Theory of Vibration - 18AE56

Old VTU Question's Answers

Module – 4

Syllabus:
Systems with Two Degrees of Freedom: Introduction, principle modes and Normal modes of vibration, coordinate coupling, generalized and principal co-ordinates, free vibration in terms of initial conditions. Geared systems. Forced Oscillations-Harmonic excitation. Applications: Vehicle suspension, Dynamic vibration absorber and Dynamics of reciprocating Engines.
Continuous Systems: Introduction, vibration of string, longitudinal vibration of rods, Torsional vibration of rods, Euler's equation for beams.

Part – A Questions

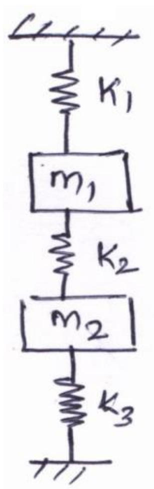
1. Briefly explain principal modes and normal modes of vibration.

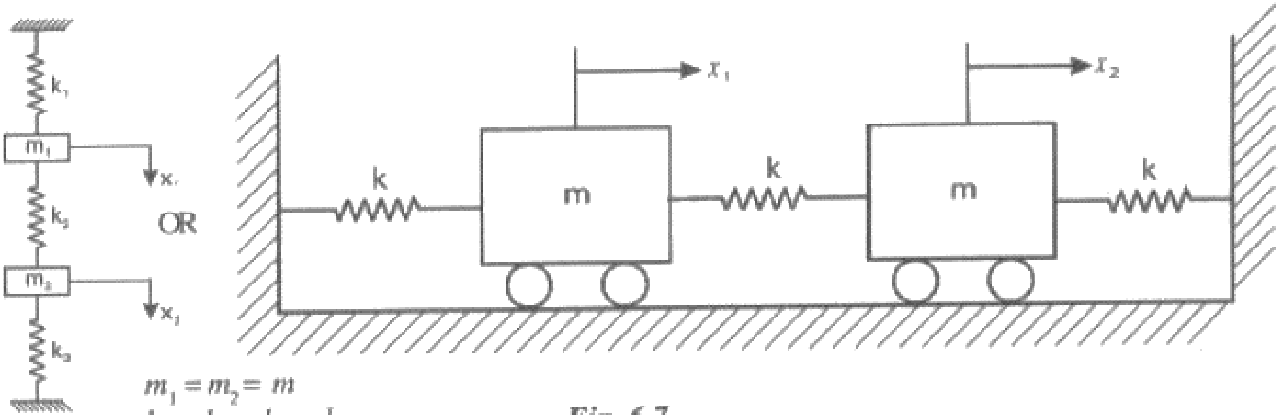
A two degrees of freedom system has two equations of motion (ie, one for each mass) and hence two natural frequencies. The natural frequencies are found by solving the frequency equation of an undamped system or the characteristic equation of a damped system. The system at its lowest or first natural frequency is called its first mode, and its immediate next higher is called the second mode, and so on. If the two masses vibrate at the same frequency and in phase, it is called a principal mode of vibration. If at the principal mode of vibration, the amplitude of one of the masses is unity, it is known as normal mode of vibration. Normal mode of vibrations are free vibrations that depend only on the mass and stiffness of the system and how they are distributed.

2. For the system shown in below fig. Determine:
(i) Equations of motion
(ii) Natural frequencies of the system
(iii) Modal vectors
(iv) Draw mode shapes
Take m1 = m2 = m; K1 = K2 = K3 = K

Or

For the system shown in below fig. i) Derive the equation of motion ii) Set up frequency equal and obtain natural frequencies of the system.





$m_1 = m_2 = m$
$k_1 = k_2 = k_3 = k$

Fig. 6.7

Solution :

i) Derive the equation of motion

The two masses m and m are defined by their positions x_1 and x_2 respectively at any time 't'

The equations of motion are obtained from Newton's second law of motion. Considering the free body diagram shown in Fig 6.8, the equations are;

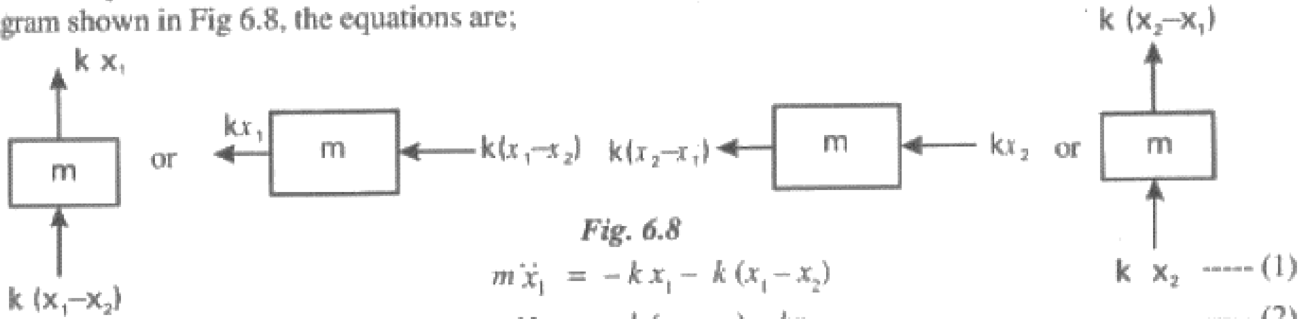


Fig. 6.8

$$m\ddot{x}_1 = -k\,x_1 - k\,(x_1 - x_2) \qquad \text{----- (1)}$$

$$m\ddot{x}_2 = -k\,(x_2 - x_1) - kx_2 \qquad \text{----- (2)}$$

Equations (1) and (2) can be re written as

$$m\ddot{x}_1 + 2k\,x_1 - k\,x_2 = 0 \qquad \text{----- (3)}$$

$$m\ddot{x}_2 - k\,x_1 + 2kx_2 = 0 \qquad \text{----- (4)}$$

Equation (3) and (4) represents the differential equations of motion of second order.

ii) Frequency equation and natural frequencies of the system.

Assuming the harmonic motion of masses m and m at the same frequency ω and the same phase angle ϕ, the solutions of equations (3) and (4) can be written as

$$x_1 = A\sin(\omega t + \phi) \qquad \text{----- (5)}$$

$$x_2 = B\sin(\omega t + \phi) \qquad \text{----- (6)}$$

Where A, B and ϕ are arbitrary constants.

Now, $\dot{x}_1 = A\omega\cos(\omega t + \phi)$; $\ddot{x}_1 = -A\omega^2\sin(\omega t + \phi)$

$\dot{x}_2 = B\omega\cos(\omega t + \phi)$; $\ddot{x}_2 = -B\omega^2\sin(\omega t + \phi)$

Substituting the values of x_1, x_2, $\ddot{x}_1$ and $\ddot{x}_2$ into the equations of (3) and (4)

$-m\omega^2 A\sin(\omega t + \phi) + 2kA\sin(\omega t + \phi) - kB\sin(\omega t + \phi) = 0$

$$\text{i.e., } (-m\omega^2 + 2k)A - kB = 0 \qquad \text{----- (7)}$$

Similarly $-m\omega^2 B\sin(\omega t + \phi) - kA\sin(\omega t + \phi) + 2kB\sin(\omega t + \phi) = 0$

$$\text{i.e., } (-m\omega^2 + 2k)B - kA = 0 \qquad \text{----- (8)}$$

Equation (7) and (8) are homogeneous linear algebraic equations in A and B. For a non trivial solution of A and B, the determinant of the coefficients of A and B must be zero.

$$\text{i.e., } \begin{vmatrix} (-m\omega^2 + 2k) & -k \\ -k & (-m\omega^2 + 2k) \end{vmatrix} = 0$$

Expanding the determinant

$$(-m\omega^2 + 2k)(-m\omega^2 + 2k) - k^2 = 0$$

$$m^2\omega^4 - 4m\omega^2 k + 4k^2 - k^2 = 0$$

$$\text{i.e., } \omega^4 - \frac{4k}{m}\omega^2 + \frac{3k^2}{m^2} = 0 \quad \text{------ (9)}$$

Equation (9) is called the frequency equation.

$$\therefore \omega^2 = + \frac{\frac{4k}{m} \pm \sqrt{\left(\frac{4k}{m}\right)^2 - 4\times 1\times \frac{3k^2}{m^2}}}{2\times 1}$$

$$\therefore \omega_1^2 = \frac{\frac{4k}{m} - \sqrt{\frac{16k^2}{m^2} - \frac{12k^2}{m^2}}}{2} = \frac{\frac{4k}{m} - \frac{2k}{m}}{2} = \frac{k}{m}$$

$$\therefore \omega_2^2 = \frac{\frac{4k}{m} + \sqrt{\frac{16k^2}{m^2} - \frac{12k^2}{m^2}}}{2} = \frac{4\frac{k}{m} + 2\frac{k}{m}}{2} = \frac{3k}{m}$$

$$\therefore \omega_1 = \sqrt{\frac{k}{m}} \text{ rad/sec} ; \omega_2 = \sqrt{\frac{3k}{m}} \text{ rad/sec}$$

ω_1 and ω_2 are the circular frequencies of first and second modes respectively.

$\therefore$ Natural frequency of first mode $f_1 = \frac{1}{2\pi}\omega_1 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ Hz

Natural frequency of second mode $f_2 = \frac{1}{2\pi}\omega_2 = \frac{1}{2\pi}\sqrt{\frac{3k}{m}}$ Hz

iii) Modal vectors

The values of A and B depend on the natural frequencies of ω_1 and ω_2

Let A_1 = Amplitude of x_1 when frequency $\omega = \omega_1$

A_2 = Amplitude of x_1 when frequency $\omega = \omega_2$

B_1 = Amplitude of x_2 when frequency $\omega = \omega_1$

B_2 = Amplitude of x_2 when frequency $\omega = \omega_2$

Using the equations (7) and (8), the amplitude ratio can be written as

$$\frac{A_1}{B_1} = \frac{k}{-m\omega_1^2 + 2k} = \frac{-m\omega_1^2 + 2k}{k} = \frac{1}{\lambda_1} \text{ and}$$

$$\frac{A_2}{B_2} = \frac{k}{-m\omega_2^2 + 2k} = \frac{-m\omega_2^2 + 2k}{k} = \frac{1}{\lambda_2} \text{ and}$$

Amplitude ratio for the first mode can be written as

$$\frac{A_1}{B_1} = \frac{k}{-m\omega_1^2 + 2k} = \frac{k}{-m\left(\frac{k}{m}\right) + 2k} = 1 = \frac{1}{\lambda_1} \quad \left\{\because \omega_1 = \sqrt{\frac{k}{m}}\right\}$$

Amplitude ratio for the second mode can be written as

$$\frac{A_2}{B_2} = \frac{k}{-m\omega_2^2 + 2k} = \frac{k}{-m\left(\frac{3k}{m}\right) + 2k} = -1 = \frac{1}{\lambda_2} \quad \left\{\because \omega_2 = \sqrt{\frac{3k}{m}}\right\}$$

Hence for the first mode, the two masses move in the same phase with equal amplitudes and for the second mode the two masses move out of phase with equal amplitudes.

If one of the amplitude is chosen equal to one or any other number then the amplitude ratio is normalized to that number. The normalized amplitude ratio is called the normal mode and is designated by $\phi_i(x)$.

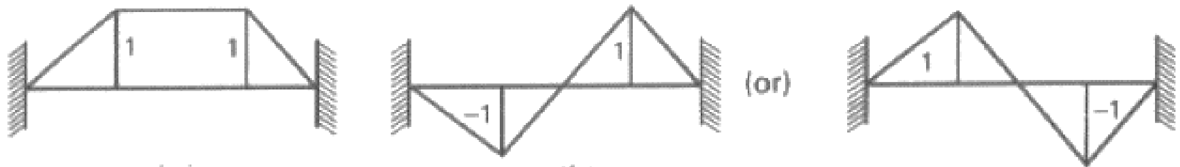
The normal modes of vibration corresponding to ω_1^2 and ω_2^2 can be expressed respectively.

$$\phi_1(x) = \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ \lambda_1 A_1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\phi_2(x) = \begin{Bmatrix} A_2 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} A_2 \\ \lambda_2 A_2 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \text{ or } \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

The vectors $\phi_1(x)$ and $\phi_2(x)$ which denote the normal mode of vibration are known as the modal vectors or eigen vectors of the system.

iv) Mode shapes



(a) (b)

Fig. 6.39

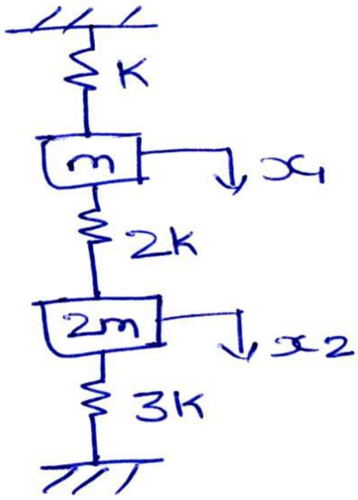
Figure 6.9(a) represents the first normal mode shape and Figure 6.9(b) represents the second normal mode shape.

