,\omega_n = \frac{1}{2\pi}\sqrt{\left[\frac{(k_1+k_2)\,a^2 - mg\,(a+b)}{m\,(a+b)^2}\right]}\ Hz$$

$$= \frac{1}{2\pi}\sqrt{\frac{(k_1+k_2)\,a^2}{m\,(a+b)^2} - \frac{g}{(a+b)}}\ Hz$$

ii) If $m, k_1, k_2\ (a+b)$ are fixed, then $\omega_n = 0$. When $\omega_n = 0$, the system will not vibrate

$$\text{i.e.,}\quad \frac{(k_1+k_2)a^2}{m(a+b)^2} - \frac{g}{(a+b)} = 0$$

$$\text{i.e.,}\quad \frac{(k_1+k_2)a^2}{m(a+b)^2} = \frac{g}{(a+b)}$$

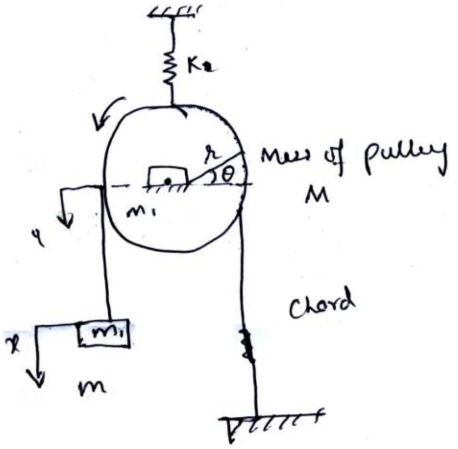
$$\text{i.e.,}\quad (k_1+k_2)\,a^2 = gm\,(a+b)$$

$$= mga + mgb$$

$$\therefore\quad b = \frac{(k_1+k_2)a^2 - mga}{mg} = a\left\{\frac{(k_1+k_2)a - mg}{mg}\right\}$$

$$\text{Hence, when } b = a\left[\frac{(k_1+k_2)a}{mg} - 1\right],\ \text{the system will not vibrate}$$

11. Using energy method, determine the natural frequency of the system shown in Fig.



Solution :

From Fig. 2.53

$x = r\theta\ ;\ \dot{x} = r\dot{\theta}\ ;\ \ddot{x} = r\ddot{\theta}$

Moment of inertia of pulley about the centre O, $I_0 = \frac{1}{2} Mr^2$

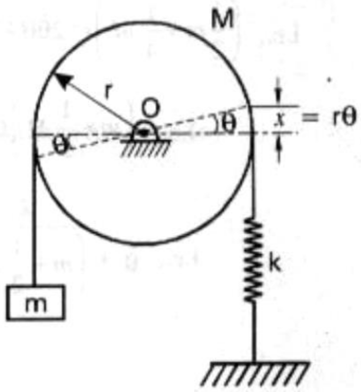


Fig. 2.53

a) Newton's method

According to Newton's second law of motion in the form of torques

$$(m\ddot{x})\,r + I_0\ddot{\theta} = -(kx)\,.\,r$$

$$\text{i.e.,}\quad (mr\ddot{\theta})\,r + \frac{1}{2}Mr^2\ddot{\theta} = -(kr\theta)\,r$$

$$\text{i.e.,}\quad \left(m + \frac{1}{2}M\right) r^2\ddot{\theta} + kr^2\theta = 0$$

$$\text{i.e.,}\quad \left(m + \frac{1}{2}M\right)\ddot{\theta} + k\theta = 0$$

$$\therefore\ \omega_n = \sqrt{\frac{k}{\left(m + \frac{1}{2}M\right)}}\ \text{rad/sec.}$$

$$\text{Natural frequency } f_n = \frac{1}{2\pi}\,\omega_n = \frac{1}{2\pi}\sqrt{\frac{k}{\left(m + \frac{1}{2}M\right)}}\ \text{Hz.}$$

b) Energy method

$$\text{Kinetic energy} = KE \text{ of mass} + KE \text{ of pulley}$$

$$= \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}I_0\dot{\theta}^2 = \frac{1}{2}m(r\dot{\theta})^2 + \frac{1}{2}\left(\frac{1}{2}Mr^2\right)\dot{\theta}^2$$

$$= \left(\frac{1}{2}m + \frac{1}{4}M\right) r^2\dot{\theta}^2$$

$$\text{Potential energy} = \frac{1}{2}kx^2 = \frac{1}{2}k(r\theta)^2 = \frac{1}{2}kr^2\theta^2$$

According to energy method, $KE + PE = \text{constant}$

$$\text{i.e.,}\quad \frac{d}{dt}(KE + PE) = 0$$

$$\text{i.e.,}\quad \frac{d}{dt}\left[\left(\frac{1}{2}m+\frac{1}{4}M\right)r^2\dot{\theta}^2+\frac{1}{2}kr^2\theta^2\right]=0$$

$$\text{i.e.,}\quad \left(\frac{1}{2}m+\frac{1}{4}M\right)r^2 2\dot{\theta}\ddot{\theta}+\frac{1}{2}kr^2.2\theta.\dot{\theta}=0$$

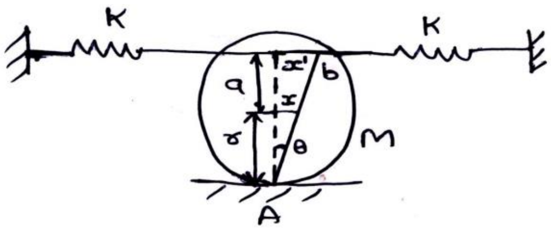
$$\text{i.e.,}\quad \left(m+\frac{1}{2}M\right)\ddot{\theta}+k\theta = 0$$

$$\text{i.e.,}\quad \ddot{\theta}+\frac{k}{\left(m+\frac{1}{2}M\right)}\theta = 0$$

$$\therefore\quad \omega_n = \sqrt{\frac{k}{\left(m+\frac{1}{2}M\right)}}\ \text{rad/sec.}$$

$$\text{Natural frequency } f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi}\sqrt{\frac{k}{\left(m+\frac{1}{2}M\right)}}\ \text{Hz.}$$

12. Find the natural frequency of the system shown in Fig, by i) Newton's method ii) Energy method.



Solution :

From Fig. 2.55, $x = r\theta$; $\dot{x} = r\dot{\theta}$; $\ddot{x} = r\ddot{\theta}$

x' = Extension of left side spring = Compression of right side spring = $(r+a)\theta$

Energy method

Kinetic energy = Rotational KE of pulley + Translational KE of pulley

$$= \frac{1}{2}I_0\dot{\theta}^2+\frac{1}{2}M\dot{x}^2 =$$

$$\frac{1}{2}\left(\frac{1}{2}Mr^2\right)\dot{\theta}^2+\frac{1}{2}M(r\dot{\theta})^2$$

$$= \frac{1}{4}Mr^2\dot{\theta}^2+\frac{1}{2}Mr^2\dot{\theta}^2 = \frac{3}{4}Mr^2\dot{\theta}^2.$$

Potential energy = PE due to left spring + PE due to right spring

$$= \frac{1}{2}k\{(r+a)\theta\}^2+\frac{1}{2}k\{(r+a)\theta\}^2 = 2\frac{1}{2}k(r+a)^2\theta^2$$

$$= k(r+a)^2\theta^2$$

According to energy method, $KE + PE$ = Constant

$$\text{i.e.,}\quad \frac{d}{dt}(KE+PE) = 0$$

i.e., $\frac{d}{dt}\left\{\frac{3}{4}Mr^2\dot{\theta}^2 + k(r+a)^2\theta^2\right\} = 0$

i.e., $\frac{3}{4}Mr^2\, 2\,\dot{\theta}\,\ddot{\theta} + k(r+a)^2\, 2\theta.\dot{\theta} = 0$

i.e., $\frac{3}{4}Mr^2\,\ddot{\theta} + k(r+a)^2\theta = 0$

i.e., $\ddot{\theta} + \frac{4k(r+a)^2}{3Mr^2}\cdot\theta = 0$

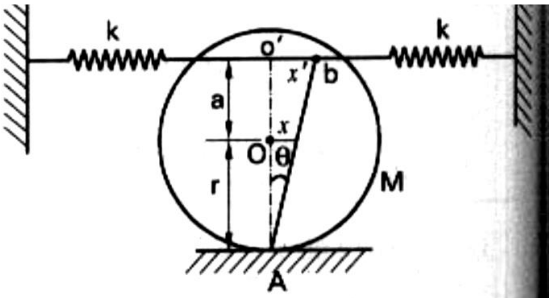


Fig. 2.55

$\therefore$ Circular frequency $\omega_n = \sqrt{\frac{4k(r+a)^2}{3Mr^2}}$ rad/sec $= \frac{(r+a)}{r}\sqrt{\frac{4k}{3M}}$ rad / sec

$\therefore$ Natural frequency $f_n = \frac{1}{2\pi}\omega_n Hz = \frac{1}{2\pi}\sqrt{\frac{4k(r+a)^2}{3Mr^2}}$ Hz. $= \frac{1}{2\pi}\cdot\frac{(r+a)}{r}\sqrt{\frac{4k}{3M}}$ Hz

(b) Newton's method

From Fig.. 2.55, $x = r\theta$; $\dot{x} = r\dot{\theta}$; $\ddot{x} = r\ddot{\theta}$

x' = Extension of leftside spring = Compression of right side spring = $(r+a)\,\theta$

Moment of inertia of pulley about A, $I_A = I_0 + Mr^2 = \frac{1}{2}Mr^2 + Mr^2$

According to Newton's second law of motion in the form of torques

$I_A\ddot{\theta} = \Sigma M$ where ΣM = Sum of restoring couples about A

i.e., $\left(\frac{1}{2}Mr^2 + Mr^2\right)\ddot{\theta} = -\{k(r+a)\theta\}(r+a) - \{k(r+a)\theta\}(r+a)$

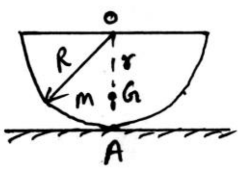
i.e., $\frac{3}{2}Mr^2\ddot{\theta} = -k(r+a)^2\theta - k\left(r+a^2\right)\theta$; i.e., $\frac{3}{2}Mr^2\ddot{\theta} = -2k(r+a)^2\theta$

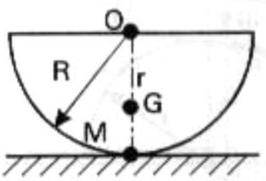
i.e., $\frac{3}{2}Mr^2\ddot{\theta} + 2k(r+a)^2\theta = 0$; i.e., $\ddot{\theta} + \frac{4k(r+a)^2}{3Mr^2}\theta = 0$

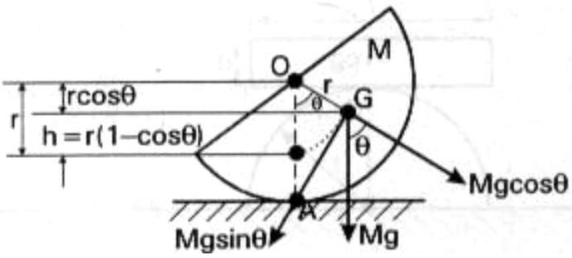
$\therefore$ Circular frequency of vibration $\omega_n = \sqrt{\frac{4k(r+a)^2}{3Mr^2}}$ rad / sec $= \frac{(r+a)}{r}\sqrt{\frac{4k}{3M}}$ rad / sec

Natural frequency of vibration $f_n = \frac{1}{2\pi}\omega_n Hz = \frac{1}{2\pi}\cdot\frac{(r+a)}{r}\sqrt{\frac{4k}{3M}}$ Hz

13. Find the natural frequency of vibration of the half solid cylinder shown in Fig, when slightly displaced from the equilibrium position and released by using i) Newton’s method ii) Energy method.







Solution :

For a small angular displacement θ, the free body diagram is as shown in Fig. 2.74 (b)

Distance of mass centre G of half solid cyclinder from $O = r = \frac{4R}{3\pi}$

Moment of inertia of half solid cylinder about A, $I_A = I_G + M \cdot GA^2$

Assume $GA = OA - OG$, which is true for small amplitudes of vibration

$$\therefore \; I_A = I_G + M\,(OA - OG)^2 = I_0 - M\,r^2 + M\,(R-r)^2 \; [\because \; OA = R \;\text{ and }\; OG = r]$$

$$= \frac{1}{2}MR^2 - M\,r^2 + M\left(R^2 + r^2 - 2Rr\right) = \frac{1}{2}MR^2 - M\,r^2 + MR^2 + Mr^2 - 2\,MRr$$

$$= \frac{3}{2}MR^2 - 2MR\frac{4R}{3\pi} = MR^2\left(\frac{3}{2} - \frac{8}{3\pi}\right) \quad \left[\because \; r = \frac{4R}{3\pi}\right]$$

(a) Newton's method

According to Newton's second law of motion in the form of torques

$I_A\,\ddot{\theta} = \Sigma M$ where ΣM = Sum of restoring couples

i.e., $$MR^2\left(\frac{3}{2} - \frac{8}{3\pi}\right)\ddot{\theta} = -\left(Mg\sin\theta\right)r$$

i.e., $$MR^2\left(\frac{9\pi - 16}{6\pi}\right)\ddot{\theta} = -Mg\theta \cdot \frac{4R}{3\pi} \quad \left(\because \; \sin\theta \cong \theta \;\text{ and }\; r = \frac{4R}{3\pi}\right)$$

i.e., $$MR^2\left(\frac{9\pi - 16}{6\pi}\right)\ddot{\theta} + Mg\cdot\frac{4R}{3\pi}\cdot\theta = 0$$

i.e., $$\ddot{\theta} + \frac{Mg\,\frac{4R}{3\pi}\cdot\theta}{MR^2\left(\frac{9\pi-16}{6\pi}\right)} = 0 \;\; ; \;\; \therefore \quad \ddot{\theta} + \frac{8g}{(9\pi - 16)\,R}\cdot\theta = 0$$

$$\therefore \quad \text{Circular frequency } \omega_n = \sqrt{\frac{8g}{(9\pi - 16)\,R}} \;\text{ rad / sec}$$

$$\text{Natural frequency } f_n = \frac{1}{2\pi}\,\omega_n = \frac{1}{2\pi}\sqrt{\frac{8g}{(9\pi - 16)\,R}} \;\; Hz$$

(b) Energy method

Potential energy $= Mgh = Mgr\,(1-\cos\theta) = Mg\left(\frac{4R}{3\pi}\right)(1-\cos\theta) \quad \left(\because \; r = \frac{4R}{3\pi}\right)$

Kinetic energy $= \frac{1}{2}\,I_A\,\dot{\theta}^2 = \frac{1}{2}\,MR^2\left(\frac{3}{2} - \frac{8}{3\pi}\right)\dot{\theta}^2 = \frac{1}{2}\,MR^2\left(\frac{9\pi-16}{6\pi}\right)\dot{\theta}^2$

According to energy method, $KE + PE$ = Constant

i.e., $$\frac{d}{dt}(KE + PE) = 0 \;\; ; \;\text{i.e.,}\; \frac{d}{dt}\left\{\frac{1}{2}MR^2\left(\frac{9\pi-16}{6\pi}\right)\dot{\theta}^2 + Mg\left(\frac{4R}{3\pi}\right)(1-\cos\theta)\right\} = 0$$

i.e., $$\frac{1}{2}MR^2\left(\frac{9\pi-16}{6\pi}\right)2\,\dot{\theta}\,\ddot{\theta} + Mg\left(\frac{4R}{3\pi}\right)(\sin\theta)\cdot\dot{\theta} = 0$$

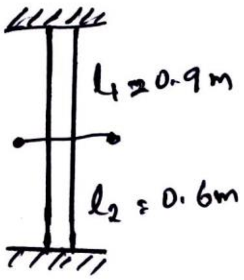
i.e., $$R\left(\frac{9\pi-16}{2}\right)\cdot\ddot{\theta} + 4g\theta = 0 \qquad [\because \; \sin\theta \cong \theta]$$

$$\therefore \quad \ddot{\theta} + \frac{8g}{R(9\pi - 16)} \cdot \theta = 0$$

$$\therefore \quad \text{Circular frequency } \omega_n = \sqrt{\frac{8g}{R(9\pi-16)}} \text{ rad/sec}$$

$$\text{Natural frequency } f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi}\sqrt{\frac{8g}{R(9\pi-16)}} \ Hz$$

14. A flywheel is mounted on a vertical shaft as shown in Fig. Both ends of shaft are fixed and diameter is 50mm. The flywheel has a mass of 500 kg and radius of gyration 0.5m find natural frequency of (i) longitudinal vibration (ii) Transverse vibration (iii) Torsional vibrations. Take E = 200 GN/m2, G = 84 GN/m2, d = 50 mm, m = 500 kg, k = 0.5m



Data :

$$d = 50 \text{ mm} = 0.05 \text{ m}$$
$$m = 500 \text{ kg}$$
$$k = 0.5 \text{ m}$$
$$E = 200 \text{ GN/m}^2 = 200 \times 10^9 \text{ N/m}^2$$
$$G = 84 \text{ GN/m}^2 = 84 \times 10^9 \text{ N/m}^2$$

Solution :

i) Longitudinal vibration

Let m_1 be the mass of flywheel carried by the portion of length l_1. $\therefore$ mass carried by the portion of length $l_2 = m - m_1$

Since extension of the portion l_1 = compression of portion l_2

$$\frac{m_1 g l_1}{AE} = \frac{(m-m_1).gl_2}{AE}$$
$$m_1 \times 0.9 = (500 - m_1)0.6$$
$$\text{i.e., } 0.9m_1 = 300 - 0.6m_1$$
$$\therefore m_1 = 200 \text{ kg.}$$
$$\therefore m_2 = 300 \text{ kg}$$
$$\therefore \delta = \frac{m_1 g l_1}{AE}$$
$$= \frac{200 \times 9.81 \times 0.9}{\frac{\pi}{4} \times 0.05^2 \times 200 \times 10^9}$$
$$= 4.496 \times 10^{-6} \text{ m}$$

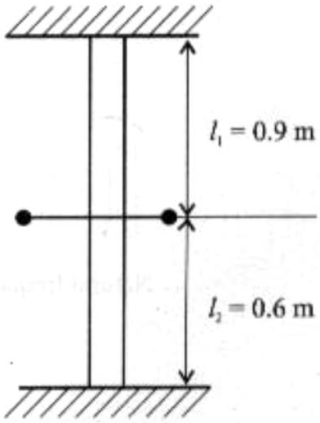


Fig. 2.80

$$\therefore \text{Circular frequency } \omega_n = \sqrt{g/\delta} = \sqrt{\frac{9.81}{4.496 \times 10^{-6}}} = 1477.14 \, rad/\sec$$

$$\therefore \text{Natural frequency } f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi} \times 1477.14 = 235.1 Hz$$

ii) Transverse vibration

$$\delta = \frac{Wl_1^3 l_2^3}{3EIl^3} = \frac{mgl_1^3 l_2^3}{3EIl^3}$$

$$= \frac{500 \times 9.81 \times 0.9^3 \times 0.6^3}{3 \times 200 \times 10^9 \times \frac{\pi}{64} \times (0.05)^4 \times 1.5^3}$$

$$(\because l = l_1 + l_2 = 0.9 + 0.6 = 1.5 \text{ m})$$

$$= 1.243 \times 10^{-3} \text{ m}$$

$$\therefore \text{Circular frequency } \omega_n = \sqrt{g/\delta} = \sqrt{\frac{9.81}{1.243 \times 10^{-3}}} = 88.838 \; rad/\sec$$

$$\therefore \text{Natural frequency } f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi} \times 88.838 = 14.14 Hz$$

iii) Torsional vibration

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{t_e}}{I}} Hz$$

where k_{t_e} = Equivalent torsional stiffness $= k_{t1} + k_{t2}$

$$k_{t_1} = \frac{GJ}{l_1} = \frac{84 \times 10^9 \times \frac{\pi}{32} \times 0.05^4}{0.9} = 57268.61$$

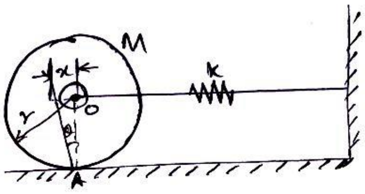
$$k_{t_2} = \frac{GJ}{l_2} = \frac{84 \times 10^9 \times \frac{\pi}{32} \times 0.05^4}{0.6} = 85902.92$$

$$I = mk^2 = 500 \times 0.5^2 = 125 \text{ kgm}^2$$

$$k_{t_e} = k_{t_1} + k_{t_2} = 57268.61 + 85902.92 = 143171.53$$

$$\therefore \text{Natural frequency } f_n = \frac{1}{2\pi}\sqrt{\frac{k_{t_e}}{I}} Hz = \frac{1}{2\pi}\sqrt{\frac{14317153}{125}} = 5.386 \text{ Hz.}$$

15. Determine the natural frequency of the system shown in Fig.



From Fig. 2.54, $x = r\theta \quad \therefore \quad \dot{x} = r\dot{\theta}$ and $\ddot{x} = r\ddot{\theta}$

Moment of inertia of pulley about A, $I_A = I_0 + Mr^2 = \frac{1}{2}Mr^2 + Mr^2$.

a) Newton's method

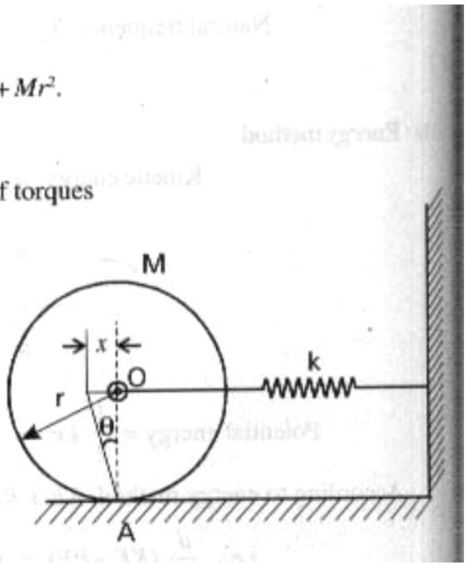
According to Newton's second law of motion in the form of torques

$$I_A \ddot{\theta} = -(kx)\, r$$

i.e., $$\left(\frac{1}{2}Mr^2 + Mr^2\right)\ddot{\theta} = -k(r\theta)\, r$$

i.e., $$\frac{3}{2}Mr^2\ddot{\theta} + kr^2\theta = 0$$

i.e., $$\ddot{\theta} + \frac{k}{\frac{3}{2}M}\theta = 0$$



i.e., $\ddot{\theta} + \frac{2k}{3M}\theta = 0$

$\therefore\ \omega_n = \sqrt{\frac{2k}{3M}}$ rad/sec ; Natural frequency $f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi}\sqrt{\frac{2k}{3m}}$ Hz.

b) Energy method

Kinetic energy = Rotational KE of pulley + Translational KE of pulley

$$= \frac{1}{2} I_0 \dot{\theta}^2 + \frac{1}{2} M \dot{x}^2 = \frac{1}{2}\left(\frac{1}{2}Mr^2\right)\dot{\theta}^2 + \frac{1}{2} M (r\dot{\theta})^2$$

$$= \frac{1}{4} Mr^2 \dot{\theta}^2 + \frac{1}{2} Mr^2 \dot{\theta}^2 = \frac{3}{4} Mr^2 \dot{\theta}^2$$

Potential energy $= \frac{1}{2} kx^2 = \frac{1}{2} k (r\theta)^2 = \frac{1}{2} kr^2\theta^2$

According to energy method, $KE + PE$ = constant

i.e., $\frac{d}{dt}(KE + PE) = 0$

i.e., $\frac{d}{dt}\left(\frac{3}{4}Mr^2\dot{\theta}^2 + \frac{1}{2}kr^2\theta^2\right) = 0$

i.e., $\frac{3}{4} Mr^2\, 2\,\dot{\theta}\ddot{\theta} + \frac{1}{2} kr^2\, 2\,\theta.\dot{\theta} = 0$

i.e., $\frac{3}{2} M\ddot{\theta} + k\theta = 0$

$\ddot{\theta} + \frac{2k}{3M}\theta = 0$

$\therefore\ \omega_n = \sqrt{\frac{2}{3}\frac{k}{M}}$ rad/sec ; Natural frequency $f_n = \frac{1}{2\pi}\sqrt{\frac{2k}{3M}}$ Hz.