3. Below fig. shows a spring mass system. Determine

(i) Equation of motion

(ii) Frequency equation and natural frequencies of the system

(iii) Modal vector and mode shapes



Solution :

i) Equation of motion

The free body diagrams are shown in Fig. 6.11.

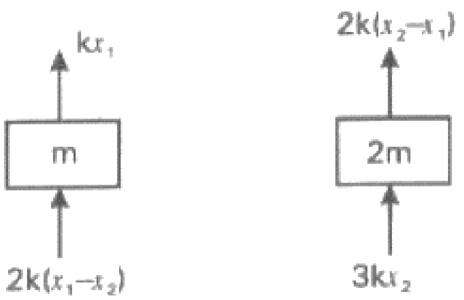


Fig. 6.11

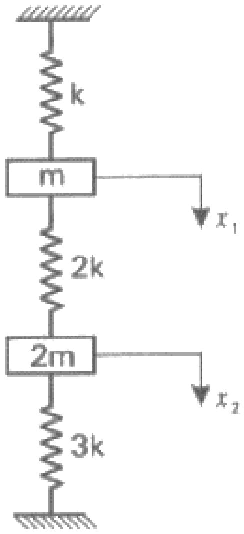


Fig. 6.10

The equations of motion are obtained from Newton's second law of motion. Considering the free body diagram shown in Fig 6.11.

$$m\ddot{x}_1 = -kx_1 - 2k\left(x_1 - x_2\right)$$

$$\therefore \quad m\ddot{x}_1 + kx_1 + 2k(x_1 - x_2) = 0$$

$$\text{i.e.,} \quad m\ddot{x}_1 + 3kx_1 - 2kx_2 = 0 \qquad \text{----- (1)}$$

$$\text{Similarly } 2m\ddot{x}_2 = -2k(x_2 - x_1) - 3kx_2$$

$$\text{i.e.,} \quad 2m\ddot{x}_2 + 2k(x_2 - x_1) + 3kx_2 = 0$$

$$\text{i.e.,} \quad 2m\ddot{x}_2 - 2kx_1 + 5kx_2 = 0 \qquad \text{----- (2)}$$

Equations (1) & (2) are called the differential equations of motion for the given system

ii) Frequency equation and natural frequencies.

Let the solution of equations (1) & (2) can be written as

$$x_1 = A \sin(\omega t + \phi)$$
$$x_2 = B \sin(\omega t + \phi)$$

where A, B and ϕ are arbitrary constants.

$$\text{Now} \quad \dot{x}_1 = A\omega \cos(\omega t + \phi)\,; \quad \ddot{x}_1 = -A\omega^2 \sin(\omega t + \phi)$$

$$\text{Similarly } \ddot{x}_2 = -B\omega^2 \sin(\omega t + \phi)$$

Substituting these values in equations (1) and (2)

$$m\left[-A\omega^2 \sin\left(\omega t + \phi\right)\right] + 3kA \sin(\omega t + \phi) - 2kB \sin(\omega t + \phi) = 0$$

$$\text{i.e.,} \quad (-m\omega^2 + 3k)A - 2kB = 0 \qquad \text{----- (3)}$$

$$\text{Similarly } 2m[-B\omega^2 \sin(\omega t + \phi)] - 2kA \sin(\omega t + \phi) + 5kB \sin(\omega t + \phi) = 0$$

$$\text{i.e.,} \quad (-2m\omega^2 + 5k)B - 2kA = 0 \qquad \text{----- (4)}$$

Equations (3) and (4) are homogenious linear algebraic equations in A and B. For a nontrivial solution of A and B, the determinant of the coefficients of A and B must be zero.

$$\therefore \begin{vmatrix} -m\omega^2 + 3k & -2k \\ -2k & -2m\omega^2 + 5k \end{vmatrix} = 0$$

Expanding the determinant

$$(3k - m\omega^2)(5k - 2m\omega^2) - 4k^2 = 0$$

i.e., $15k^2 - 6km\omega^2 - 5km\omega^2 + 2m^2\omega^4 - 4k^2 = 0$

i.e., $2m^2\omega^4 - 11km\omega^2 - 11k^2 = 0$

$$\text{i.e.,} \quad \omega^4 - 5.5\frac{k}{m}\omega^2 - 5.5\frac{k^2}{m^2} = 0 \qquad \text{----- (5)}$$

Equation (5) is called the frequency equation.

$$\text{Now,} \quad \omega^2 = + \frac{5.5\frac{k}{m} \pm \sqrt{\left(5.5\frac{k}{m}\right)^2 - 4\times 1\times 5.5\frac{k^2}{m^2}}}{2\times 1}$$

$$\omega_1^2 = \frac{5.5\frac{k}{m} - 2.872\frac{k}{m}}{2} = 1.314\frac{k}{m}$$

$$\omega_2^2 = \frac{5.5\frac{k}{m} + 2.872\frac{k}{m}}{2} = 4.186\frac{k}{m}$$

$$\therefore \quad \omega_1 = 1.146\sqrt{\frac{k}{m}} \text{ rad/sec}$$

$$\omega_2 = 2.046\sqrt{\frac{k}{m}} \text{ rad/sec}$$

ω_1 and ω_2 are the circular frequencies of first and second modes respectively.

Natural frequency of first mode $f_1 = \frac{1}{2\pi}\omega_1 = \frac{1}{2\pi}\times 1.146\sqrt{\frac{k}{m}} = 0.182\sqrt{\frac{k}{m}}$

Natural frequency of second mode $f_2 = \frac{1}{2\pi}\omega_2 = \frac{1}{2\pi}\times 2.046\sqrt{\frac{k}{m}} = 0.325\sqrt{\frac{k}{m}}\ Hz$

iii) Modal vectors and mode shapes

Let A_1 = Amplitude of x_1 when frequency $\omega = \omega_1$
A_2 = Amplitude of x_1 when frequency $\omega = \omega_2$
B_1 = Amplitude of x_2 when frequency $\omega = \omega_1$
B_2 = Amplitude of x_2 when frequency $\omega = \omega_2$

Using the equations (3) and (4), the amplitude ratio can be written as

$$\frac{A_1}{B_1} = \frac{2k}{-m\omega_1^2 + 3k} = \frac{-2m\omega_1^2 + 5k}{2k}$$

$$\frac{A_2}{B_2} = \frac{2k}{-m\omega_2^2 + 3k} = \frac{-2m\omega_2^2 + 5k}{2k}$$

Amplitude ratio of first mode can be written as

$$\frac{A_1}{B_1} = \frac{2k}{-m\omega_1^2 + 3k} = \frac{2k}{(-m)\left(1.314\frac{k}{m}\right) + 3k} \quad \left(\because \omega_1^2 = 1.314\frac{k}{m}\right)$$

$$= 1.186 = \frac{1}{\lambda_1}$$

Amplitude ratio of second mode can be written as

$$\frac{A_2}{B_2} = \frac{2k}{-m\omega_2^2 + 3k} = \frac{2k}{(-m)\left(4.186\frac{k}{m}\right) + 3k} \quad \left(\because \omega_2^2 = 4.186\frac{k}{m}\right)$$

$$= -1.686 = \frac{1}{\lambda_2}$$

The normal modes of vibration corresponding to ω_1^2 and ω_2^2 can be expressed respectively.

$$\phi_1(x) = \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ \lambda_1 A_1 \end{Bmatrix} = \begin{Bmatrix} 1.186 \\ 1.0 \end{Bmatrix}$$

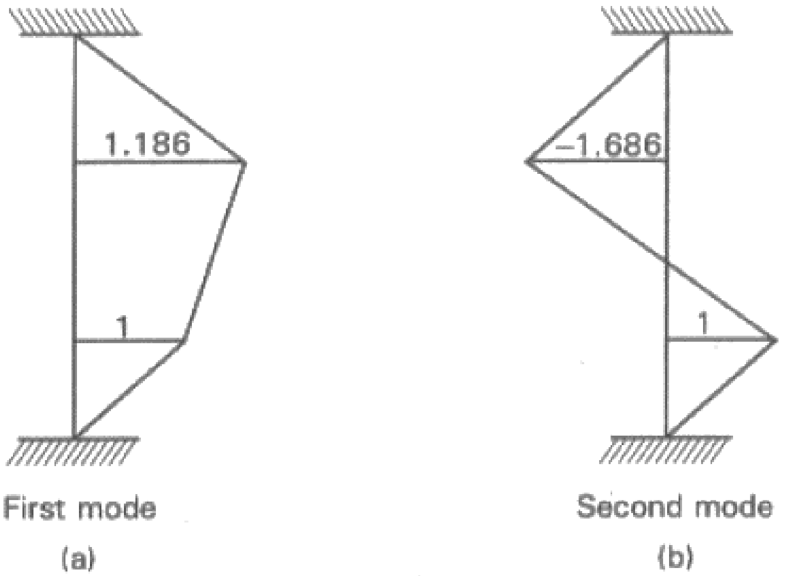


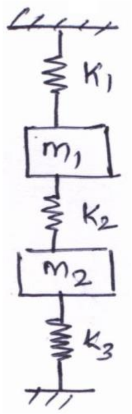
Fig. 6.12

$$\phi_2(x) = \begin{Bmatrix} A_2 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} A_2 \\ \lambda_2 A_2 \end{Bmatrix} = \begin{Bmatrix} -1.686 \\ 1.0 \end{Bmatrix}$$

The vectors $\phi_1(x)$ and $\phi_2(x)$ which denote the normal mode of vibration are known as the modal vectors or eigen vectors of the system.

The mode shapes for first mode and second mode are as shown in Fig 6.12 (a) and 6.12 (b) respectively. In the first normal mode, the two masses move in phase. In the second normal mode the two masses move in out of phase with each other.

4. Below fig shows spring mass system. If the mass m1 is displaced 20 mm from its static equilibrium position and released, determine the resulting displacements x (t) 1 and x (t) 2 of the masses.



Given $k_1 = k_2 = k_3 = k$ and $m_1 = m_2$

Solution :

The free body diagrams are shown in figure 6.15 (*b*)

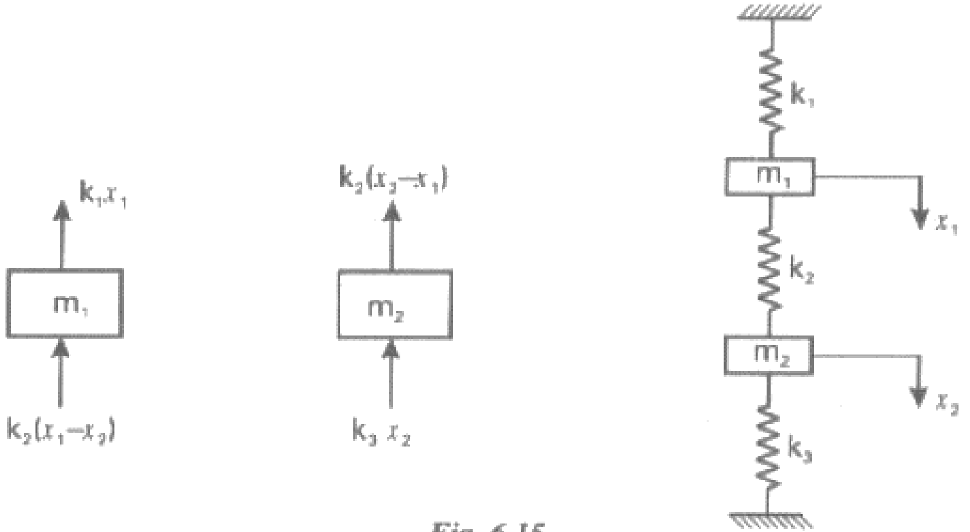


Fig. 6.15

The equations of motion are obtained from Newton's second law of motion. Considering the free body diagram shown in Fig 6.15*b*.

$$m\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2)$$

i.e., $m_1\ddot{x}_1 + (k_1 + k_2)\,x_1 - k_2x_2 = 0$ ------ (1)

Similarly $m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1)$

i.e., $m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)\,x_2 = 0$ ------ (2)

Let the solution of equations (1) and (2) be,

$$x_1 = A\sin(\omega t + \phi)$$

$$x_2 = B\sin(\omega t + \phi)$$

Where A, B and ϕ are arbitrary constants.

Now, $\dot{x}_1 = A\omega\cos(\omega t + \phi)$; $\ddot{x}_1 = -A\omega^2\sin(\omega t + \phi)$

Similarly $\ddot{x}_2 = -B\omega^2\sin(\omega t + \phi)$

Substituting these values in equation (1) and (2)

$m_1\{-A\omega^2\sin(\omega t + \phi)\} + (k_1 + k_2)\,A\sin(\omega t + \phi) - k_2\,B\sin(\omega t + \phi) = 0$

i.e., $\{-m_1\omega^2 + (k_1 + k_2)\}\,A - k_2B = 0$ ------ (3)

Similarly, $m_2\{-B\omega^2 \sin(\omega t + \phi)\} - k_2 A \sin(\omega t + \phi) + (k_2 + k_3) B \sin\omega t = 0$

i.e., $\{-m_2\omega^2 + (k_2 + k_3)\}B - k_2 A = 0$ ----- (4)

For a nontrivial solution of A and B, the determinant of the coefficients of A and B must be zero

i.e., $$\begin{vmatrix} -m_1\omega^2 + (k_1 + k_2) & -k_2 \\ -k_2 & -m_2\omega^2 + (k_2 + k_3) \end{vmatrix} = 0$$

i.e., $\{-m_1\omega^2 + (k_1 + k_2)\}\{-m_2\omega^2 + (k_2 + k_3)\} - k_2^2 = 0$

i.e., $m_1 m_2 \omega^4 - \{(k_1 + k_2) m_2 + (k_2 + k_3) m_1\}\omega^2 + \{(k_1 + k_2)(k_2 + k_3) - k_2^2\} = 0$ ----- (5)

Equation (5) is the frequency equation.