16. Determine the natural frequency of the simple pendulum i) Neglecting the mass of rod and ii) Considering the mass of rod.

i) Neglecting the mass of rod

Consider the simple pendulum shown in Fig. 2.21(a). For the displaced position, the weight of bob can be resolved into two parts one in the tangential direction and the other along the string. It is the tangential force which restores the mass again in equilibrium position.

Let m = Mass of bob

l = Length of rod

I_0 = Mass moment of inertia about 0

$= ml^2$

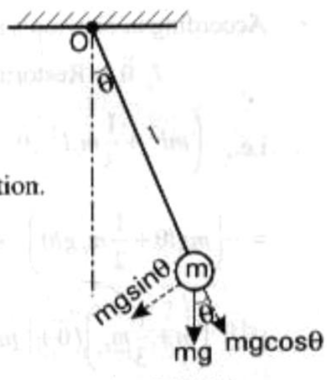


Fig. 2.21 (a)

a) Newton's Method

According to Newton's second law of motion in the form of torque equation.

$I_0\ddot{\theta}$ = Restoring torque

$= -(mg\sin\theta)\, l = -mgl\,\theta$

since θ is small $\sin\theta \approx \theta$

i.e., $ml^2\ddot{\theta} + mgl\,\theta = 0$

i.e., $\ddot{\theta} + \frac{g}{l}\theta = 0$. It is the equation of motion

$\therefore\ \omega_n = \sqrt{\frac{g}{l}}$ rad/sec. ; Natural frequency $f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ Hz.

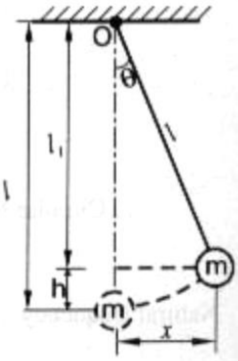
b) Energy Method

From Fig. 2.21 (b) $\cos\theta = \frac{l_1}{l}$ $\therefore$ $l_1 = l\cos\theta$; $x = l\theta$

Kinetic energy $= \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m(l\dot{\theta})^2$

Potential energy $= mgh = mg(l - l_1)$
$= mg(l - l\cos\theta) = mgl(1-\cos\theta)$

According to energy method $KE + PE =$ Constant $\therefore \frac{d}{dt}(KE+PE) = 0$



i.e., $\frac{d}{dt}\left\{\frac{1}{2}m(l\dot{\theta})^2 + mgl(1-\cos\theta)\right\} = 0$

i.e., $\frac{1}{2}ml^2 2\dot{\theta}\ddot{\theta} + mgl(\sin\theta)\dot{\theta} = 0$

Fig. 2.21 (b)

i.e., $\ddot{\theta} + \frac{g}{l}\theta = 0$ Since θ is small, $\sin\theta \approx \theta$

$\therefore \omega_n = \sqrt{\frac{g}{l}}$ rad/sec; Natural frequency $f_n = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$ Hz.

ii) Considering the mass of rod

Let m_r be the mass of rod and it acts through the centre of the rod halfway from both ends as shown in Fig. 2.21(c).

a) Newton's Method

According to Newton's second law of motion in the form of torque equation

$$I_o\ddot{\theta} = \text{Restoring torque}$$

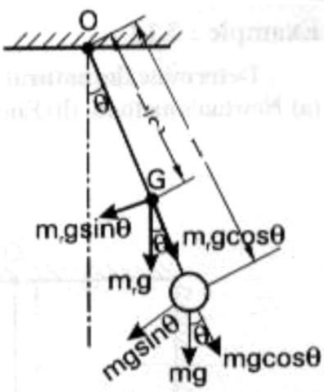
i.e., $\left(ml^2 + \frac{1}{3}m_r l^2\right)\ddot{\theta} = -(mg\sin\theta)\,l - (m_r g\sin\theta)\,\frac{l}{2}$

$= -\left(mgl\theta + \frac{1}{2}m_r gl\theta\right)$ since θ is small $\sin\theta \approx \theta$

i.e., $\left(m + \frac{1}{3}m_r\right)l\ddot{\theta} + \left(m + \frac{1}{2}m_r\right)g\theta = 0$

$$\therefore \ddot{\theta} + \frac{\left(m + \frac{1}{2}m_r\right)}{\left(m + \frac{1}{3}m_r\right)}\cdot\frac{g}{l}\theta = 0$$

$$\therefore \omega_n = \sqrt{\frac{\left(m + \frac{1}{2}m_r\right)}{\left(m + \frac{1}{3}m_r\right)}\frac{g}{l}} \text{ rad/sec}$$



Natural frequency of free vibration $f_n = \frac{1}{2\pi}\sqrt{\frac{\left(m + \frac{1}{2}m_r\right)}{\left(m + \frac{1}{3}m_r\right)}\frac{g}{l}}$ Hz.

Fig. 2.21 (c)

b) Energy Method

Kinetic energy $= KE$ of mass $+ KE$ of rod $= \frac{1}{2}m(\dot{\theta}l)^2 + \frac{1}{2}\cdot\frac{1}{3}m_r(\dot{\theta}l)^2$

$$\text{Potential energy} = PE \text{ of mass} + PE \text{ of rod}$$

$$= mgl(1-\cos\theta) + m_r g \frac{l}{2}(1-\cos\theta)$$

Since $KE + PE = \text{constant}$

$$\frac{d}{dt}(KE+PE) = 0$$

i.e., $$\frac{1}{2}ml^2\, 2\,\dot{\theta}\ddot{\theta} + \frac{1}{6}m_r l^2\, 2\dot{\theta}\ddot{\theta} + mgl(\sin\theta)\dot{\theta} + m_r g\frac{l}{2}(\sin\theta)\dot{\theta} = 0$$

i.e., $$\left(m+\frac{1}{3}m_r\right)l\ddot{\theta} + \left(m+\frac{m_r}{2}\right)g\theta = 0 \quad (\because \sin\theta \approx \theta)$$

$$\therefore \quad \ddot{\theta} + \frac{\left(m+\frac{m_r}{2}\right)}{\left(m+\frac{m_r}{3}\right)}\cdot\frac{g}{l}\theta = 0$$

Hence $$\omega_n = \sqrt{\frac{\left(m+\frac{m_r}{2}\right)}{\left(m+\frac{m_r}{3}\right)}\frac{g}{l}} \text{ rad/sec and } f_n = \frac{1}{2\pi}\sqrt{\left(\frac{m+m_r/2}{m+m_r/3}\right)\frac{g}{l}} \text{ Hz.}$$

17. What are the types of damping? Explain any two types of damping.

The following are the common types of damping.

(i) Viscous damping

(ii) Coulomb damping

(iii) Solid or structural damping

(iv) Slip or interfacial damping.

(i) Viscous damping

It is the most common type of damping. When a system is allowed to vibrate in a liquid or viscous medium, the damping is called as viscous damping. The resisting force experiences in viscous damping is proportional to the velocity. i.e., $F \alpha v \therefore F = cv = c\dot{x}$ where c is the damping constant or coefficient of viscous damping. Two important types of viscous dampers commonly used are

(a) Fluid dash pot (b) Eddy current damping.

(iii) Solid or structural damping

This type of damping is due to the internal friction of the molecules. Due to vibratory motion, materials are cyclically stressed and energy is dissipated due to intermolecular friction. For most structural materials like steel, aluminium, the energy dissipated is found to be a function of amplitude only over a wide frequency range. Experiments show that for elastic materials for loading and unloading conditions a loop is formed on stress-strain curve. The area of this loop represents the energy dissipated due to molecular friction per cycle per unit volume. The size of the loop depends upon the material of the vibrating body, frequency and the amount of dynamic stress. This loop is called hysteresis loop (Figure. 3.7a). Hence this damping is also called as hysteresis damping.

(iv) Slip or interfacial damping

Machine elements are connected by means of various types of joints. Energy of vibration is dissipated by microscopic slip on the interfaces of machine parts in contact under fluctuating loads. Slip also occurs on the interfaces of machine elements, forming various types of joints. The energy dissipated per cycle depends upon the coefficient of friction, the pressure between the contacting parts and the amplitude. The amount of damping depends upon the energy dissipated per cycle. At zero pressure there is no energy loss since no energy is dissipated in friction but there is large slip. At very high pressure also no energy loss since there is no slip. Therefore there is an optimum value of pressure for which the energy dissipated is maximum. The effective damping is larger for larger the energy dissipation. Figure 3.9 shows the variation of slip damping with contact pressure.

18. Derive the equations of motion for damped free vibration with usual notations. Formulate and discuss the response of a critically damped and over damped system.

or

Set up the differential equation for a spring mass damper system and obtain the complete solution for the over damped condition.

or

If x(t) represents general response for an damped free vibration system, then obtain the solution for critically damped system and also plot the response and hence give its applications.

$x(t) = A_1 e^{(-\xi+\sqrt{\xi^2-1})\omega_n t} + A_2 e^{(-\xi-\sqrt{\xi^2-1})\omega_n t}$ where ξ = damping ratio.

or

Derive the equation for damped free vibration and solve for critical damping system.

Consider a spring carrying a mass at one end and the other end of which is fixed. A damper is provided between the mass and the rigid support as shown in Fig 3.10.