Since $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$, equation (5) becomes

$$m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$$

i.e., $$\omega^4 - 4\frac{k}{m}\omega^2 + 3\left(\frac{k}{m}\right)^2 = 0$$

$$\therefore \quad \omega^2 = \frac{4\dfrac{k}{m} \pm \sqrt{\left(4\dfrac{k}{m}\right)^2 - 4\times 1\times 3\left(\dfrac{k}{m}\right)^2}}{2\times 1}$$

Hence $\omega_1 = \sqrt{\dfrac{k}{m}}$ rad/sec = Circular frequency of first mode

$\omega_2 = \sqrt{\dfrac{3k}{m}}$ rad/sec = Circular frequency of second mode

Let A_1 = Amplitude of x_1 when frequency $\omega = \omega_1$

A_2 = Amplitude of x_1 when frequency $\omega = \omega_2$

B_1 = Amplitude of x_2 when frequency $\omega = \omega_1$

B_2 = Amplitude of x_2 when frequency $\omega = \omega_2$

using the equations (3) and (4) the amplitude ratio can be written as

$$\frac{A_1}{B_1} = \frac{k_2}{-m_1\omega_1^2 + (k_1 + k_2)} = -\frac{m_2\omega_1^2 + (k_2 + k_3)}{k_2} = \frac{1}{\lambda_1}$$

$$\frac{A_2}{B_2} = \frac{k_2}{-m_1\omega_2^2 + (k_1 + k_2)} = -\frac{m_2\omega_2^2 + (k_2 + k_3)}{k_2} = \frac{1}{\lambda_2}$$

$\therefore$ Amplitude ratio of first mode $\dfrac{A_1}{B_1} = \dfrac{k}{-m.\dfrac{k}{m} + (k + k)} = 1 = \dfrac{1}{\lambda_1}$

$\therefore$ Amplitude ratio of second mode $\dfrac{A_2}{B_2} = \dfrac{k}{-m.\dfrac{3k}{m} + (k + k)} = -1 = \dfrac{1}{\lambda_2}$

The general solution of the equation of motion is,

$$x_1(t) = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2)$$

$$= A_1 \sin\left(\sqrt{\frac{k}{m}}t+\phi_1\right) + A_2 \sin\left(\sqrt{\frac{3k}{m}}t+\phi_2\right) \quad \text{------ (6)}$$

$$x_2(t) = B_1 \sin(\omega_1 t + \phi_1) + B_2 \sin(\omega_2 t + \phi_2)$$

$$= \lambda_1 A_1 \sin\left(\sqrt{\frac{k}{m}}t+\phi_1\right) + \lambda_2 A_2 \sin\left(\sqrt{\frac{3k}{m}}t+\phi_2\right)$$

$$= A_1 \sin\left(\sqrt{\frac{k}{m}}t+\phi_1\right) - A_2 \sin\left(\sqrt{\frac{3k}{m}}t+\phi_2\right) \quad \text{------ (7)}$$

The constants A_1, A_2, ϕ_1 and ϕ_2 can be obtained from the following initial conditions :

$$x_1(0) = 20 \text{ mm} ; x_2(0) = 0; \quad \dot{x}_1(0) = 0; \quad \dot{x}_2(0) = 0$$

$$\therefore \quad x_1(0) = 20 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \quad \text{------ (8)}$$

$$x_2(0) = 0 = A_1 \sin\phi_1 - A_2 \sin\phi_2 \quad \text{------ (9)}$$

$$\dot{x}_1(t) = A_1\sqrt{\frac{k}{m}} \cos\left(\sqrt{\frac{k}{m}}t+\phi_1\right) + A_2\sqrt{\frac{3k}{m}} \cos\left(\sqrt{\frac{3k}{m}}t+\phi_2\right)$$

$$\therefore \quad \dot{x}_1(0) = 0 = A_1\sqrt{\frac{k}{m}} \cos\phi_1 + A_2\sqrt{\frac{3k}{m}} \cos\phi_2 \quad \text{------ (10)}$$

Similarly, $\dot{x}_2(0) = 0 = A_1\sqrt{\frac{k}{m}} \cos\phi_1 - A_2\sqrt{\frac{3k}{m}} \cos\phi_2$ ------ (11)

Adding the equations (8) and (9), we get

$$2A_1 \sin\phi_1 = 20$$

$$\text{i.e.,} \quad A_1 \sin\phi_1 = 10; \quad \therefore \quad A_1 = \frac{10}{\sin\phi_1}$$

Substracting the equation (9) from (8), we get

$$2 A_2 \sin\phi_2 = 20$$

$$\text{i.e.,} \quad A_2 \sin\phi_2 = 10; \quad \therefore \quad A_2 = \frac{10}{\sin\phi_2}$$

Adding the equations (10) and (11), we get $2 A_1 \sqrt{\frac{k}{m}} \cos\phi_1 = 0$

$$\text{i.e.,} \quad \cos\phi_1 = 0 \quad \therefore \quad \phi_1 = 90^\circ$$

Substracting the equation (11) from (10), we get

$$2 A_2 \sqrt{\frac{3k}{m}} \cos\phi_2 = 0$$

$$\text{i.e.,} \quad \cos\phi_2 = 0 \quad \therefore \quad \phi_2 = 90^\circ$$

Hence, $A_1 = \frac{10}{\sin\phi_1} = \frac{10}{\sin 90} = 10$ mm

$$A_2 = \frac{10}{\sin\phi_2} = \frac{10}{\sin 90} = 10 \text{ mm}$$

Thus the motions of the masses are

$$x_1(t) = 10\sin\left(\sqrt{\frac{k}{m}}t + 90^\circ\right) + 10\sin\left(\sqrt{\frac{3k}{m}}t + 90^\circ\right)$$

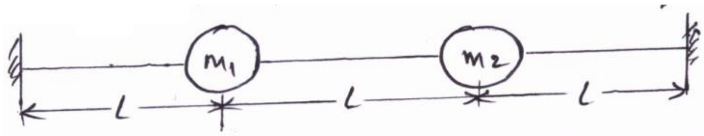
$$= 10\cos\sqrt{\frac{k}{m}}t + 10\cos\sqrt{\frac{3k}{m}}t \quad \text{mm}$$

$$x_2(t) = 10\sin\left(\sqrt{\frac{k}{m}}t + 90^\circ\right) - 10\sin\left(\sqrt{\frac{3k}{m}}t + 90^\circ\right)$$

$$= 10\cos\sqrt{\frac{k}{m}}t - 10\cos\sqrt{\frac{3k}{m}}t \quad \text{mm}$$

For the first principal mode of vibration, the two masses move in the same direction with equal amplitudes. For the second principal mode of vibration, the two masses move in opposite directions with equal amplitudes.

5. Below fig. shows a system subjected to vibration. Find an expression for the natural frequency.



Solution :

Assume tension in string as T and it remains constant for small oscillations. Figure 6.20a shows the displaced position of the masses due to the oscillations.

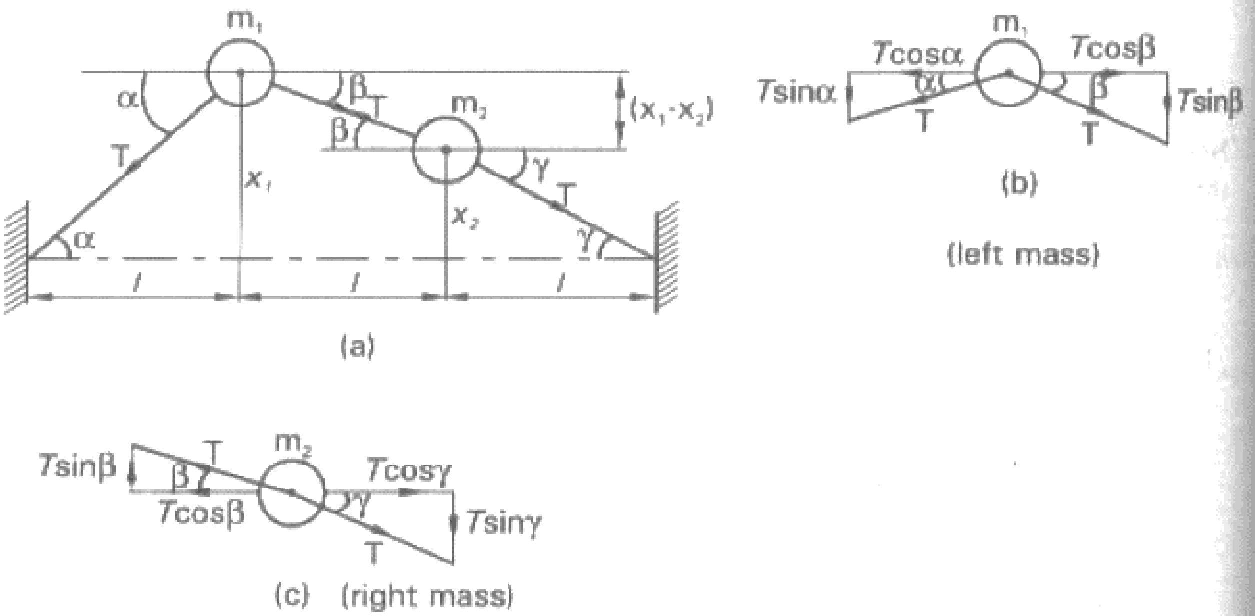


Fig. 6.20

From Fig. 6.20a

$$\sin\alpha = \frac{x_1}{l}\ ;\ \sin\beta = \frac{x_1 - x_2}{l}\ ;\ \sin\gamma = \frac{x_2}{l}$$

Consider left mass

The free body diagram of left mass is shown in Fig. 6.20 (b)

From Newton's second law of motion, the equation of motion for the left mass is,

$$m_1 \ddot{x}_1 = -T\sin\alpha - T\sin\beta$$

i.e., $m_1 \ddot{x}_1 + T\sin\alpha + T\sin\beta = 0$

i.e., $m_1 \ddot{x}_1 + T\left(\frac{x_1}{l}\right) + T\left(\frac{x_1 - x_2}{l}\right) = 0$

$$m_1 \ddot{x}_1 + \frac{2T}{l}x_1 - \frac{T}{l}x_2 = 0 \qquad \text{----- (1)}$$

Consider right mass

The free body diagram of right mass is shown in Fig. 6.20(c)

From Newton's second law of motion, the equation of motion for the right mass is,

$$m_2 \ddot{x}_2 = -T\sin\gamma + T\sin\beta$$

i.e., $m_2 \ddot{x}_2 + T\sin\gamma - T\sin\beta = 0$

i.e., $m_2 \ddot{x}_2 + \mathrm{T}\frac{x_2}{l} - T\left(\frac{x_1 - x_2}{l}\right) = 0$

$$m_2 \ddot{x}_2 - \frac{T}{l}x_1 + \frac{2T}{l}x_2 = 0 \qquad \text{----- (2)}$$

Let the solution of equations (1) and (2) be

$$x_1 = A\sin(\omega t + \phi)$$

$$x_2 = B\sin(\omega t + \phi)$$

where A, B and ϕ are arbitrary constants .

Now $\ddot{x}_1 = -A\omega^2\sin(\omega t + \phi)$; $\ddot{x}_2 = -\mathrm{B}\omega^2\sin(\omega t + \phi)$

Substituting these values in equations (1) and (2),

$$m_1(-A\omega^2)\sin(\omega t + \phi) + \frac{2T}{l}A\sin(\omega t + \phi) - \frac{T}{l}B\sin(\omega t + \phi) = 0$$

i.e. $\left(-m_1\omega^2 + \frac{2T}{l}\right)A - \frac{T}{l}B = 0$ ----- (3)

Similarly $m_2(-B\omega^2)\sin(\omega t + \phi) - \frac{T}{l}A\sin(\omega t + \phi) + \frac{2T}{l}B\sin(\omega t + \phi) = 0$

i.e., $\left(-m_2\omega^2 + \frac{2T}{l}\right)B - \frac{T}{l}A = 0$ ----- (4)

For a nontrivial solution of A and B, the determinant of the coefficients A and B must be zero.

i.e., $\begin{vmatrix} -m_1\omega^2 + \frac{2T}{l} & -\frac{T}{l} \\ -\frac{T}{l} & -m_2\omega^2 + \frac{2T}{l} \end{vmatrix} = 0$

$$\text{i.e., } \left(-m_1\omega^2 + \frac{2T}{l}\right)\left(-m_2\omega^2 + \frac{2T}{l}\right) - \frac{T^2}{l^2} = 0$$

$$+ m_1 m_2 \omega^4 - m_1\omega^2 \frac{2T}{l} - \frac{2T}{l} m_2\omega^2 + \frac{4T^2}{l^2} - \frac{T^2}{l^2} = 0$$

$$\text{i.e., } \omega^4 - \left(\frac{2T}{m_2 l} + \frac{2T}{m_1 l}\right)\omega^2 + \frac{3T^2}{m_1 m_2 l^2} = 0 \qquad \text{----- (5)}$$

Equation (5) is the frequency equation of the given system.

$$\omega^2 = + \frac{\left(\frac{2T}{m_2 l} + \frac{2T}{m_1 l}\right) \pm \sqrt{\left\{\frac{2T}{m_2 l} + \frac{2T}{m_1 l}\right\}^2 - 4\times 1\times \frac{3T^2}{m_1 m_2 l^2}}}{2}$$

$$\text{Circular frequency of Ist mode } \omega_1 = \left[\frac{\left(\frac{2T}{m_2 l} + \frac{2T}{m_1 l}\right) - \sqrt{\left\{\frac{2T}{m_2 l} + \frac{2T}{m_1 l}\right\}^2 - \frac{4\times 1\times 3T^2}{m_1 m_2 l^2}}}{2}\right]^{1/2} \frac{\text{rad}}{\text{sec}}$$

$$\text{Circular frequency of 2nd mode } \omega_2 = \left[\frac{\left(\frac{2T}{m_2 l} + \frac{2T}{m_1 l}\right) + \sqrt{\left\{\frac{2T}{m_2 l} + \frac{2T}{m_1 l}\right\}^2 - \frac{4\times 1\times 3T^2}{m_1 m_2 l^2}}}{2}\right]^{1/2} \frac{\text{rad}}{\text{sec}}$$

Note :

i) If $m_1 = m_2 = m$, then the frequency equation is

$$\omega^4 - \left(\frac{2T}{ml} + \frac{2T}{ml}\right)\omega^2 + \frac{3T^2}{m^2 l^2} = 0$$

$$\text{i.e., } \omega^4 - \frac{4T}{ml}\omega^2 + \frac{3T^2}{m^2 l^2} = 0$$

ii) If $m_1 = m$ and $m_2 = 2m$, then the frequency equation is

$$\omega^4 - \left(\frac{2T}{2ml} + \frac{2T}{ml}\right)\omega^2 + \frac{3T^2}{2m^2 l^2} = 0$$

$$\text{i.e., } \omega^4 - \frac{3T}{ml}\omega^2 + \frac{3T^4}{2m^2 l^2} = 0$$

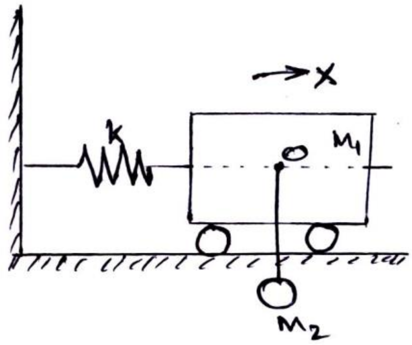
iii) If $m_1 = m_2 = 2m$, then the frequency equation is

$$\omega^4 - \left(\frac{2T}{2ml} + \frac{2T}{2ml}\right)\omega^2 + \frac{3T^2}{4m^2 l^2} = 0$$

i.e., $$\omega^4 - \frac{2T}{ml}\omega^2 + \frac{3T^2}{4m^2l^2} = 0$$

iv) Procedure for amplitude ratio, mode shape etc. refer Example 6.5.

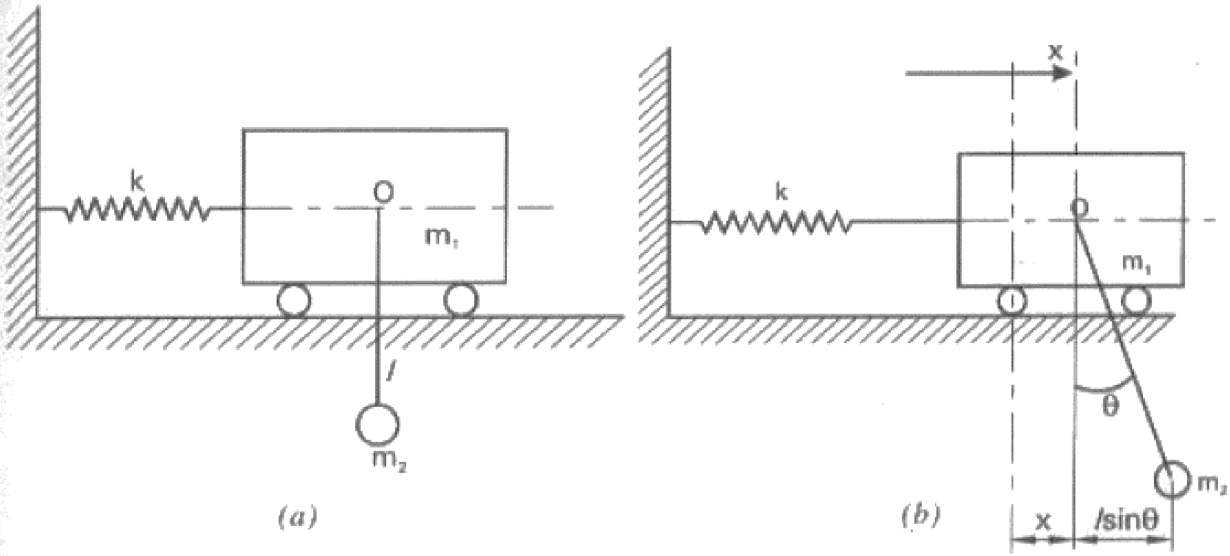
6. Determine the differential equation, natural frequency and the amplitude ratio of the system shown in below fig.

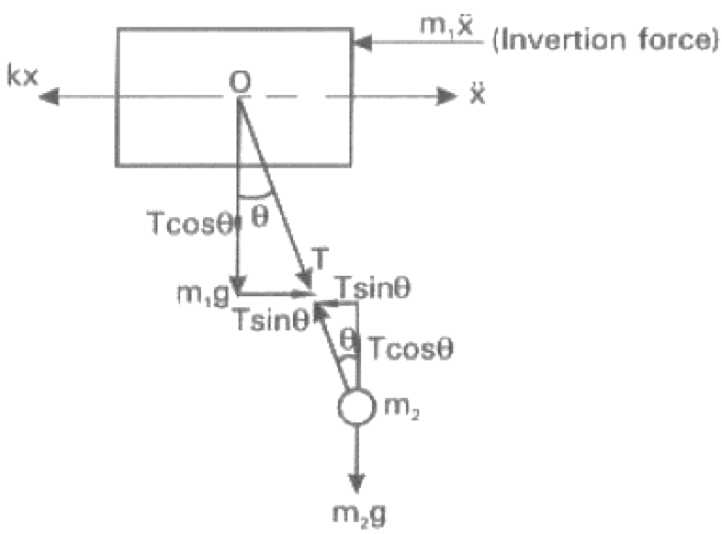


Solution :

i) **Differential equation**

Assume the rod is stiff and initially vertical. Let the system be given a small displacement and the displaced position is shown in Fig 6.29b. The free body diagrams are shown in Fig 6.29c. For small angle $\sin\theta \approx \theta$ and $\cos\theta \approx 1$.





Consider mass m_1

From the Newton's second law of motion

$$m_1\ddot{x} = -kx + T\sin\theta$$

$$\text{i.e., } m_1\ddot{x} + kx - T\theta = 0$$

From Figure 6.29 c

$$T\cos\theta = m_2 g\text{ ; i.e., } T = m_2 g$$

$$\therefore\ m_1\ddot{x} + kx - m_2 g\theta = 0 \quad \text{----(1)}$$

Consider mass m_2

Displacement of mass $m_2 = x + l\sin\theta = x + l\theta$

From Newton's second law of motion

$$m_2(\ddot{x} + l\ddot{\theta}) = -T\sin\theta = -T\theta$$

$$\therefore\ m_2\ddot{x} + m_2 l\ddot{\theta} + m_2 g\theta = 0$$

$$\text{i.e., } \ddot{x} + l\ddot{\theta} + g\theta = 0 \quad \text{----(2)}$$

Equations (1) and (2) are the differential equations of motion.

ii) **Natural frequency**

Let the solution of equations (1) and (2) be

$$x = A\sin(\omega t + \phi)$$

$$\theta = \psi\sin(\omega t + \phi)$$

$$\therefore\ \ddot{x} = -A\omega^2\sin(\omega t + \phi)\ ;\ \ddot{\theta} = -\psi\omega^2\sin(\omega t + \phi)$$

Substituting these values in equations (1) and (2)

$m_1(-A\omega^2)\sin(\omega t + \phi) + kA\sin(\omega t + \phi) - m_2 g\,\psi\sin(\omega t + \phi) = 0$

$$\text{i.e., } (-m_1\omega^2 + k)A - m_2 g\psi = 0 \quad \text{----(3)}$$

Similarly $(-A\omega^2)\sin(\omega t + \phi) + l(-\psi\omega^2)\sin(\omega t + \phi) + g\,\psi\sin(\omega t + \phi) = 0$

$$\text{i.e., } -\omega^2 A + (g - l\omega^2)\psi = 0 \quad \text{-----(4)}$$

For a non trivial solution of A and ψ determinant of the coefficients of A and ψ must be zero

$$\text{i.e., } \begin{vmatrix} -m_1\omega^2 + k & -\omega^2 \\ -m_2 g & g - l\omega^2 \end{vmatrix} = 0$$

i.e., $(-m_1\omega^2 + k)(g - l\omega^2) - m_2 g\omega^2 = 0$

$+ m_1 l\omega^4 - kl\omega^2 - m_1 g\omega^2 + kg - m_2 g\omega^2 = 0$

i.e., $m_1 l\omega^4 - (m_1 g + m_2 g + kl)\,\omega^2 + kg = 0$

$$\text{i.e., } \omega^4 - \left(\frac{g}{l} + \frac{m_2 g}{m_1 l} + \frac{k}{m_1}\right)\omega^2 + \frac{kg}{m_1 l} = 0 \quad \text{-----(5)}$$

Equation (5) is called the frequency equation

$$\therefore\ \omega^2 = \frac{\left(\frac{g}{l} + \frac{m_2 g}{m_1 l} + \frac{k}{m_1}\right) \pm \sqrt{\left(\frac{g}{l} + \frac{m_2 g}{m_1 l} + \frac{k}{m_1}\right)^2 - \frac{4kg}{m_1 l}}}{2}$$

$$\therefore\ \omega_1^2 = \frac{1}{2}\left[\frac{g}{l} + \frac{m_2 g}{m_1 l} + \frac{k}{m_1}\right] - \frac{1}{2}\left[\left(\frac{g}{l} + \frac{m_2 g}{m_1 l} + \frac{k}{m_1}\right)^2 - \frac{4kg}{m_1 l}\right]^{1/2}$$

$$\omega_2^2 = \frac{1}{2}\left[\frac{g}{l}+\frac{m_2 g}{m_1 l}+\frac{k}{m_1}\right]+\frac{1}{2}\left[\left(\frac{g}{l}+\frac{m_2 g}{m_1 l}+\frac{k}{m_1}\right)^2-\frac{4kg}{m_1 l}\right]^{1/2}$$

Using ω_1^2 and ω_2^2 frequency of first and second mode can be calculated.

iii) Amplitude ratio

Using the equations (3) and (4) the amplitude ratio can be written as,

$$(-m_1\omega_1^2 + k)A_1 - m_2 g\psi_1 = 0$$

$$\frac{A_1}{\psi_1} = \frac{m_2 g}{-m_1\omega_1^2 + k}$$

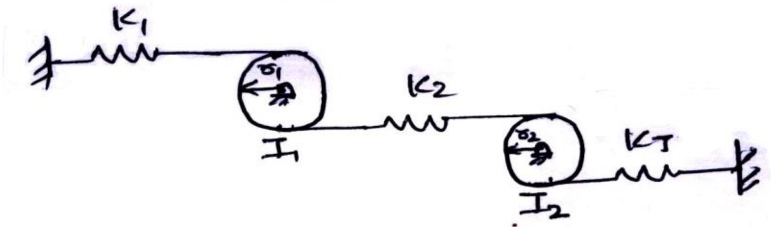
Similarly $-\omega_1^2 A_1 + (g - l\omega_1^2)\psi_1 = 0$

$$\therefore \quad \frac{A_1}{\psi_1} = \frac{g - l\omega_1^2}{\omega_1^2}$$

$$\therefore \quad \frac{A_1}{\psi_1} = \frac{m_2 g}{-m_1\omega_1^2 + k} = \frac{g - l\omega_1^2}{\omega_1^2}$$

$$\text{and} \quad \frac{A_2}{\psi_2} = \frac{m_2 g}{-m_1\omega_2^2 + k} = \frac{g - l\omega_2^2}{\omega_2^2}$$

7. Determine the natural frequency of the system shown in below fig



or

Determine the natural frequencies of the system shown in below fig

K1 = 40 kN/m

K2 = 50 kN/m

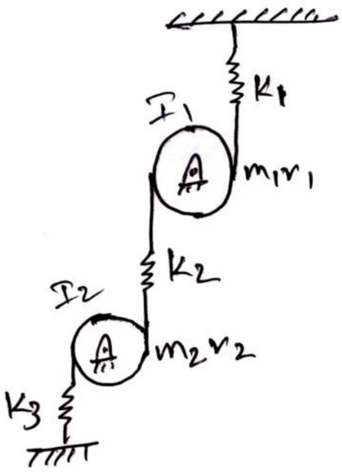
K3 = 60 kN/m

m1 = 10 kg

m2 = 12 kg

r1 = 0.1m

r2 = 0.11m.