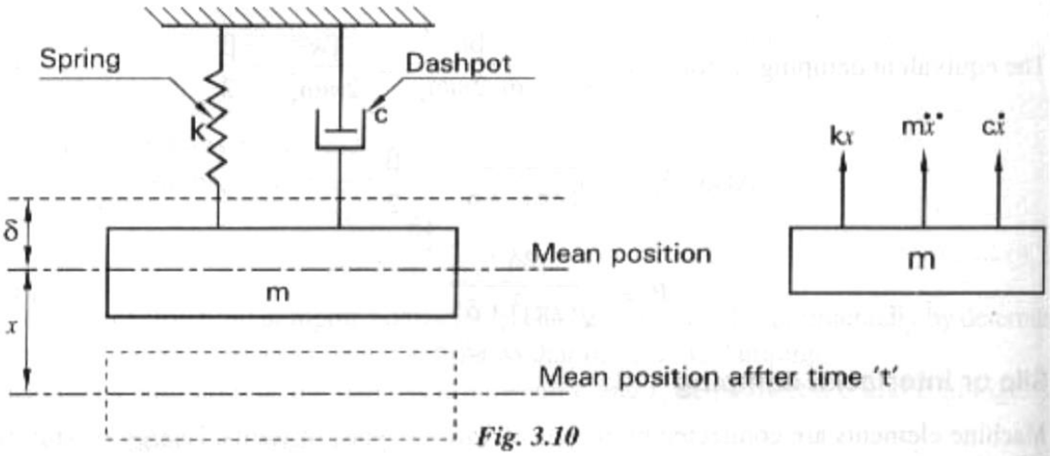


Fig. 3.10

Let k = Stiffness of spring = $\frac{mg}{\delta}$

c = damping coefficient (damping force per unit velocity)

ω_n = Frequency of natural undamped vibrations

x = displacement of mass from mean positions at time 't'

$\dot{x}$ = velocity of mass at time 't'

$\ddot{x}$ = Acceleration of mass at time 't'

kx = spring force

$m\ddot{x}$ = Inertia force

$c\dot{x}$ = Damping force

m = Mass suspended from the spring

w = Weight of body = mg

δ = static deflection of the spring

Let the body be displaced by a distance x in the downward direction from mean position. Now the forces acting on the body are,

(i) Accelerating force in the direction of motion (i.e., downwards) = $m\frac{d^2x}{dt^2} = m\ddot{x}$

(ii) Damping force or friction force in the opposite direction of motion (i.e., upwards) = $c\frac{dx}{dt} = c\dot{x}$

(iii) Spring force in the opposite direction of motion (i.e., upwards) = kx

For the dynamic equilibrium of the body, the sum of inertia force and external forces in any direction should be zero.

Here the external forces are damping force and spring force. The magnitude of inertia force is same as that of accelerating force but it acts in the opposite direction of accelerating force.

$\therefore$ Inertia force $= m\ddot{x}$, in the opposite direction of motion (i.e., upwards).

Therefore the equation of motion can be written as,

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \text{----- (3.3.1)}$$

Equation 3.3.1 is the differential equation of the system and it can also be written as

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \qquad \text{----- (3.3.2)}$$

A system having the equation of motion as given by equation 3.3.2 is said to be a single degree of freedom damped vibrating system.

To determine the natural frequency (ω_n)

Put $c = 0$

Hence the equation 3.3.2 becomes $\ddot{x} + 0 + \frac{k}{m}.x = 0$

$$\therefore \quad \omega_n = \sqrt{\frac{k}{m}}, \text{ rad/sec} \qquad \text{----- (3.3.3)}$$

To determine the critical damping coefficient

Equation 3.3.2 is the differential equation of second order. Assuming the solution is of the form,

$$x = e^{\alpha t}, \qquad \text{----- (3.3.4)}$$

$$\dot{x} = \alpha e^{\alpha t} \text{ and } \ddot{x} = \alpha^2 e^{\alpha t}$$

Substituting these values in equation 3.3.2, the equation becomes

$$\alpha^2 e^{\alpha t} + \frac{c}{m}\alpha e^{\alpha t} + \frac{k}{m}e^{\alpha t} = 0$$

$$\text{i.e., } \left(\alpha^2 + \frac{c}{m}\alpha + \frac{k}{m}\right)e^{\alpha t} = 0$$

$$\text{i.e., } \alpha^2 + \frac{c}{m}\alpha + \frac{k}{m} = 0 \qquad \text{----- (3.3.5)}$$

$$\therefore \quad \alpha = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{k}{m}}}{2\times 1} = \frac{1}{2}\left[-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}\right]$$

$$\therefore \quad \alpha_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

$$\therefore \quad \alpha_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)} \qquad \text{----- (3.3.6)}$$

The solution given by equation 3.3.6 takes one of the three forms, depending on whether the quantity $\left[\left(\frac{c}{2m}\right)^2 - \frac{k}{m}\right]$ is zero or positive or negative. If this quantity is zero, then

$$\left(\frac{c}{2m}\right)^2 = \frac{k}{m}; \quad \text{i.e.,} \quad \frac{c}{2m} = \sqrt{\frac{k}{m}} = \omega_n$$

$$\therefore \quad c = 2m\omega_n$$

The **critical damping coefficient** c_c is defined as the value of damping coefficient c for which the mathematical term $\left[\left(\frac{c}{2m}\right)^2 - \frac{k}{m}\right]$ in equation 3.3.6 is equal to zero

$$\therefore \quad \text{Critical damping coefficient } c_c = 2m\omega_n = 2\sqrt{mk} \qquad \text{----- (3.3.7)}$$

The ratio of actual damping coefficient to critical damping coefficient is called the **damping factor** or damping ratio. It is denoted by ξ (zeta)

$\therefore$ Damping factor or Damping ratio

$$\xi = \frac{c}{c_c} \qquad \text{----- (3.3.8)}$$

$$\text{We have,} \quad \alpha = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

$$= -\frac{c}{c_c}\omega_n \pm \sqrt{\left(\frac{c}{c_c}\omega_n\right)^2 - \omega_n^2} \quad [\because c_c = 2m\omega_n \text{ and } \omega_n = \sqrt{k/m}]$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} = \left[-\xi \pm \sqrt{\xi^2 - 1}\right]\omega_n$$

$$\therefore \quad \alpha_1 = \left\{-\xi + \sqrt{\xi^2 - 1}\right\}\omega_n$$

$$\alpha_2 = \left\{-\xi - \sqrt{\xi^2 - 1}\right\}\omega_n \qquad \text{----- (3.3.9)}$$

The most general solution for the differential equation 3.3.2 is given by,

$$x = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \qquad \text{----- (3.3.10)}$$

where A_1 and A_2 are two arbitrary constants which are to be determined from the initial conditions of motion of the mass. α_1 and α_2 are its two roots.

Therefore equation 3.3.10 can be written as

$$x = A_1 e^{\left\{-\xi + \sqrt{\xi^2 - 1}\right\}\omega_n t} + A_2 e^{\left\{-\xi - \sqrt{\xi^2 - 1}\right\}\omega_n t} \qquad \text{----- (3.3.11)}$$

Depending upon whether ξ is greater or equal or less than one, these roots may be real or equal or complex conjugate

(i) If $\xi > 1$, the roots are real and negative. The damping is known as over damping.

(ii) If $\xi = 1$, the roots are real and equal. The damping is known as critical damping.

(iii) If $\xi < 1$, the roots are imaginary and both the roots are complex conjugates. The damping is known as under damping.

(i) Over damped system

If $\xi > 1$, then the roots α_1 and α_2 are real but negative. This is a case of over damping or large damping and the mass moves slowly to the equilibrium position. The motion is not periodic (i.e., aperiodic). In actual practice the over damped vibrations are avoided. From equation 3.3.11 the general solution of the motion is,

$$x = A_1 e^{\left\{-\xi + \sqrt{\xi^2 - 1}\right\}\omega_n t} + A_2 e^{\left\{-\xi - \sqrt{\xi^2 - 1}\right\}\omega_n t}$$

The values of constants A_1 and A_2 can be determined from initial conditions.

i.e., when $t = 0$; displacement $x = x_0$ and velocity $v = v_0$

$$\therefore \quad x_{(0)} = x_0 = A_1 + A_2 \qquad \text{------ (3.3.12)}$$

$$\text{Now,} \quad \dot{x}_{(0)} = A_1\left[-\xi + \sqrt{\xi^2 - 1}\right]\omega_n + A_2\left[-\xi - \sqrt{\xi^2 - 1}\right]\omega_n = v_0 \qquad \text{------ (3.3.13)}$$

From equation 3.3.12 and 3.3.13 the values of A_1 and A_2 can be determined.

The value of displacement x goes on decreasing with time. The system is non-vibratory in nature. The characteristics of this type of motion are shown in Fig 3.11a and b

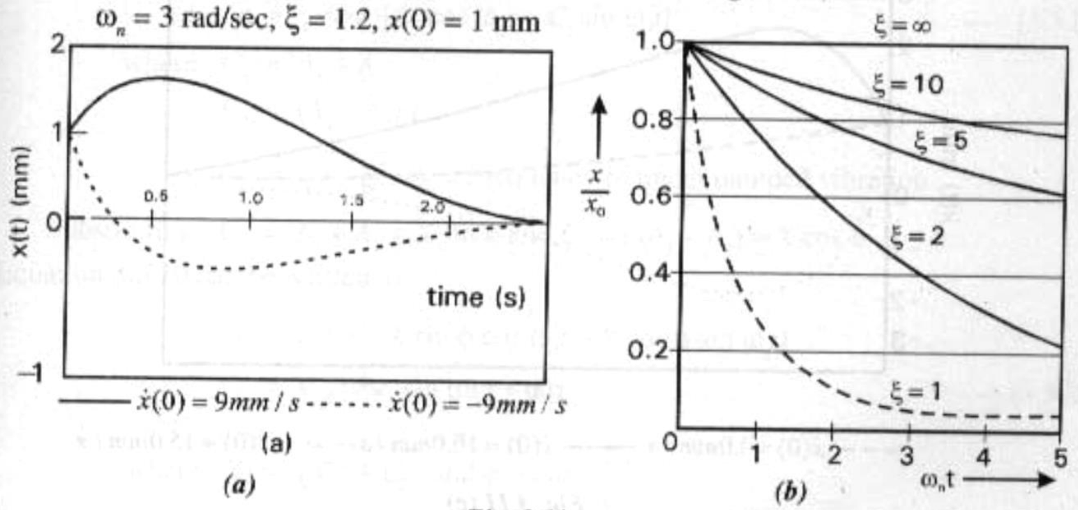


Fig. 3.11

(ii) Critically damped system [Fig 3.11(c)]

If $\xi = 1$, then the radical becomes zero and the roots α_1 and α_2 are real and equal. It is the ca of critical damping i.e., when the frequency of damped vibration is zero, critical damping w occur. The motion in critical damping is not periodic (i.e., it is aperiodic). This type of damping also avoided because the mass moves back rapidly to its equilibrium position.