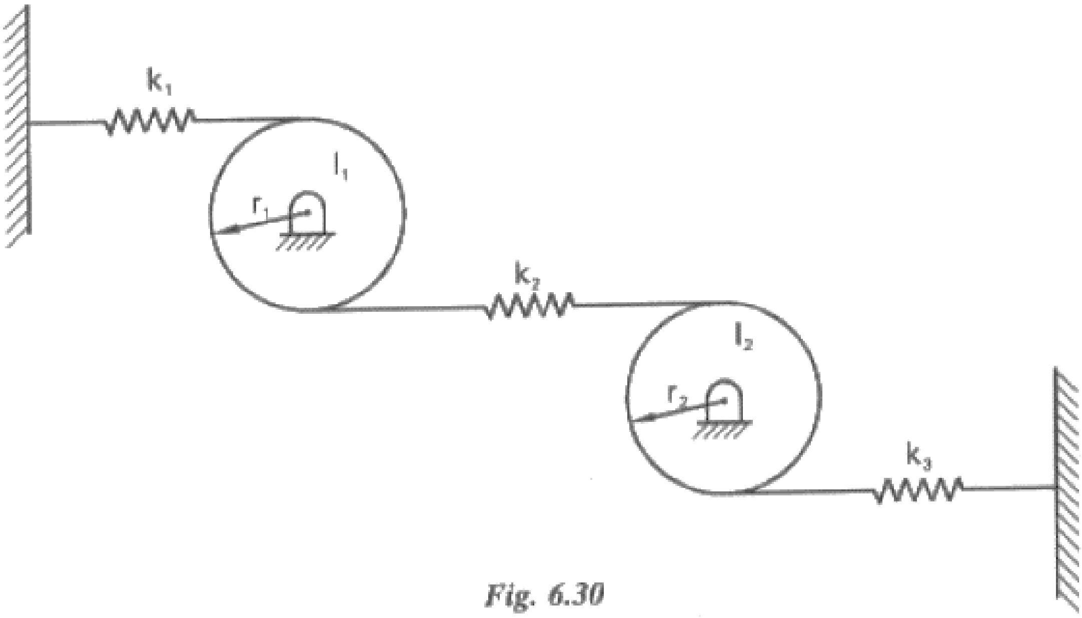


Fig. 6.30

Solution :

The free body diagrams are shown in Fig 6.31.

The torque equation is $I\ddot{\theta} = \Sigma T$

$$\therefore \quad I_1\ddot{\theta}_1 = -(k_1 r_1 \theta_1) r_1 - k_2 (r_1 \theta_1 - r_2\theta_2) r_1$$

i.e., $I_1\ddot{\theta}_1 + (k_1 r_1^2 + k_2 r_1^2)\theta_1 - k_2 r_1 r_2 \theta_2 = 0$ ---- (1)

$$\text{Similarly } I_2\ddot{\theta}_2 = -k_2 (r_2\theta_2 - r_1\theta_1) r_2 - (k_3 r_2 \theta_2) r_2$$

i.e., $I_2\ddot{\theta}_2 + (k_2 r_2^2 + k_3 r_2^2)\theta_2 - k_2 r_1 r_2 \theta_1 = 0$ ---- (2)

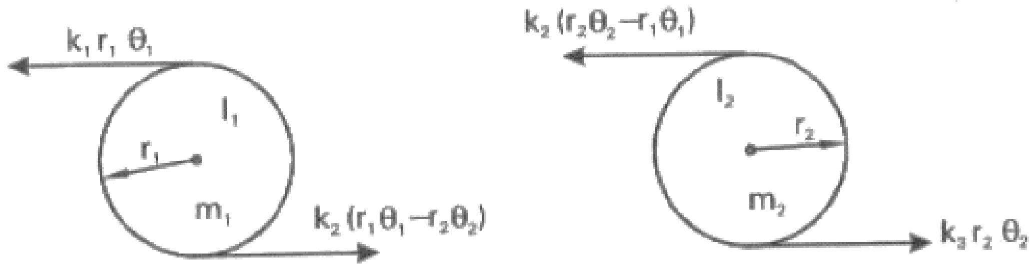


Fig. 6.31

Let the solutions of equations (1) and (2) be

$$\theta_1 = \psi_1 \sin(\omega t + \phi)$$

$$\theta_2 = \psi_2 \sin(\omega t + \phi)$$

$$\therefore \quad \ddot{\theta}_1 = -\psi_1\omega^2 \sin(\omega t + \phi); \quad \ddot{\theta}_2 = -\psi_2\omega^2 \sin(\omega t + \phi)$$

Substituting these values in equations (1) and (2)

$$I_1(-\psi_1\omega^2)\sin(\omega t + \phi) + (k_1 r_1^2 + k_2 r_1^2)\psi_1 \sin(\omega t + \phi) - k_2 r_1 r_2 \psi_2 \sin(\omega t + \phi) = 0$$

i.e., $\{-I_1\omega^2 + (k_1 r_1^2 + k_2 r_1^2)\}\psi_1 - k_2 r_1 r_2 \psi_2 = 0$ ----- (3)

$$\text{Similarly } I_2(-\psi_2\omega^2)\sin(\omega t + \phi) + (k_2 r_2^2 + k_3 r_2^2)\psi_2 \sin(\omega t + \phi) - k_2 r_1 r_2 \psi_1 \sin(\omega t + \phi) = 0$$

i.e., $\{-I_2\omega^2 + (k_2 r_2^2 + k_3 r_2^2)\}\psi_2 - k_2 r_1 r_2 \psi_1 = 0$ ----- (4)

For a nontrivial solution of ψ_1 and ψ_2 determinant of the coefficients of ψ_1 and ψ_2 must be zero.

$$\text{i.e.,}\quad \begin{vmatrix} -I_1\omega^2 + (k_1 r_1^2 + k_2 r_1^2) & -k_2 r_1 r_2 \\ -k_2 r_1 r_2 & -I_2\omega^2 + (k_2 r_2^2 + k_3 r_2^2) \end{vmatrix} = 0$$

i.e., $\{-I_1\omega^2 + (k_1 r_1^2 + k_2 r_1^2)\}\ \{-I_2\omega^2 + (k_2 r_2^2 + k_3 r_2^2)\} - (k_2 r_1 r_2)^2 = 0$

i.e., $I_1 I_2 \omega^4 - I_1\omega^2 (k_2 r_2^2 + k_3 r_2^2) - I_2\omega^2 (k_1 r_1^2 + k_2 r_1^2) + (k_1 r_1^2 + k_2 r_1^2)(k_2 r_2^2 + k_3 r_2^2) - (k_2 r_1 r_2)^2 = 0$

i.e., $I_1 I_2 \omega^4 - (I_1 k_2 r_2^2 + I_1 k_3 r_2^2 + I_2 k_1 r_1^2 + I_2 k_2 r_1^2)\,\omega^2$

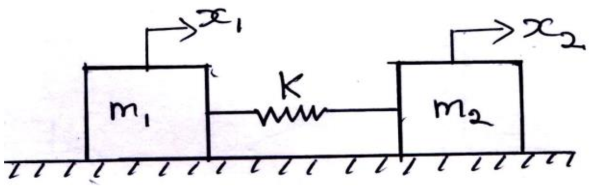
$$+ k_1 k_2 r_1^2 r_2^2 + k_1 k_3 r_1^2 r_2^2 + k_2^2 r_1^2 r_2^2 + k_2 k_3 r_1^2 r_2^2 - k_2^2 r_1^2 r_2^2 = 0$$

$$\text{i.e.,}\quad \omega^4 - \left(\frac{k_2}{I_2} r_2^2 + \frac{k_3}{I_2} r_2^2 + \frac{k_1}{I_1} r_1^2 + \frac{k_2}{I_1} r_1^2\right)\omega^2 + \frac{\left(k_1 k_2 r_1^2 r_2^2 + k_1 k_3 r_1^2 r_2^2 + k_2 k_3 r_1^2 r_2^2\right)}{I_1 I_2} = 0$$

$$\therefore\quad \omega^2_{1,2} = \frac{1}{2}\left[\frac{k_2 r_2^2 + k_3 r_2^2}{I_2} + \frac{k_1 r_1^2 + k_2 r_1^2}{I_1}\right] \pm \frac{1}{2}\left[\left\{\frac{k_2 r_2^2 + k_3 r_2^2}{I_2} + \frac{k_1 r_1^2 + k_2 r_1^2}{I_1}\right\}^2 - 4\frac{\left(k_1 k_2 r_1^2 r_2^2 + k_1 k_3 r_1^2 r_2^2 + k_2 k_3 r_1^2 r_2^2\right)}{I_1 I_2}\right]^{1/2}$$

Using $\omega^2_{1,2}$ frequency of first and second mode can be calculated.

8. For the system shown in below fig. find the natural frequencies and amplitude ratios. Given m1 = 10 kg, m2 = 15 kg, K = 320 N/m.



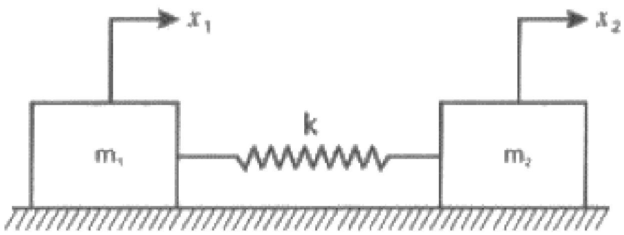


Fig. 6.43

Solution :

(i) **Equation of motion**

The free body diagrams are shown in **Fig.6.44**

Fig. 6.44

The equations of motion are obtained from Netwon's second law of motion. Considering the free body diagram shown in **Fig. 6.44**

$$m_1 \ddot{x}_1 = -k(x_1 - x_2)$$

$$m_1 \ddot{x}_1 + kx_1 - kx_2 = 0 \quad \text{---(1)}$$

Similarly,

$$m_2 \ddot{x}_2 = -k(x_2 - x_1)$$

$$m_2 \ddot{x}_2 - kx_1 + kx_2 = 0 \quad \text{---(2)}$$

Equations (1) and (2) are called the differential equations of motion for the given system.

(ii) Frequency equation and natural frequencies

Let the solution of equations (1) and (2) can be written as

$$x_1 = A \sin(\omega t + \phi) \text{ and } x_2 = B \sin(\omega t + \phi)$$

where, A, B and ϕ are arbitrary constants.