Since $\xi = 1$, the two roots of the equation 3.3.9, α_1 and α_2 are equal to each other

$$\text{i.e.,} \ \alpha_1 = \alpha_2 = -\omega_n \qquad \text{----- (3.3.1}$$

Now the solution of equation 3.3.2 for $\alpha_1 = \alpha_2$ is given by,

$$x = A_1 e^{\alpha_1 t} + A_2 t e^{\alpha_2 t} = A_1 e^{-\omega_n t} + A_2 t e^{-\omega_n t}$$

$$= [A_1 + A_2 t]\, e^{-\omega_n t} \qquad \text{----- (3.3.1}$$

The values of constants A_1 and A_2 can be determined from initial conditions.

$$\text{i.e., when } t = 0;\ x = x_0 \text{ and } \dot{x} = v_0 = 0$$

$$\therefore x_{(0)} = x_0 = A_1$$

$$\dot{x} = (A_1 + A_2 t)(-\omega_n)\, e^{-\omega_n t} + A_2\, e^{-\omega_n t}$$

$$= A_2\, e^{-\omega_n t} - (A_1 + A_2 t)\,\omega_n\, e^{-\omega_n t}$$

$$\text{Since when } t = 0,\ x_0 = A_1 \text{ and } v_0 = 0 = \dot{x}_{(0)}$$

$$\dot{x}_{(0)} = v_0 = 0 = A_2 - x_0\,\omega_n$$

$$\therefore A_2 = x_0\,\omega_n$$

Substituting the values of A_1 and A_2 in equation 3.3.15

$$x = [x_0 + x_0\,\omega_n t]\, e^{-\omega_n t} = x_0\,[1 + \omega_n t]\, e^{-\omega_n t} \qquad \text{----- (3.3.16}$$

$$\text{If } v_0 \neq 0, \text{ then } A_2 = v_0 + x_0\omega_n$$

$$\therefore x = \{x_0 + (v_0 + x_0\omega_n)t\}\, e^{-\omega_n t} \qquad \text{----- (3.3.17}$$

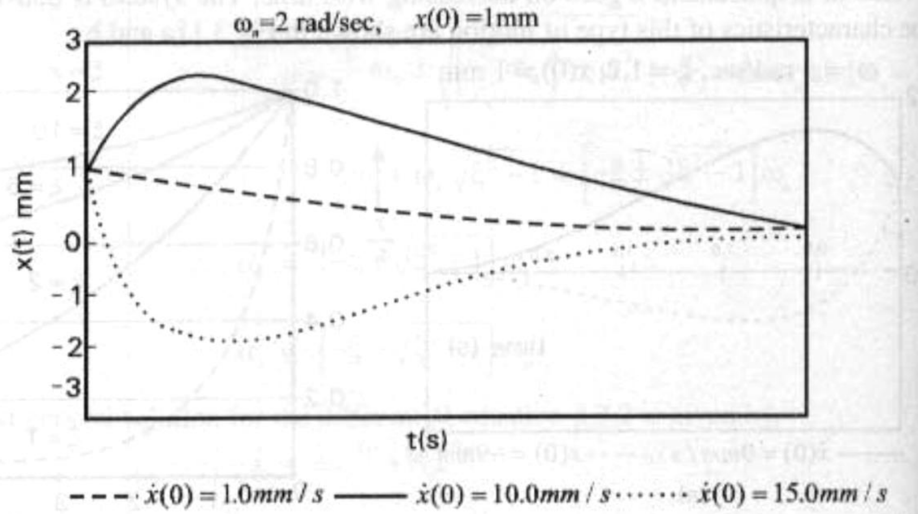


Fig. 3.11 (c)

(iii) Under damped system

For this system $\xi < 1$. Therefore the term under the roots becomes negative. The two roots α_1 and α_2 are thus known as complex conjugate. This is the most general condition that occurs in all physical systems and is known as under damping. This is the only case which leads to oscillatory motion.

Since the two roots of equation 3.3.9, α_1 and α_2 are complex conjugate i.e., imaginary, their values can be written as

$$\alpha_1 = \left\{-\xi + i\sqrt{1-\xi^2}\right\}\omega_n \qquad \text{----- (3.3.18)}$$

$$\text{and} \quad \alpha_2 = \left\{-\xi - i\sqrt{1-\xi^2}\right\}\omega_n$$

Hence the general expression becomes,

$$x = A_1 e^{\left\{-\xi+i\sqrt{1-\xi^2}\right\}\omega_n t} + A_2 e^{\left\{-\xi-i\sqrt{1-\xi^2}\right\}\omega_n t}$$

$$= e^{-\xi\omega_n t}\left[A_1 e^{i\sqrt{1-\xi^2}\omega_n t} + A_2 e^{-i\sqrt{1-\xi^2}\omega_n t}\right] \qquad \text{----- (3.3.19)}$$

According to Euler's Theorem

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Hence equation 3.3.19 can be written as

$$x = e^{-\xi\omega_n t}\,[A_1\cos\sqrt{1-\xi^2}\,\omega_n t + A_1 i\sin\sqrt{1-\xi^2}\,\omega_n t +$$

$$A_2\cos\sqrt{1-\xi^2}\,\omega_n t - A_2 i\sin\sqrt{1-\xi^2}\,\omega_n t]$$

$$= e^{-\xi\omega_n t}\,[(A_1 + A_2)\cos\sqrt{1-\xi^2}\,\omega_n t + (A_1 - A_2)\,i\sin\sqrt{1-\xi^2}\,\omega_n t]$$

$$\text{i.e.,} \quad x = e^{-\xi\omega_n t}\,[C_1\cos\omega_d t + C_2\sin\omega_d t] \qquad \text{----- (3.3.20)}$$

where $C_1 = A_1 + A_2$

$C_2 = (A_1 - A_2)\,i$

$\omega_d = \sqrt{1-\xi^2}\;\omega_n$ = Frequency of under damped vibration

Substituting $C_1 = A_1 + A_2 = X\sin\phi$ and $C_2 = i\,(A_1 - A_2) = X\cos\phi$

Equation 3.3.20 can be written as

$$x = e^{-\xi\omega_n t}\ [X \sin\phi \cos\omega_d t + X \cos\phi \sin\omega_d t]$$

$$= X e^{-\xi\omega_n t} \sin(\omega_d t + \phi) \qquad \text{----- (3.3.21)}$$

where $X = \sqrt{C_1^2 + C_2^2}$ and $\phi = \tan^{-1}\dfrac{C_1}{C_2}$

The values of C_1 and C_2 can be obtained, if the values of A_1 and A_2 are known. The values of A_1 and A_2 can be found from the initial conditions.

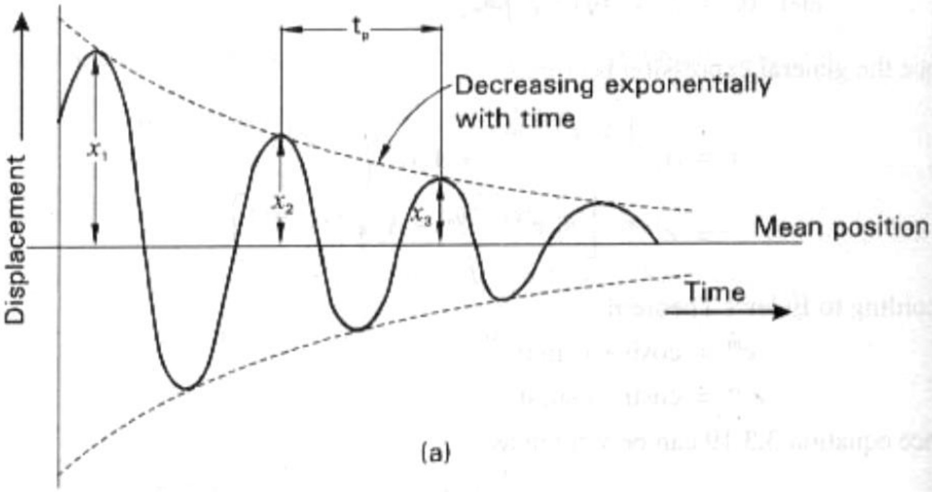
Equation 3.3.21 indicates that the system oscillates with frequency ω_d. As ξ is less than 1, ω_d is always less than ω_n. The solution consists of three terms

→ X which is constant

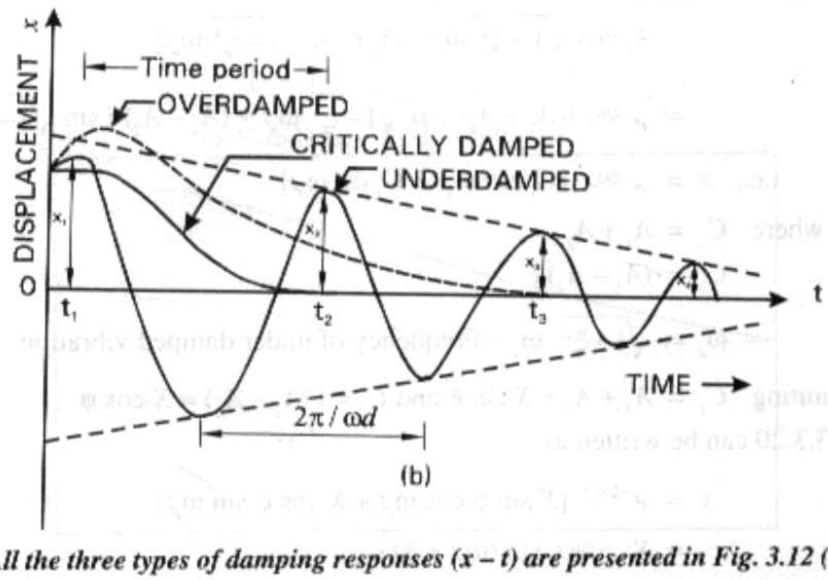
→ $e^{-\xi\omega_n t}$ which decreases with time and finally $e^{-\alpha} = 0$ when $t \to \infty$

→ $\sin(\omega_d t + \phi)$ which represents a repetition of motion.

Thus the resultant motion is oscillatory with decreasing amplitude having frequency of ω_d and ultimately the motion dies down with time as shown in Fig 3.12(a).



(a)



(b)

All the three types of damping responses (x – t) are presented in Fig. 3.12 (b)

Figure 3.13 shows the variation of damped frequency with the damping factor ξ.

19. Define logarithmic decrement, Derive an expression for logarithmic decrement.
or

Define logarithmic decrement and ST it can be expressed as $\delta = \frac{1}{n}\log\left(\frac{u_0}{x_1}\right)$ **where n cycles, u_0 is the initial amplitude and x_n is the amplitude after n cycles.**

It is defined as the natural logarithm of the ratio of any two successive amplitudes on the same side of the mean position in an under damped system. It is denoted by δ (delta). The ratio of any two successive amplitudes in an under damped system is always constant.

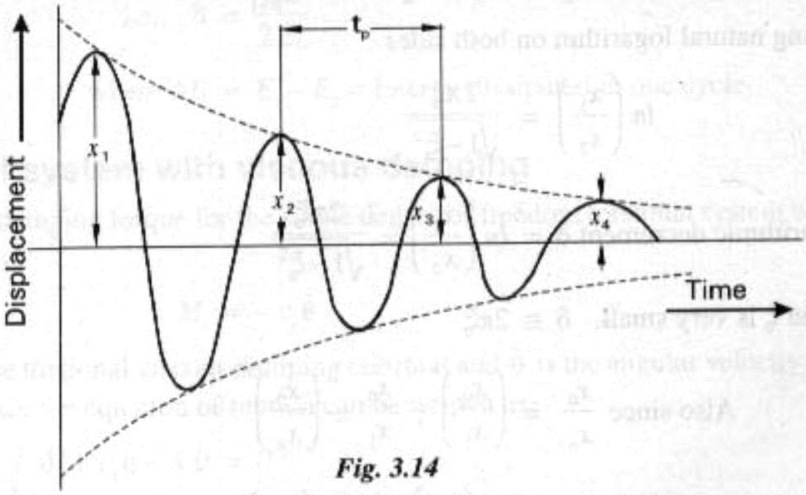


Fig. 3.14

Equation 3.3.21 gives the displacement of an under damped system

i.e., $x = X e^{-\xi\omega_n t} \sin(\omega_d t + \phi)$

Equation 3.3.21 is an equation of harmonic motion in which $X e^{-\xi\omega_n t}$ is the amplitude and ω_d is the angular frequency. When $\sin(\omega_d t + \phi)$ is equal to one, the amplitude is maximum. Also the amplitude will go on decreasing exponentially with time.