Now, $\dot{x}_1 = A\omega \cos(\omega t + \phi)$; $\ddot{x}_1 = -A\omega^2 \sin(\omega t + \phi)$

$\dot{x}_2 = B\omega \cos(\omega t + \phi)$; $\ddot{x}_2 = -B\omega^2 \sin(\omega t + \phi)$

Substituting these values in equation (1) and (2)

$$(m_1)(-A\omega^2) \sin(\omega t + \phi) + kA \sin(\omega t + \phi) - kB \sin(\omega t + \phi) = 0$$

i.e., $\left(-m_1\omega^2 + k\right) A - kB = 0$ ---(3)

Similarly,

$$(m_2)(-B\omega^2) \sin(\omega t + \phi) - k\,A \sin(\omega t + \phi) + k\,B \sin(\omega t + \phi) = 0$$

i.e., $\left(-m_2\omega^2 + k\right) B - kA = 0$ ---(4)

Equations (3) and (4) are homogeneous, linear algebraic equations in A and B. For a non trivial solution of A and B, the determinant of the coefficient of A and B must be zero.

$$\begin{vmatrix} (-m_1\omega^2 + k) & -k \\ -k & (-m_2\omega^2 + k) \end{vmatrix} = 0$$

Expanding the determinant

$$(-m_1\omega^2 + k)(-m_2\omega^2 + k) - k^2 = 0$$

i.e., $+ m_1 m_2 \omega^4 - m_1 k\omega^2 - m_2 k\omega^2 + k^2 - k^2 = 0$

i.e., $m_1 m_2 \omega^4 - (m_1 + m_2) k\omega^2 = 0$

i.e., $\omega^4 - \left(\dfrac{m_1 + m_2}{m_1 m_2}\right) k\omega^2 = 0$ ---(5)

Equation (5) is called the frequency equation

Equation (5) can be written as

$$\omega^2 \left[\omega^2 - \left(\frac{m_1 + m_2}{m_1\, m_2}\right) k\right] = 0$$

i.e., $\omega^2 = 0$ or $\omega^2 - \left(\dfrac{m_1 + m_2}{m_1\, m_2}\right) k = 0$

$$\therefore \quad \omega_1 = 0 \quad \text{and} \quad \omega_2 = \sqrt{k\left(\frac{m_1 + m_2}{m_1\ m_2}\right)} = \sqrt{320\left(\frac{10+15}{10\times 15}\right)} = 7.303 \text{ rad / sec}$$

$$\therefore \text{ Natural frequency of first mode } f_1 = \frac{1}{2\pi}\ \omega_1 = 0$$

$$\text{Natural frequency of second mode } f_2 = \frac{1}{2\pi}\ \omega_2 = \frac{1}{2\pi}\sqrt{k\left(\frac{m_1 + m_2}{m_1\ m_2}\right)},\ \text{Hz}$$

$$= \frac{1}{2\pi}\sqrt{320\left(\frac{10+15}{10\times 15}\right)} = 1.162 \text{ Hz}$$

(iii) Amplitude ratio and mode shapes

Using the equations (3) and (4), the ratio of amplitudes of motion can be written as

$$\frac{A_1}{B_1} = \frac{k}{-m_1\ \omega_1^2 + k} = \frac{-m_2\ \omega_1^2 + k}{k}$$

$$\frac{A_2}{B_2} = \frac{k}{-m_1\ \omega_2^2 + k} = \frac{-m_2\ \omega_2^2 + k}{k}$$

$$\therefore \text{ Amplitude ratio of first mode } \frac{A_1}{B_1} = \frac{k}{-m_1\ \omega_1^2 + k} = \frac{k}{0+k} = 1$$

$$\text{Amplitude ratio of second mode } \frac{A_2}{B_2} = \frac{k}{-m_1\ \omega_2^2 + k} = \frac{320}{-(10)\ (7.303)^2 + 320} = -1.5$$

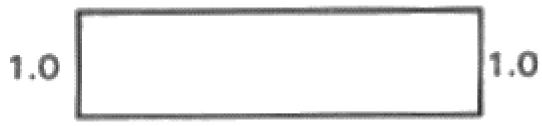
The normal mode of vibration corresponding to ω_1^2 can be expressed as

$$\phi_1(x) = \begin{Bmatrix} A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ \lambda_1 A_1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

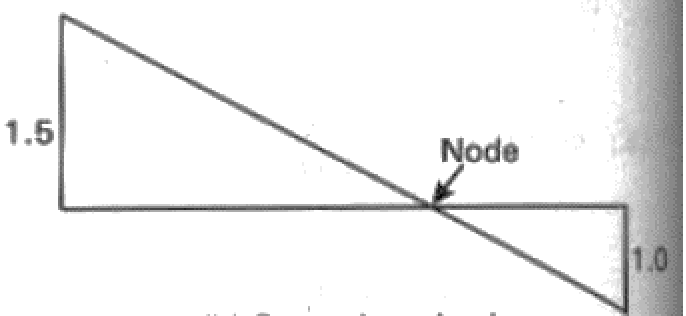
The normal mode of vibration corresponding to ω_2^2 can be expressed as

$$\phi_2(x) = \begin{Bmatrix} A_2 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} A_2 \\ \lambda_2\ A_2 \end{Bmatrix} = \begin{Bmatrix} 1.5 \\ -1 \end{Bmatrix} \text{ or } \begin{Bmatrix} -1.5 \\ 1 \end{Bmatrix}$$

$\phi_1(x)$ and $\phi_2(x)$ are known as the modal vectors or eigen vectors of the system. The mode shapes for first mode and second mode are as shown in **Fig. 6.45** (a) and (b) respectively.



(a) First mode shape



(b) Second mode shape

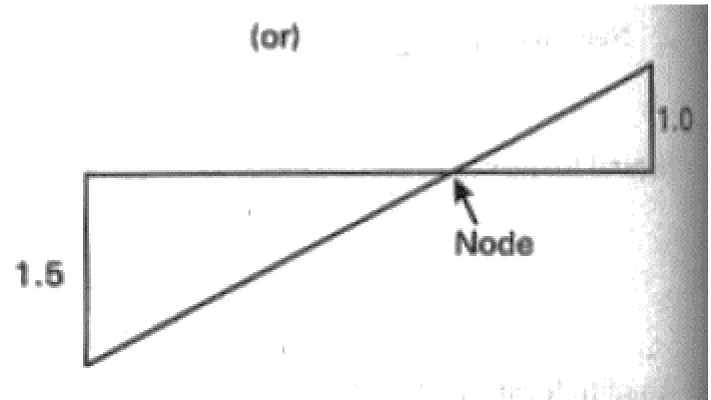
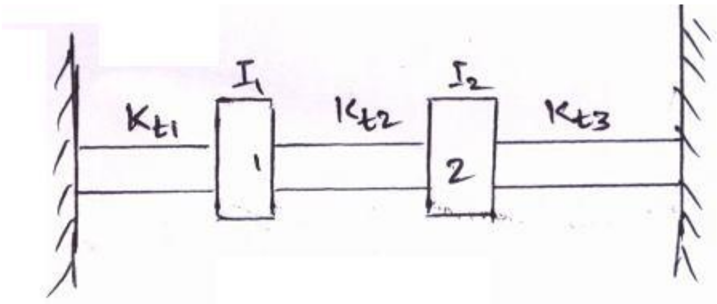


Fig. 6.45

9. Determine the frequency equation and the general solution of the two degrees of freedom torsional system shown in below fig.



Solution :

Let k_t = Torsional stiffness

I_1 = mass moment of inertia of rotor 1 = $\frac{1}{2} m_1 r_1^2$

I_2 = mass moment of inertia of rotor 2 = $\frac{1}{2} m_2 r_2^2$

Let θ_1 and θ_2 be the angular displacements of rotors I_1 and I_2 respectively.

The free body diagrams of the two rotors are as shown in Fig. 6.48.

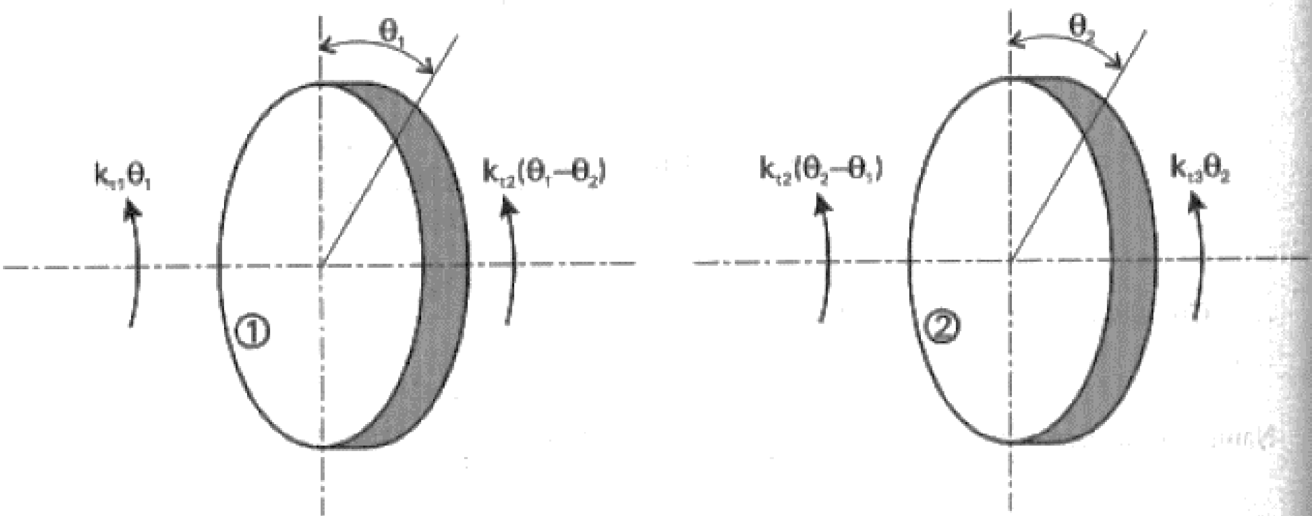


Fig. 6.48

From Newton's second law of motion (Torque equation)

$$I\ddot{\theta} = \Sigma T \text{ or } \Sigma M$$

Consider first rotor

$$I_1\ddot{\theta}_1 = -k_{t_1}\theta_1 - k_{t_2}(\theta_1-\theta_2) \quad \text{if } \theta_1>\theta_2$$

$$\text{i.e., } I_1\ddot{\theta}_1 + (k_{t_1}+k_{t_2})\theta_1 - k_{t_2}\theta_2 = 0 \qquad \text{---(1)}$$

Consider the second rotor

$$I_2\ddot{\theta}_2 = -k_{t3}\theta_2 - k_{t2}(\theta_2-\theta_1)$$

$$I_2\ddot{\theta}_2 + (k_{t_2}+k_{t_3})\theta_2 - k_{t_2}\theta_1 = 0 \qquad \text{---(2)}$$

Equations (1) and (2) are the differential equation of motion of the system.

Assuming the harmonic motions of I_1 and I_2 at the same circular frequency ω and at the same phase angle ϕ, the solutions of equations (1) and (2) can be written as

$$\theta_1 = \psi_1 \sin(\omega t+\phi) \text{ and } \theta_2 = \psi_2 \sin(\omega t+\phi)$$

$$\therefore\ \ddot{\theta}_1 = -\psi_1\omega^2\sin(\omega t+\phi) \text{ and } \ddot{\theta}_2 = -\psi_2\omega^2\sin(\omega t+\phi)$$

Substituting these values in equations (1) and (2)

$$I_1(-\psi_1\omega^2)\sin(\omega t+\phi) + (k_{t_1}+k_{t_2})\psi_1\sin(\omega t+\phi) - k_{t_2}\psi_2\sin(\omega t+\phi) = 0$$

$$\text{i.e., } \{-I_1\omega^2 + (k_{t_1}+k_{t_2})\}\psi_1 - k_{t_2}\psi_2 = 0 \qquad \text{---(3)}$$

$$\text{Similarly, } I_2(-\psi_2\omega^2)\sin(\omega t+\phi) + (k_{t_2}+k_{t_3})\psi_2\sin(\omega t+\phi) - k_{t_2}\psi_1\sin(\omega t+\phi) = 0$$

$$\text{i.e., } \{-I_2\omega^2 + (k_{t_2}+k_{t_3})\}\psi_2 - k_{t_2}\psi_1 = 0 \qquad \text{---(4)}$$

For a non trivial solution of ψ_1 and ψ_2 determinant of the coefficients of ψ_1 and ψ_2 must be zero.

$$\text{i.e., } \begin{vmatrix} -I_1\omega^2 + (k_{t_1}+k_{t_2}) & -k_{t_2} \\ -k_{t_2} & -I_2\omega^2 + (k_{t_2}+k_{t_3}) \end{vmatrix} = 0$$

$$\text{i.e., } \{-I_1\omega^2 + (k_{t_1}+k_{t_2})\}\{-I_2\omega^2 + (k_{t_2}+k_{t_3})\} - k_{t2}^2 = 0$$

$$\text{i.e., } I_1I_2\omega^4 - I_1\omega^2(k_{t_2}+k_{t_3}) - I_2\omega^2(k_{t_1}+k_{t_2}) + (k_{t_1}+k_{t_2})(k_{t_2}+k_{t_3}) - k_{t2}^2 = 0$$

$$\text{i.e., } I_1I_2\omega^4 - \{I_1(k_{t_2}+k_{t_3}) + I_2(k_{t_1}+k_{t_2})\}\omega^2 + k_{t_1}k_{t_2} + k_{t_1}k_{t_3} + k_{t2}^2 + k_{t_2}k_{t_3} - k_{t2}^2 = 0$$

$$\text{i.e., } \omega^4 - \left[\frac{k_{t_2}+k_{t_3}}{I_2} + \frac{k_{t_1}+k_{t_2}}{I_1}\right]\omega^2 + \left(\frac{k_{t_1}k_{t_2} + k_{t_1}k_{t_3} + k_{t_3}k_{t_1}}{I_1I_2}\right) = 0 \qquad \text{---(5)}$$

$$\therefore\ \omega^2 = +\frac{1}{2}\left[\frac{k_{t_2}+k_{t_3}}{I_2} + \frac{k_{t_1}+k_{t_2}}{I_1}\right] \pm \frac{1}{2}\left[\left(\frac{k_{t_2}+k_{t_3}}{I_2} + \frac{k_{t_1}+k_{t_2}}{I_1}\right)^2 - 4\left(\frac{k_{t_1}k_{t_2} + k_{t_2}k_{t_3} + k_{t_3}k_{t_1}}{I_1I_2}\right)\right]^{\frac{1}{2}} \qquad \text{---(6)}$$

Equation (5) is the frequency equation of the system.

From equation (6) the two values of circular frequency ω_1 and ω_2 can be obtained.

Using the equations (3) and (4) the amplitude ratio can be written as

$$\frac{\psi_{11}}{\psi_{21}} = \frac{k_{t_2}}{-I_1\omega_1^2 + (k_{t_1}+k_{t_2})} = \frac{-I_2\omega_1^2 + (k_{t_2}+k_{t_3})}{k_{t_2}} = \frac{1}{\lambda_1} \qquad \text{---(7)}$$

$$\frac{\psi_{12}}{\psi_{22}} = \frac{k_{t_2}}{-I_1\omega_2^2 + (k_{t_1}+k_{t_2})} = \frac{-I_2\omega_2^2 + (k_{t_2}+k_{t_3})}{k_{t_2}} = \frac{1}{\lambda_2} \qquad \text{---(8)}$$

The normal modes of vibration corresponding to ω_1^2 and ω_2^2 can be expressed respectively.

$$\phi_1(\theta) = \begin{Bmatrix}\psi_{11}\\ \psi_{21}\end{Bmatrix} = \begin{Bmatrix}\psi_{11}\\ \lambda_1\psi_{11}\end{Bmatrix} \qquad \text{---- (9)}$$

$$\phi_2(\theta) = \begin{Bmatrix}\psi_{12}\\ \psi_{22}\end{Bmatrix} = \begin{Bmatrix}\psi_{12}\\ \lambda_2\psi_{12}\end{Bmatrix} \qquad \text{---- (10)}$$

The vectors $\phi_1(\theta)$ and $\theta_2(\theta)$ which denote the normal modes of vibration are known as the modal vectors or eigen vectors of the system.

The general solution of the equation of motion is composed of two harmonic motions of frequencies ω_1 and ω_2, they are the fundamental and first harmonic.

$$\therefore\ \theta_1 = \psi_{11}\sin(\omega_1 t + \phi_1) + \psi_{12}\sin(\omega_2 t + \phi_2) \qquad \text{----- (11)}$$

$$\theta_2 = \psi_{21}\sin(\omega_1 t + \phi_1) + \psi_{22}\sin(\omega_2 t + \phi_2)$$

$$= \lambda_1\psi_{11}\sin(\omega_1 t + \phi_1) + \lambda_2\psi_{12}\sin(\omega_2 t + \phi_2) \qquad \text{----- (12)}$$

where ψ_{11}, ψ_{12}, ϕ_1 and ϕ_2 are arbitrary constants. These constants can be evaluated from the four initial conditions $\theta_1(0)$, $\dot{\theta}_1(0)$, $\theta_2(0)$ and $\dot{\theta}_2(0)$.

If the arbitrary constant ψ_{12} is zero in equation (11) and (12), then the first mode will exist. Hence the equation becomes

$$\theta_1 = \psi_{11}\sin(\omega_1 t + \phi_1)$$

$$\theta_2 = \lambda_1\psi_{11}\sin(\omega_1 t + \phi_1) \qquad \text{----- (13)}$$

If the arbitrary constant ψ_{11} is zero in equation (11) and (12), then the second mode will exist. Hence the equation becomes,

$$\theta_1 = \psi_{12}\sin(\omega_2 t + \phi_2)$$

$$\theta_2 = \lambda_2\psi_{12}\sin(\omega_2 t + \phi_2) \qquad \text{----- (14)}$$

Fig. 6.49 shows the mode shape diagram.

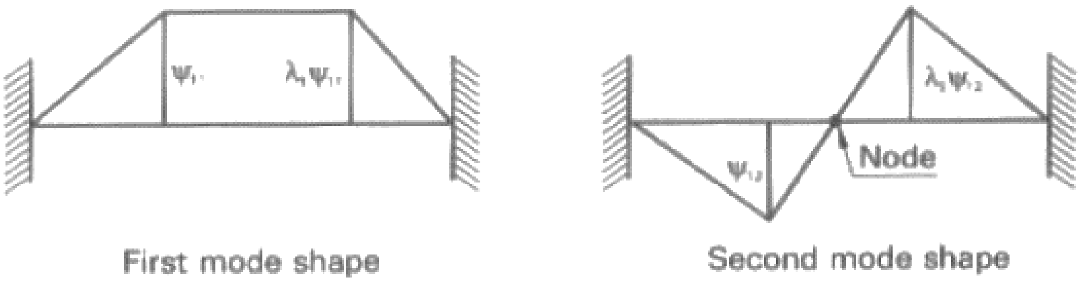
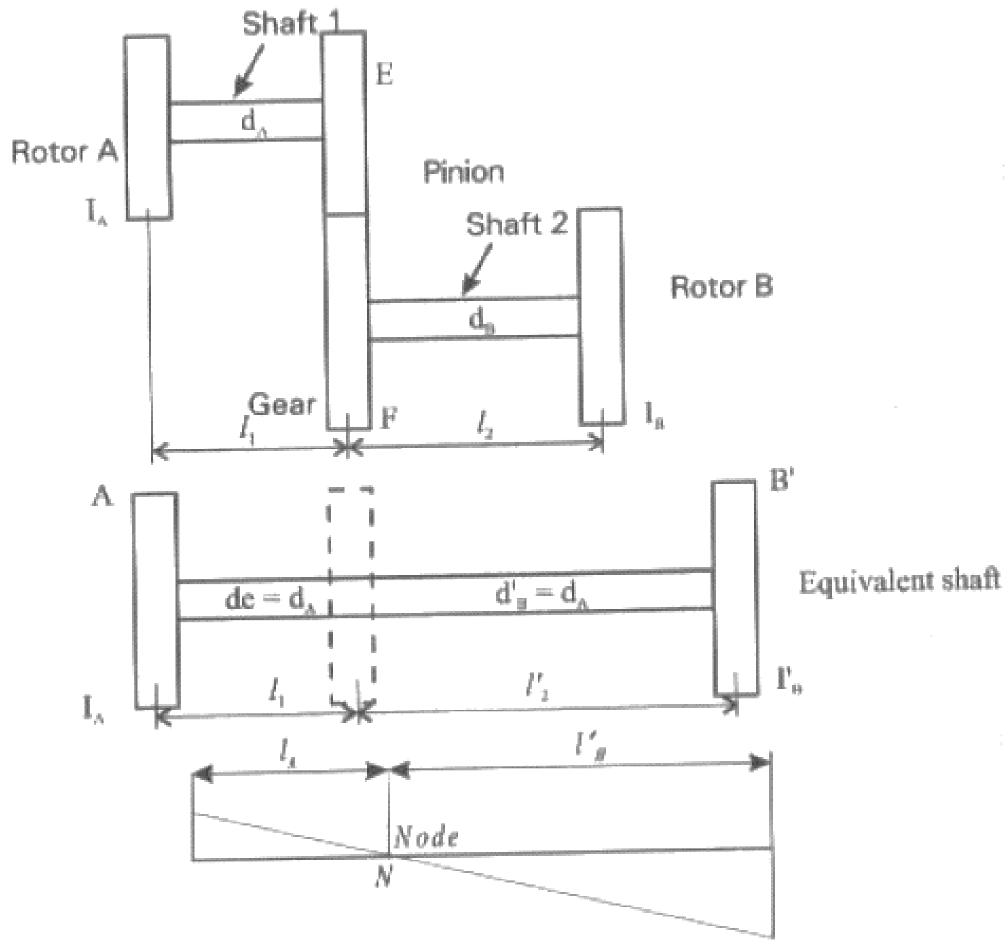


Fig. 6.49

10. Prove the angular displacements of the two rotors are inversely proportional to their inertias.

The Fig 6.60 shows a geared system in which rotor *A* on one shaft is connected through pinion '*E*' and gear wheel *F* to the rotor '*B*' on the second shaft. This system is replaced by an equivalent system as shown below.



While replacing an equivalent shaft, the following assumptions are made,

i) The inertia of the gears and the shafts is negligible

ii) The load is within elastic limits of gear teeth i.e., they are rigid.

iii) No backlash or slip occurs in the gear drive.

Let d_A and d_B = Diameter of the shafts 1 and 2

l_1 and l_2 = Length of shafts 1 and 2

I_A and I_B = Mass moment of inertia of the rotors A and B

ω_A and ω_B = Angular speed of the rotors A and B

$$G = \text{Gear ratio} = \frac{\text{Speed of pinion } E}{\text{Speed of gear } F} = \frac{\omega_A}{\omega_B}$$

d_e = Diameter of the equivalent shaft = d_A

l_e = Length of the equivalent shaft

I_B' = Mass moment of inertia of the equivalent rotor B'.

The systems are equivalent preferably,

(i) $K.E$ of the original system is equal to that of equivalent system

(ii) Strain energy of the original system is equal to the strain energy of the equivalent system

(A) Equating Kinetic energies

$K.E$ of original system = $K.E$ of section l_1 + $K.E$ of section l_2

$K.E$ of equivalent system = $K.E.$ of section l_1 + $K.E$ of section l_2'

$\therefore$ $K.E$ of section l_2 = $K.E$ of section l_2'

i.e., $\frac{1}{2} I_B \omega_B^2 = \frac{1}{2} I'_B \omega_B^{2\prime}$

$$\therefore \quad I_B' = I_B \cdot \left(\frac{\omega_B}{\omega_B'}\right)^2 \quad \text{But} \quad \omega_B' = \omega_A$$

$$\therefore \quad I_B' = I_B \left(\frac{\omega_B}{\omega_A}\right)^2 \quad \text{since Gear ratio } G = \frac{\omega_A}{\omega_B}$$

$$\therefore \quad I_B' = \frac{I_B}{G^2} \qquad \text{----- (6.5.1)}$$

(B) Equating strain energies

Strain Energy of section l_2' = Strain Energy of section l_2

i.e., $\frac{1}{2} T_B' \theta_B' = \frac{1}{2} T_B \theta_B$ $\qquad \frac{T}{J} = \frac{G\theta}{l}$

$$\therefore \quad \frac{GJ'_B}{l_2'} \theta'_B . \theta'_B = \frac{GJ_B}{l_2} . \theta_B . \theta_B \qquad \therefore \quad \theta = \frac{Tl}{GJ}$$

$$\therefore \quad l_2' = l_2 \cdot \left(\frac{\theta'_B}{\theta_B}\right)^2 \cdot \frac{J'_B}{J_B} = l_2 \left(\frac{\omega'_B}{\omega_B}\right)^2 \left(\frac{d'_B}{d_B}\right)^4 \quad \because \quad \theta = \omega t\,;\, J = \frac{\pi}{32} d^4$$

$$= l_2 \left(\frac{\omega_A}{\omega_B}\right)^2 \cdot \left(\frac{d_A}{d_B}\right)^4 \qquad \because \quad \omega_B' = \omega_A\,;\, d_B' = d_A$$

$$\therefore \quad l_2' = l_2 \cdot \left(\frac{d_A}{d_B}\right)^4 . G^2 \qquad \text{----- (6.5.2)}$$

$$\therefore \quad \text{Equivalent length } \; l_e = l_1 + l_2 \cdot \left(\frac{d_A}{d_B}\right)^4 . G^2 \qquad \text{----- (6.5.3)}$$

The natural frequency of torsional vibration of a geared system which has been reduced to two rotor system is calculated as explained below :

Let the node of the equivalent system be N as shown in figure 6.41 Distance of the node from rotor A is l_A and from rotor B' is l_B'.

Since $f_{n_A} = f'_{n_B}$

$$\frac{1}{2\pi} \sqrt{\frac{GJ}{l_A I_A}} = \frac{1}{2\pi} \sqrt{\frac{GJ}{l'_B I'_B}}$$

$$\therefore \quad l_A I_A = l_B' I_B' \qquad \text{----- (6.5.4)}$$

$$\text{Also} \quad l_A + l_B' = l_e \qquad \text{----- (6.5.5)}$$

From equations 6.5.4 and 6.5.5, the values of l_A and l_B' can be calculated.

When the inertia of the gearing is taken into consideration, then an additional rotor [shown dotted in Fig 6. 60] must be introduced to the equivalent system at a distance l_1 from the rotor A

The mass moment of inertia of this rotor is given by $I_E' = I_E + \frac{I_F}{G^2}$ where I_E and I_F are the mass moment of inertia of the pinion and gear wheel respectively. The system then becomes a three rotor system.

11. Design a dynamic vibration absorber and show that in order to reduce the amplitude of main system, exciting force must be equal to spring force of absorber system.

or

With help of suitable sketches illustrate the working of: (i) the Dynamic Vibration Absorber and (ii) Dynamics of reciprocating engines.

6.6.2 Dynamic Vibration Absorber

(VTU, June/July 2009, May/June 2010, June/July 2011, Dec. 2012, Dec. 2013 / Jan. 2014, June / July 2014, Dec., 2014 / Dec., 2015)

A dynamic vibration absorber is a single degree of freedom system, to which is attached another single degree freedom system as an auxiliary system, thus it will transform the whole system into a two degree of freedom system, having two natural frequencies of vibration. One of the natural frequencies is set above the excitation frequency while the other is set below it so that the main mass of the entire system will have very small amplitude of vibration instead of very large amplitude under the given excitation. Fig. 6.75(a) shows a schematic sketch of a spring mounted dynamic absorber. The equivalent system of the dynamic absorber is shown in Fig. 6.75(b).