Now, maximum amplitude $x = X e^{-\xi\omega_n t}$ ------ (3.4.1)

Let x_1 be the maximum amplitude when the time is t_1 and x_2 be the maximum amplitude when the time is t_2

$$\therefore\ x_1 = X e^{-\xi\omega_n t_1} \quad \text{and}$$

$$x_2 = X e^{-\xi\omega_n t_2}$$

$$\therefore\ \frac{x_1}{x_2} = \frac{X e^{-\xi\omega_n t_1}}{X e^{-\xi\omega_n t_2}} = e^{-\xi\omega_n t_1 - (-\xi\omega_n t_2)} = e^{\xi\omega_n (t_2 - t_1)} \qquad \text{------ (3.4.2)}$$

Where $(t_2 - t_1)$ is the period of oscillation or time difference between two consecutive amplitudes

$$\therefore\ t_2 - t_1 = t_p = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$$

Substituting the value of t_p in equation 3.4.2,

$$\frac{x_1}{x_2} = e^{\xi\omega_n \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}} = e^{\frac{2\pi\xi}{\sqrt{1-\xi^2}}} \qquad \text{------ (3.4.3)}$$

Similarly it can be proved $\frac{x_2}{x_3} = e^{\frac{2\pi\xi}{\sqrt{1-\xi^2}}}$ and so on

$$\text{Hence } \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} \cdots\cdots\cdots \frac{x_n}{x_{n+1}} = e^{\frac{2\pi\xi}{\sqrt{1-\xi^2}}}$$

Now taking natural logarithm on both sides

$$ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\therefore \text{ Logarithmic decrement } \delta = ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \qquad \text{----- (3.4.4)}$$

when ξ is very small, $\delta \cong 2\pi\xi$

$$\text{Also since } \frac{x_0}{x_n} = \left(\frac{x_0}{x_1}\right)^n; \quad \frac{x_0}{x_1} = \left(\frac{x_0}{x_n}\right)^{1/n}$$

$$\therefore \text{ Logarithmic decrement } \delta = ln\left(\frac{x_0}{x_1}\right) = \frac{1}{n} ln\left(\frac{x_0}{x_n}\right) \qquad \text{----- (3.4.5)}$$

20. Derive an expression for energy dissipated in viscous damping.

$$\text{Logarithmic decrement, } \delta = ln\left(\frac{x_1}{x_2}\right); \quad \therefore \frac{x_1}{x_2} = e^{\delta}$$

$$\text{i.e., } \frac{x_2}{x_1} = e^{-\delta} = 1 - \delta + \frac{\delta^2}{2!} - \frac{\delta^3}{3!} + \cdots\cdots\cdots$$

Let E_1 be the vibrational energy at amplitude x_1

$$E_1 = \frac{1}{2} k x_1^2$$

Similarly E_2 be the vibrational energy at amplitude x_2

$$E_2 = \frac{1}{2} k x_2^2$$

$$\therefore \frac{E_1 - E_2}{E_1} = -\frac{E_2}{E_1} = 1 - \left(\frac{x_2}{x_1}\right)^2 = 1 - (e^{-\delta})^2 = 1 - e^{-2\delta}$$

$$= -\left[1 - 2\delta + \frac{(2\delta)^2}{2!} - \frac{(2\delta)^3}{3!} + \cdots\cdots\cdots\right]$$

$$\text{i.e., } \frac{\Delta E}{E_1} = 2\delta - \frac{(2\delta)^2}{2!} + \frac{(2\delta)^3}{3!}$$

Since δ is small, higher powers can be neglected

$$\frac{\Delta E}{E_1} = 2\delta$$

$$\text{i.e., } \delta = \frac{\Delta E}{2E_1} \qquad \text{----- (3.5.1)}$$

where $\Delta E = E_1 - E_2$ = Energy dissipated in one cycle.

21. Vibrating system consist of a mass of 50 kg a spring a stiffen 30 kN/m and a damper. Damping is 20% the critical value. Determine: (i) Damping factor (ii) Critical damping coefficient (iii) Logarithmic decrement (iv) Ratio of two consecutive amplitude (v) Natural frequency of free vibration (vi) Natural frequency of damped vibration.

Data :

$$m = 50 \text{ kg} ; k = 30 \text{ kN/m} = 30{,}000 \text{ N/m};$$
$$c = 0.2\, c_c$$

Solution :

(i) Damping factor $\xi = \frac{c}{c_c} = \frac{0.2c_c}{c_c} = 0.2$

(ii) Critical damping coefficient $c_c = 2m\omega_n = 2\sqrt{km} = 2\sqrt{30{,}000 \times 50} = 2449.5$ N-sec/m

(iii) Logarithmic decrement $\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2\pi \times 0.2}{\sqrt{1-0.2^2}} = 1.28255$

(iv) Ratio of two consecutive amplitudes

$$\text{Also } \delta = \ln \frac{x_n}{x_{n+1}}$$

$$\therefore \frac{x_n}{x_{n+1}} = e^{\delta} = e^{1.28255} = 3.6058$$

(v) Natural frequency of free vibration

Circular frequency of free vibration $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{30{,}000}{50}} = 24.495$ rad / sec

Natural frequency of free vibration $f_n = \frac{1}{2\pi}\omega_n \quad Hz = \frac{1}{2\pi} \times 24.495 = 3.9\,Hz$

(vi) Natural frequency of damped vibration or Damped free vibration

$$f_d = \frac{1}{2\pi}\omega_d = \frac{1}{2\pi} \times \omega_n\sqrt{1-\xi^2} = \frac{1}{2\pi} \times 24.495 \times \sqrt{1-0.2^2} = 3.82\,Hz$$

22. A mass of 2 kg is supported as an isolator having a spring scale of 2940 N/m and viscous damping. If the amplitude of tree vibration of the mass falls to one half of its original value in 1.5 seconds, determine the damping coefficient of the isolator.

Data : $k = 2940$ N/m ; $m = 2$ kg

Solution :

Displacement of an a under damped vibrating system is given by

$$x = X e^{-\xi\omega_n t} \sin(\omega_d t + \phi)$$

$\therefore$ Maximum displacement $x = X e^{-\xi\omega_n t}$ when $\sin(\omega_d t + \phi) = 1$

Natural frequency of undamped system $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2940}{2}} = 38.34$ rad/sec

Let x_A be the maximum amplitude of vibration when time is t_A and x_B be the maximum amplitude of vibration when time is t_B

$$\therefore x_A = X e^{-\xi\omega_n t_A} = X e^{-38.34\xi t_A}$$

$$x_B = X e^{-\xi\omega_n t_B} = X e^{-38.34\xi t_B}$$

$$\therefore \frac{x_A}{x_B} = \frac{Xe^{-38.34\xi t_A}}{Xe^{-38.34\xi t_B}} = e^{(-38.34\xi t_A + 38.34\xi t_B)}$$

$$= e^{38.34\xi(t_B - t_A)}$$

Since the amplitude falls to one half its original value in 1.5 secs.

$$x_B = 0.5x_A \text{ and } t_B - t_A = 1.5$$

$$\therefore \frac{x_A}{0.5x_A} = e^{38.34\xi \times 1.5}$$

$$\text{i.e., } 2 = e^{57.51\xi}$$

Taking natural logarithm on both sides

$$ln2 = ln\, e^{57.51\xi}$$

$$\text{i.e., } 0.69315 = 57.51\xi$$

$$\therefore \text{Damping factor } \xi = 0.012$$

$$\text{Critical damping coefficient } c_c = 2m\omega_n = 2\sqrt{km} = 2\sqrt{2940 \times 2} = 153.36 \text{ N-sec/m}$$

$$\text{Damping factor } \xi = \frac{c}{c_c}$$

$$\text{i.e., } 0.012 = \frac{c}{153.36}$$

$$\therefore \text{Damping coefficient } c = 1.84 \text{ N-sec/m.}$$

23. The mass of a single degree damped vibrating system is 7.5 kg makes 24 free oscillations in 14 seconds when disturbed from its equilibrium position. The amplitude of vibration reduced to 0.25 of its initial value after 5 oscillations. Determine: (i) Spring stiffness (ii) Logarithmic decrement (iii) Damping factor.

Data : $m = 7.5\,kg$; $x_5 = 0.25\,x_0$

Solution :

(i) Stiffness of spring

$$\text{Natural frequency of free oscillation } f_n = \frac{24}{14} = 1.714\,Hz$$

$$\text{Also } f_n = \frac{1}{2\pi}\omega_n$$

$$\therefore \omega_n = 2\pi f_n = 2\pi \times 1.714 = 10.77 \text{ rad/sec}$$

$$\text{Now, } \omega_n = \sqrt{\frac{k}{m}}$$

$$\therefore \omega_n^2 = \frac{k}{m}$$

$$\text{i.e., } 10.77^2 = \frac{k}{7.5}; \therefore \text{ Stiffness of spring } k = 870.13 \text{ N/m}$$

(ii) Logarithmic decrement

$$\text{We have } \frac{x_0}{x_5} = \frac{x_0}{0.25x_0} = 4$$

$$\text{Also } \frac{x_0}{x_5} = \frac{x_0}{x_1} \cdot \frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdot \frac{x_3}{x_4} \cdot \frac{x_4}{x_5} = \left(\frac{x_0}{x_1}\right)^5 \left(\because \frac{x_0}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5}\right)$$

$$\text{i.e., } 4 = \left(\frac{x_0}{x_1}\right)^5$$

$$\therefore \frac{x_0}{x_1} = (4)^{1/5} = 1.3195$$

$$\text{Hence logarithmic decrement } \delta = ln\frac{x_0}{x_1} = ln\,1.319 = 0.277$$

$$\text{OR } \delta = \frac{1}{n} ln\left(\frac{x_0}{x_5}\right) = \frac{1}{5} ln(4) = 0.277$$

(iii) Damping factor

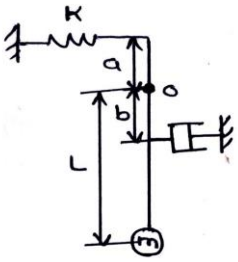
$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}; \quad \therefore \quad \delta^2 = \frac{4\pi^2\xi^2}{1-\xi^2}$$

$$\text{i.e., } 1-\xi^2 = \frac{4\pi^2\xi^2}{\delta^2}$$

$$\text{i.e., } 1-\xi^2 = \frac{4\pi^2\xi^2}{0.277^2}$$

$$\therefore \text{ Damping factor } \xi = 0.044$$

24. A Pendulum is pivoted at point O as shown in Fig. If the mass of the rod is negligible and for small oscillation. Find : i) Critical damping coefficient ii) Damped natural frequency.



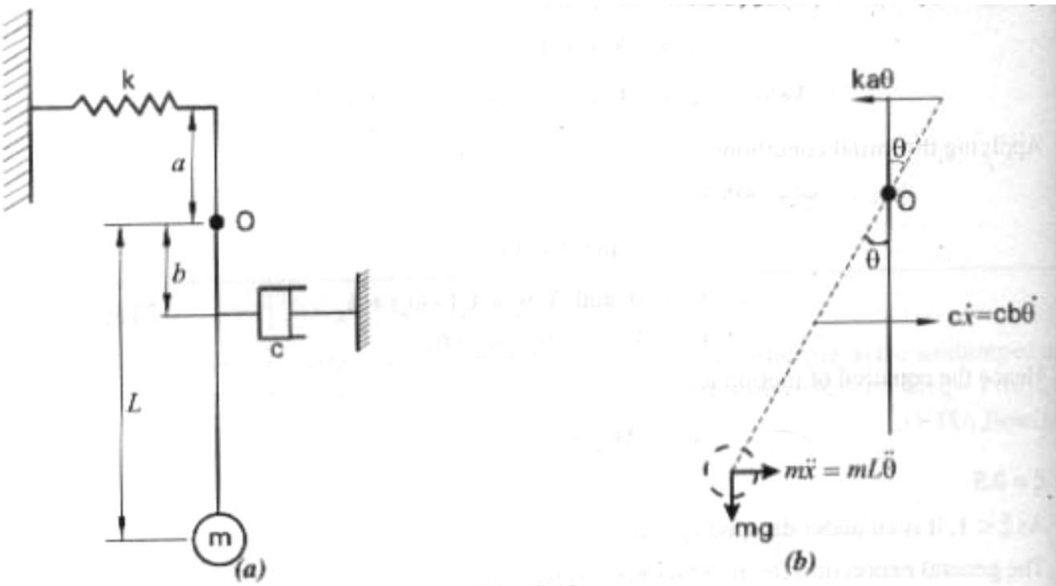


Fig. 3.18

Solution :

Moment of inertia of mass about O, $I_o = mL^2$

For a small oscillation the free body diagram of the system is shown in Fig. 3.18(b)

From Newton's second law of motion in the form of torques

$$I_o \ddot{\theta} = \Sigma M \text{ where } \Sigma M = \text{sum of restoring couples about O}$$

$$I_o \ddot{\theta} = -ka^2\theta - cb^2\dot{\theta} - mgL\theta$$

$$\text{i.e., } mL^2\ddot{\theta} = -ka^2\theta - cb^2\dot{\theta} - mgL\theta$$

$$\text{i.e., } mL^2\ddot{\theta} + cb^2\dot{\theta} + (ka^2 + mgL)\theta = 0$$

$$\therefore \ddot{\theta} + \left(\frac{cb^2}{mL^2}\right)\dot{\theta} + \left(\frac{ka^2 + mgL}{mL^2}\right)\theta = 0 \qquad \text{----- (i)}$$

Equation (i) is the differential equation of motion for the given system.

The roots of this equation are

$$\alpha_{1,2} = \frac{-\frac{cb^2}{mL^2} \pm \sqrt{\left(\frac{cb^2}{mL^2}\right)^2 - 4\left(\frac{ka^2+mgL}{mL^2}\right)}}{2}$$

$$\therefore \alpha_1 = -\frac{cb^2}{2mL^2} + \sqrt{\left(\frac{cb^2}{2mL^2}\right)^2 - \left(\frac{ka^2+mgL}{mL^2}\right)}$$

$$\alpha_2 = -\frac{cb^2}{2mL^2} - \sqrt{\left(\frac{cb^2}{2mL^2}\right)^2 - \left(\frac{ka^2+mgL}{mL^2}\right)}$$

(i) Critical damping coefficient

If the sytem is critically damped then the radical must be zero

$$\text{i.e., } \left(\frac{cb^2}{2mL^2}\right)^2 - \frac{ka^2+mgL}{mL^2} = 0$$

At critical damping $c = c_c$

$$\therefore \left(\frac{c_c b^2}{2mL^2}\right)^2 = \frac{ka^2+mgL}{mL^2}$$

$$\therefore \frac{c_c b^2}{2mL^2} = \sqrt{\frac{ka^2+mgL}{mL^2}}$$

$$\text{i.e., } c_c = \frac{2mL^2}{b^2}\sqrt{\frac{ka^2+mgL}{mL^2}}$$

$$\therefore \text{ Critical damping coefficient } c_c = \frac{2mL^2}{b^2}\sqrt{\frac{ka^2+mgL}{mL^2}} \text{ N.sec/m}$$

(ii) Natural frequency of damped vibration

The general expression for torsional damped system is,

$$I_0\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0$$

$$\text{i.e., } \ddot{\theta} + \frac{c_t}{I}\dot{\theta} + \frac{k_t}{I}\theta = 0$$

$$\text{i.e., } \ddot{\theta} + 2\omega_n\xi\dot{\theta} + \omega_n^2\theta = 0 \quad \text{----- (ii)}$$

$$\left[\because \omega_n = \sqrt{\frac{k_t}{I}}\ rad/\sec, \xi = \frac{c_t}{c_{tc}} = \frac{c_t}{2I\omega_n}, \therefore c_t = 2I\omega_n\xi\right]$$

Comparing the equations (i) and (ii)

$$2\xi\omega_n = \frac{cb^2}{mL^2} \quad \text{----- (iii)}$$

$$\omega_n^2 = \frac{ka^2+mgL}{mL^2} \quad \text{----- (iv)}$$

From equation (iii),

$$\xi^2 = \left(\frac{cb^2}{2mL^2}\right)^2 \cdot \frac{1}{\omega_n^2}$$

$$= \left(\frac{cb^2}{2mL^2}\right)^2 \cdot \frac{mL^2}{ka^2+mgL}$$

$$= \left(\frac{cb^2}{2}\right)^2 \cdot \frac{1}{mL^2(ka^2+mgL)}$$

$$\text{Damped frequency } \omega_d = \sqrt{1-\xi^2}\,.\omega_n$$

$$\therefore \omega_d^2 = (1-\xi^2)\,\omega_n^2$$

$$= \left[1-\left(\frac{cb^2}{2}\right)^2 \cdot \frac{1}{mL^2\left(ka^2+mgL\right)}\right]\left(\frac{ka^2+mgL}{mL^2}\right)$$

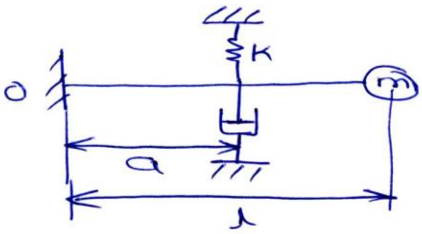
$$= \frac{ka^2+mgL}{mL^2}-\left(\frac{cb^2}{2mL^2}\right)^2$$

$$\therefore \quad \omega_d = \sqrt{\frac{ka^2+mgL}{mL^2}-\left(\frac{cb^2}{2mL^2}\right)^2} \text{ rad/sec}$$

$\therefore$ Natural frequency of damped vibration $f_d = \frac{1}{2\pi}\omega_d \, Hz$

$$= \frac{1}{2\pi}\sqrt{\frac{ka^2+mgL}{mL^2}-\left(\frac{cb^2}{2mL^2}\right)^2} \; Hz$$

25. Obtain the differential equation of motion for the system shown in Fig. and hence find (i) Critical damping coefficient (ii) Natural frequency of damped oscillation



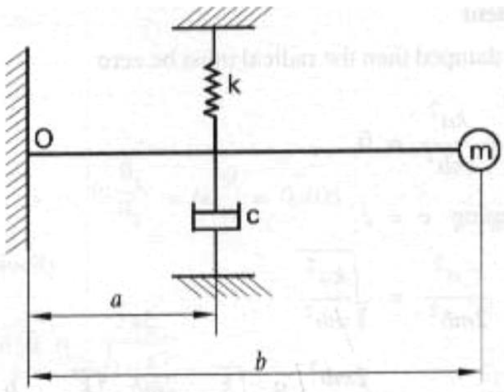


Fig. 3.20

Solution :

Moment of inertia of mass about O, $I_o = mb^2$

For a small angular displacement of θ in the downward direction,

Spring force $= ka\theta$ ↑

Damping force $= c\dot{x} = ca\dot{\theta}$ ↑

From Newton' second law of motion in the form of forques

$$I_0\ddot{\theta} = \Sigma M \quad \text{where } \Sigma M = \text{Sum of restoring couples about O}$$

$$\text{i.e.,} \quad mb^2\ddot{\theta} = (-ca\dot{\theta})\,a-(ka\theta)\,a$$

$$\text{i.e.,} \quad mb^2\ddot{\theta} + ca^2\dot{\theta} + ka^2\theta = 0$$

$$\text{i.e.,} \quad \ddot{\theta} + \frac{ca^2}{mb^2}\dot{\theta} + \frac{ka^2}{mb^2}\theta = 0 \qquad \text{----- (i)}$$

Equation (i) is the differential equation of motion.

The roots of this equation are,

$$\alpha_{1,2} = \frac{-\frac{ca^2}{mb^2} \pm \sqrt{\left(\frac{ca^2}{mb^2}\right)^2 - \frac{4ka^2}{mb^2}}}{2}$$

$$\therefore \quad \alpha_1 = -\frac{ca^2}{2mb^2} + \sqrt{\left(\frac{ca^2}{2mb^2}\right)^2 - \frac{ka^2}{mb^2}}$$

$$\alpha_2 = -\frac{ca^2}{2mb^2} - \sqrt{\left(\frac{ca^2}{2mb^2}\right)^2 - \frac{ka^2}{mb^2}}$$

(i) Critical damping coefficient

If the system is critically damped then the radical must be zero

$$\text{i.e.,} \quad \left(\frac{ca^2}{2mb^2}\right)^2 - \frac{ka^2}{mb^2} = 0$$

At critical damping $c = c_c$

$$\therefore \quad \frac{c_c a^2}{2mb^2} = \sqrt{\frac{ka^2}{mb^2}}$$

$$\text{i.e.,} \quad c_c = \frac{2mb^2}{a^2} \cdot \frac{a}{b}\sqrt{\frac{k}{m}} = \frac{2mb}{a}\sqrt{\frac{k}{m}} = 2\frac{b}{a}\sqrt{km}$$

$\therefore$ Critical damping coefficient $c_c = 2\dfrac{b}{a}\sqrt{km}$ N-sec/m.

(i) Natural frequency of damped vibration

Damped frequency ω_d = Radical with negative sign

$$\therefore \quad \omega_d = \sqrt{\left(\frac{ka^2}{mb^2}\right) - \left(\frac{ca^2}{2mb^2}\right)^2} \text{ rad/sec}$$

Proof : When $a = b$

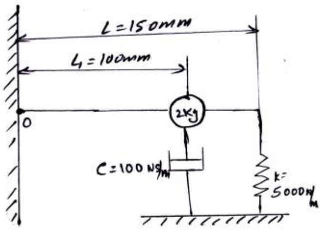
$$\omega_d = \sqrt{\left(\frac{kb^2}{mb^2}\right) - \left(\frac{cb^2}{2mb^2}\right)^2} = \sqrt{\left(\frac{k}{m}\right) - \left(\frac{c}{2m}\right)^2}$$

$$= \sqrt{\omega_n^2 - \left(\frac{c\omega_n}{c_c}\right)^2} \quad \left(\because c_c = 2m\omega_n, 2m = \frac{c_c}{\omega_n}\right)$$

$$= \sqrt{\omega_n^2 - \omega_n^2\xi^2} = \omega_n\sqrt{1-\xi^2} \text{ rad/sec} \quad \left(\because \frac{c}{c_c} = \xi\right)$$

= Damped natural frequency of a standard system.

26. Obtain the differential equation of motion for the system shown in Fig. and hence find (i) Critical damping co-efficient. (ii) Damping ratio. (iii) Natural frequency of damped oscillations (iv) Natural frequency of undamped vibration.



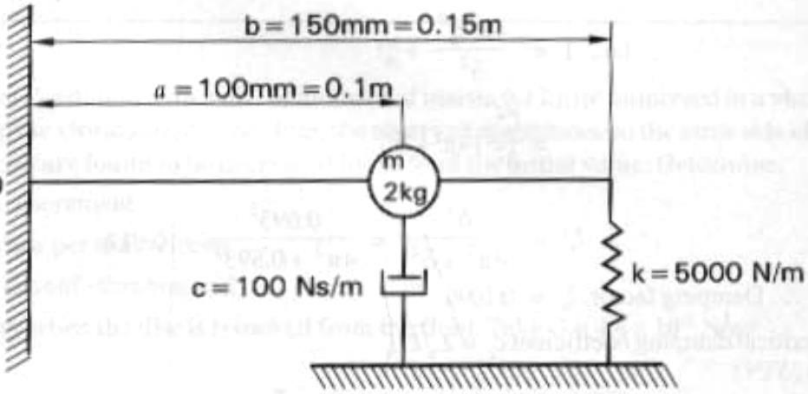


Fig. 3.22

Solution :

Moment of inertia of mass about O, $I_o = ma^2$

For a small angular displacement of θ in the downward direction

$$\text{Spring force} = kb\theta \uparrow$$

$$\text{Damping force} = c\dot{x} = ca\dot{\theta} \uparrow$$

From Newton' second law of motion in the form of torques

$$I_0\ddot{\theta} = \Sigma M \quad \text{where } \Sigma M = \text{Sum of restoring couples about O}$$

$$\text{i.e.,} \; ma^2\ddot{\theta} = (-ca\dot{\theta})\,a - (kb\theta)\,b$$

$$\text{i.e.,} \; ma^2\ddot{\theta} + ca^2\dot{\theta} + kb^2\theta = 0$$

$$\therefore \; \ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{kb^2}{ma^2} = 0 \quad \text{---(i)}$$

Equation (i) is the differential equation of motion. The roots of this equation are,

$$\alpha_{1,2} = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\frac{kb^2}{ma^2}}}{2}$$

$$\therefore \; \alpha_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{kb^2}{ma^2}}\; ; \; \alpha_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{kb^2}{ma^2}}$$

(i) Critical damping coefficient

If the system is critically damped then the radical must be zero

$$\text{i.e.,} \; \left(\frac{c_c}{2m}\right)^2 = \frac{kb^2}{ma^2} \quad [\because \text{For critical damping } c = c_c]$$

$$\text{i.e.,} \; \frac{c_c}{2m} = \frac{b}{a}\sqrt{\frac{k}{m}}$$

$$\therefore \; c_c = \sqrt{\frac{k}{m}}\,.2m\frac{b}{a} = 2\frac{b}{a}\sqrt{km} \text{ N-sec/m}$$

Hence critical damping coefficient

$$c_c = \frac{2\times 0.15}{0.10}\sqrt{5000\times 2} = 300 \text{ N-sec/m.}$$

(ii) Damping ratio

$$\xi = \frac{c}{c_c} = \frac{100}{300} = 0.333$$

(iii) Natural frequency of damped vibration

Damped frequency ω_d = Radical with negative sign

$$\therefore \omega_d = \sqrt{\left(\frac{kb^2}{ma^2}\right) - \left(\frac{c}{2m}\right)^2} \text{ rad/sec}$$

$$= \sqrt{\frac{5000 \times 0.15^2}{2 \times 0.1^2} - \left(\frac{100}{2 \times 2}\right)^2} = 70.71 \text{ rad/sec}$$

$\therefore$ Natural frequency of damped oscillations

$$f_d = \frac{1}{2\pi}\omega_d \text{ Hz} = \frac{1}{2\pi} \times 70.71 = 11.254 \text{ Hz.}$$

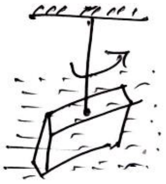
(iv) Natural frequency of undamped vibration

$$\omega_d = \sqrt{1-\xi^2}.\omega_n \quad ; \text{ i.e., } 70.71 = \sqrt{1-0.333^2}.\omega_n$$

$\therefore$ Circular frequency of undamped vibration $\omega_n = 75$ rad/sec

Natural frequency of undamped vibration $f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi} \times 75 = 11.94 Hz$

27. A thin plate of area A and weight W is attached to the end of a spring and allowed to oscillate in a viscous fluid as shown in Fig. If f_1 is the frequency of the system in air and f_2 in the liquid. Show that $\alpha = \frac{2\pi W}{gA}\sqrt{f_1^2 - f_2^2}$, where the damping force $F_d = \alpha 2AV$, V being velocity.



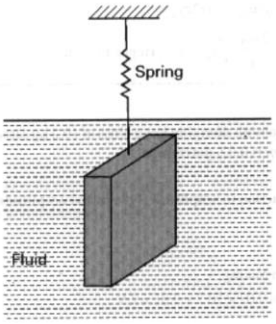


Fig. 3.24

$$\text{Damping force } F_d = 2\eta A v$$

$$\text{i.e., } c\dot{x} = 2\eta A v$$

$$\therefore c = 2\eta A$$

Frequency of free vibration in air $f_n = \frac{1}{2\pi}\omega_n$ Hz where $\omega_n = \sqrt{\frac{k}{m}}$

Frequency of damped free vibration in liquid

$$f_d = \frac{1}{2\pi}\omega_d = \frac{\omega_n\sqrt{1-\xi^2}}{2\pi} = \frac{2\pi f_n\sqrt{1-\xi^2}}{2\pi}$$

$$\therefore f_d = f_n\sqrt{1-\xi^2}$$

$$\text{Hence } 1-\xi^2 = \left(\frac{f_d}{f_n}\right)^2$$

$$\text{i.e., } \xi^2 = 1 - \frac{f_d^2}{f_n^2} = \frac{f_n^2 - f_d^2}{f_n^2}$$

$$\text{Also } \xi^2 = \left(\frac{c}{c_c}\right)^2 = \left(\frac{2\eta A}{2m\omega_n}\right)^2 = \left(\frac{\eta A}{m\omega_n}\right)^2$$

$$\therefore \frac{f_n^2 - f_d^2}{f_n^2} = \left(\frac{\eta A}{m\omega_n}\right)^2$$

$$\text{i.e., } f_n^2 - f_d^2 = \left(\frac{f_n \eta A}{m\omega_n}\right)^2$$

$$\text{i.e., } \sqrt{f_n^2 - f_d^2} = \frac{f_n \eta A}{m\omega_n} = \frac{f_n \eta A}{m 2\pi f_n}$$

$$\therefore \eta = \frac{2\pi m}{A}\sqrt{f_n^2 - f_d^2} \text{. Hence proved.}$$

28. A vibrating system is defined by following parameters: M = 3 kg, K = 100 N/m, C = 3 N-sec/m. Determine: (i) Damping factor (ii) Natural frequency of damped vibration (iii) Logarithmic decrement (iv) ration of two consecutive amplitudes and (v) the number of cycles after which the original amplitude is reduced to 20%.

Solution :

(i) Damping factor

Circular frequency of undamped vibration $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{3}} = 5.7735$ rad/sec

Critical damping coefficient $c_c = 2m\omega_n = 2 \times 3 \times 5.7735 = 34.641$ N.sec/m

$\therefore$ Damping factor $\xi = \frac{c}{c_c} = \frac{3}{34.641} = 0.0866$

(ii) Natural frequency of damped vibration

Circular frequency of damped vibration $\omega_d = \sqrt{1-\xi^2}\,.\omega_n = \sqrt{1-0.0866^2}\,(5.7735) = 5.7518$ rad/sec

Natural frequency of damped vibration $f_d = \frac{1}{2\pi}\,\omega_d = \frac{1}{2\pi} \times 5.7518 = 0.9154$ Hz

(iii) Logarithmic decrement

$$\text{Logarithmic decrement } \delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2\pi \times 0.0866}{\sqrt{1-0.0866^2}} = 0.5462$$

(iv) Ratio of two successive amplitudes

$$\text{Logarithmic decrement } \delta = ln\,\frac{x_n}{x_{n+1}}$$

$\therefore$ Ratio of two successive amplitudes $\frac{x_n}{x_{n+1}} = e^{\delta} = e^{0.5462} = 1.7266$

(v) Number of cycles after which the original amplitude is below 20 %

$$\text{Logarithmic decrement } \delta = \frac{1}{n}\,ln\left(\frac{x_0}{x_n}\right)$$

$$\text{i.e., } 0.5462 = \frac{1}{n}\,ln\left(\frac{x_0}{0.2x_0}\right)$$

$$\text{i.e., } n = 2.9466 \simeq 3$$

$\therefore$ Number of cycles after which the original amplitude is below 20 % is 3

29. A body of 5 kg is supported on a spring of stiffness 200 N/m and has dashpot connected to it, which produces a resistance of 0.002 N at a velocity of 1 cm/sec. In what ratio will the amplitude of vibration be reduced after 5 cycles.

Similar problem:

A mass of 5kg is supported on a spring of stiffness 1960 N/m. The dashpot attached to the system produces a resistance of 1.96 N at a velocity of 1m/sec. Determine,

(i) Natural frequency of free vibration

(ii) Damping resistance

(iii) Critical damping resistance

(iv) Damping factor

(v) Logarithmic decrement

(vi) Ratio of any two successive amplitude

(vii) Amplitude ratio after 5 cycles.

(viii) Decrease in amplitude after 5 complete cycles or oscillations.

(ix) Amplitude after 5 cycles if initial amplitude is 10 mm

Data : $m = 5$ kg, $k = 1960$ N/m, $F_d = 1.96$ N ; $\dot{x} = 1$m/sec ; $x_0 = 10$ mm

Solution :

(i) Natural frequency of free vibration

$$\text{Circular frequency } \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1960}{5}} = 19.8 \quad \text{rad / sec}$$

$$\text{Natural frequency } f_n = \frac{1}{2\pi}\,\omega_n = \frac{1}{2\pi} \times 19.8 = 3.15\ Hz$$

(ii) Damping resistance

Resisting force $F_d = c\dot{x}$; i.e., $1.96 = c \times 1$

$\therefore$ Damping resistance $c = 1.96$ N.sec/m

(iii) Critical damping resistance

$$c_c = 2\,m\,\omega_n = 2 \times 5 \times 19.8 = 198\ N.\text{sec/m}$$

(iv) Damping factor

$$\xi = \frac{c}{c_c} = \frac{1.96}{198} = 9.9 \times 10^{-3}$$

(v) Logarithmic decrement

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2 \times \pi \times 9.9 \times 10^{-3}}{\sqrt{1-\left(9.9 \times 10^{-3}\right)^2}} = 0.0622$$

(vi) Ratio of any two successive amplitude

$$\delta = ln\left(\frac{x_n}{x_{n+1}}\right)$$

$$\therefore \quad \frac{x_n}{x_{n+1}} = e^{\delta} = e^{0.0622} = 1.0642$$

(vii) Amplitude ratio after 5 cycles

$$\delta = \frac{1}{n}\, ln\left(\frac{x_0}{x_5}\right)$$

$$\text{i.e.,} \quad 0.0622 = \frac{1}{5} \ln\left(\frac{x_0}{x_5}\right)$$

$$\therefore \quad \frac{x_0}{x_5} = e^{(0.0622 \times 5)} = 1.365$$

(viii) Decrease in amplitude after 5 complete oscillations

$$\frac{x_0}{x_5} = 1.365$$

$$\therefore \quad x_5 = \frac{x_0}{1.365} = 0.7326 \; x_0$$

(ix) Amplitude after 5 cycles, if $x_0 = 10$ mm

$$x_5 = 0.7326 \; x_0 = 0.7326 \times 10 = 7.326 \text{ mm}$$