Spring mass system $k_1 - m_1$ is the main system and spring mass system $k_2 - m_2$ is the auxiliary or absorber system. The equation of motion for this system is,

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2) + F_0 \sin \omega t$$

i.e., $m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F_0 \sin \omega t$ ----- 6.6.10

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$$

i.e., $m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$ ----- 6.6.11

Let the solution of equations 6.6.10 and 6.611 be

$$x_1 = A \sin \omega t; \quad x_2 = B \sin \omega t$$

$$\therefore \quad \ddot{x}_1 = -A\omega^2 \sin \omega t; \; \ddot{x}_2 = -B\omega^2 \sin \omega t$$

Substituting these values in equation 6.6.10 and 6.6.11.

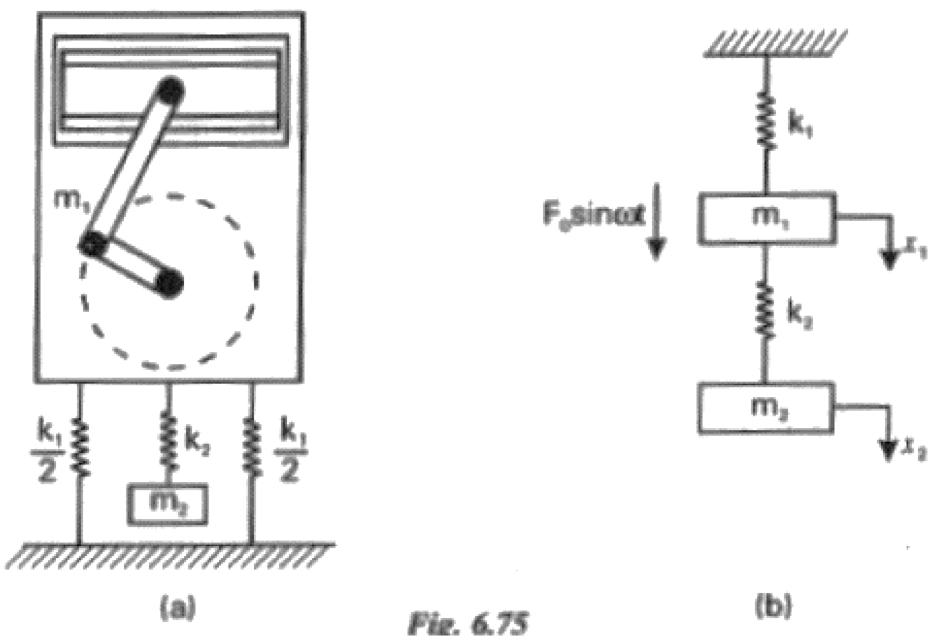


Fig. 6.75

$m_1(-A\,\omega^2 \sin \omega t) + (k_1 + k_2)(A \sin \omega t) - k_2 B \sin \omega t = F_0 \sin \omega t$

i.e., $[-m_1\omega^2 + (k_1 + k_2)]\,A - k_2 B = F_0$ ----- 6.6.12

Similarly $m_2(-B\omega^2 \sin \omega t) + k_2 B \sin \omega t - k_2 A \sin \omega t = 0$

i.e., $(-m_2\omega^2 + k_2)\,B - k_2 A = 0$ ----- 6.6.13

It can be written in matrix form

$$\begin{vmatrix} -m_1\omega^2 + (k_1 + k_2) & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{vmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} F_0 \\ O \end{Bmatrix} \qquad \text{----- 6.6.14}$$

Equation 6.6.14 can be solved for the amplitudes A and B

$$A = \frac{\begin{vmatrix} F_0 & -k_2 \\ O & -m_2\omega^2 + k_2 \end{vmatrix}}{\begin{vmatrix} -m_1\omega^2 + (k_2 + k_1) & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{vmatrix}}$$

$$= \frac{F_0\left(k_2 - m_2\omega^2\right)}{m_1 m_2 \omega^4 - (m_1 k_2 + m_2 k_2 + m_2 k_1)\omega^2 + k_1 k_2} \qquad \text{----- 6.6.15}$$

$$\text{Similarly } B = \frac{\begin{vmatrix} -m_1\omega^2 + (k_1 + k_2) & F_0 \\ -k_2 & O \end{vmatrix}}{\begin{vmatrix} -m_1\omega^2 + (k_1 + k_2) & -k_2 \\ -k_2 & \left(-m_2\omega^2 + k_2\right) \end{vmatrix}}$$

$$= \frac{F_0 k_2}{m_1 m_2 \omega^4 - (m_1 k_2 + m_2 k_2 + m_2 k_1)\omega^2 + k_1 k_2} \qquad \text{----- 6.6.16}$$

To make the equations 6.6.15 and 6.6.16 in the dimensionless form, divide the numerators and denominators by $k_1 k_2$

Let $A_{st} = \dfrac{F_0}{k_1}$ = Zero frequency deflection

$\omega_1 = \sqrt{\dfrac{k_1}{m_1}}$ = Natural frequency of the main system alone.

$\omega_2 = \sqrt{\dfrac{k_2}{m_2}}$ = Natural frequency of the absorber system alone.

$\mu = \dfrac{m_2}{m_1}$ = Ratio of the absorber mass to the main mass

$$\therefore \quad \frac{A}{A_{st}} = \frac{1-\frac{\omega^2}{\omega_2^2}}{\frac{\omega^4}{\omega_1^2\omega_2^2} - \left[(1+\mu)\frac{\omega^2}{\omega_1^2} + \frac{\omega^2}{\omega_2^2}\right] + 1} \quad \text{----- 6.6.17}$$

$$\frac{B}{A_{st}} = \frac{1}{\frac{\omega^4}{\omega_1^2\omega_2^2} - \left[(1+\mu)\frac{\omega^2}{\omega_1^2} + \frac{\omega^2}{\omega_2^2}\right] + 1} \quad \text{----- 6.6.18}$$

when $\omega = \omega_2$, $A = 0$ i.e., when the excitation frequency is equal to the natural frequency of the absorber, then the main system amplitude is zero even though it is excited by a harmonic force. It is the principle of an undamped dynamic vibration absorber, because, if a main system has undesirable vibrations, then a secondary absorber system having its natural frequency equal to the operating frequency can be coupled to the main system to reduce its amplitude to zero.

$$\text{Now when } \omega = \omega_2,\ B = \frac{-A_{st}}{\mu\frac{\omega_2^2}{\omega_1^2}} = \frac{-\frac{F_0}{k_1}}{\frac{m_2}{m_1}\cdot\frac{k_2}{m_2}\cdot\frac{m_1}{k_1}} = -\frac{F_0}{k_2}$$

$$\therefore \quad F_0 = -Bk_2 \quad \text{----- 6.6.19}$$

i.e., the spring force Bk_2 is equal and opposite to the exciting force on the main mass, resulting in no motion of the system. Since the main system vibrations have been reduced to zero and these vibrations have been taken by the absorber system, it is called vibration absorber. This undamped dynamic vibration absorber is also called as Frahm Vibration Absorber.

For more effectiveness the operating frequency of the absorber must be equal to the natural frequency of the main system, i.e., $\omega_2 = \omega_1$ or $k_2/m_2 = k_1/m_1$. Under this condition the absorber is called tuned absorber.

$\therefore$ For a tuned absorber,

$$\frac{A}{A_{st}} = \frac{1-\frac{\omega^2}{\omega_2^2}}{\frac{\omega^4}{\omega_2^4} - (2+\mu)\frac{\omega^2}{\omega_2^2} + 1} \quad \text{----- 6.6.20}$$

$$\frac{B}{A_{st}} = \frac{1}{\frac{\omega^4}{\omega_2^4} - (2+\mu)\frac{\omega^2}{\omega_2^2} + 1} \quad \text{----- 6.6.21}$$

The amplitude ratio $\frac{A}{A_{st}}$ will be infinite if the denominator of the above equations are zero

i.e., $\frac{\omega^4}{\omega_2^4} - (2+\mu)\frac{\omega^2}{\omega_2^2} + 1 = 0$

$$\therefore \left(\frac{\omega}{\omega_2}\right)^2 = \left(1+\frac{\mu}{2}\right) \pm \frac{1}{2}\sqrt{(2+\mu)^2 - 4}$$

$$\therefore \left(\frac{\omega}{\omega_2}\right)^2 = \left(1+\frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}} \qquad \text{------ 6.6.22}$$

Using the equation 6.6.22 the two resonant frequencies can be calculated. Also on the basis of equation 6.6.22 a plot has been shown in Fig. 6.76. It shows the effect of mass ratio on the natural frequencies of the system. For each mass ratio there are two natural frequencies which are above and below the natural frequency of the main system. For smaller value of mass ratio $\frac{m_2}{m_1}$, the two values of frequency are found closer to unity i.e., $\omega = \omega_2$.

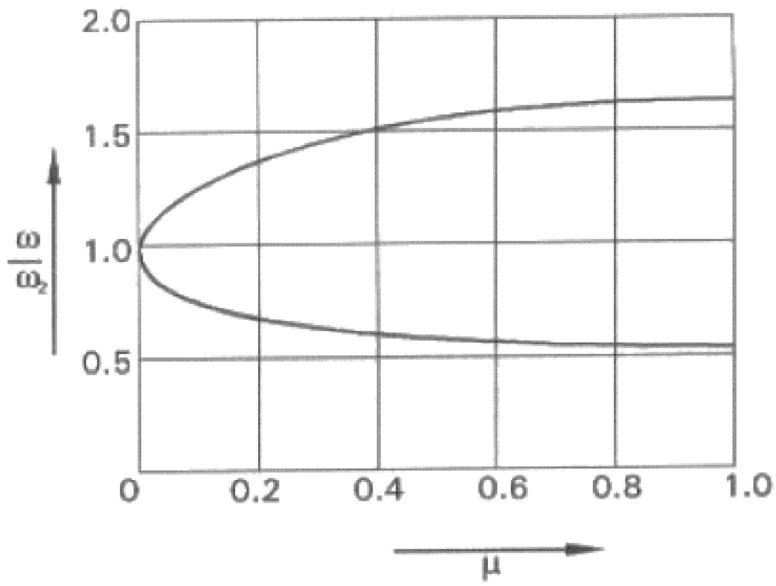
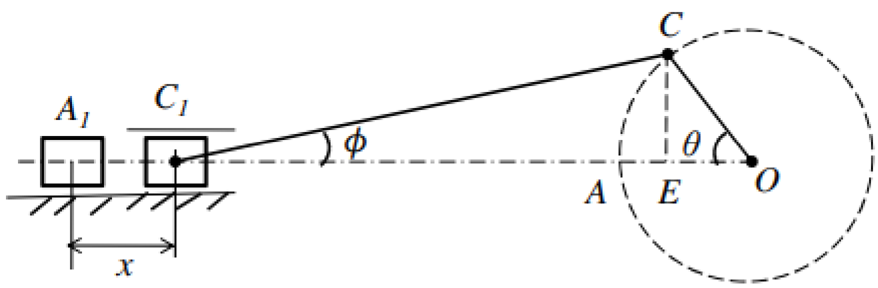


Fig. 6.76

Dynamics of reciprocating engines

- Effect of static and kinetic forces of reciprocating parts
- Static forces arise due to weight of reciprocating parts as well as due to variation of fluid pressure on account of expansion or compression. (I.C engines)
- Due to reciprocating or to and fro motion, each member is subjected to varying acceleration at its different positions. This leads to varying kinetic forces from instant to instant.
- The above variation of fluid pressure and kinetic forces for every position of crank leads to non-uniform development of torque and work. This necessitates use of flywheel in reciprocating engines to limit fluctuation of speed.



Velocity and acceleration of piston:

- x=displacement of piston from inner dead centre
- r=radius of crank
- l= length of connecting rod
- $n=l/r$

$$x = r\left[(1-\cos\theta) + \left(n - \sqrt{n^2 - \sin^2\theta}\right)\right]$$

If the connecting rod were infinitely long, then

$n - \sqrt{n^2 - \sin^2\theta}$ will become zero.

$x = r(1-\cos\theta)$ a S.H.M

Velocity of piston :

$$v = \frac{dx}{dt} = \omega r\left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right)$$

$$= \omega r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) \text{ for } n = 4 \text{ or } 5 \text{ as } \sin^2\theta \le 1$$

if n is infinitely large,

$v = \omega r \sin\theta$ as in S.H.M

Acceleration of piston

$$f = \frac{dv}{dt} = \omega^2 r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$at\ \theta = 0^o$

$$f = \omega^2 r\left(1 + \frac{1}{n}\right)$$

$at\ \theta = 180^o$

$$f = \omega^2 r(-1 + \frac{1}{n})$$

Part – B Questions

1. Determine an expression for the general solution for lateral vibration of string.

2. Derive one dimensional wave equation for lateral vibration of a string.

3. Derive one dimensional wave equation for critical vibration of a string.

4. Derive the general solution for vibration a string

5. Determine an expression for the free longitudinal vibration of a uniform bar of length l, one end of which is fixed and the other end is free.

6. Derive the governing differential equation for transverse vibration of a beam.

7. Derive an expression for torsional vibration of a uniform shaft.

8. Derive the frequency equation of longitudinal vibrations for a free-free beam with zero initial displacement.

9. Derive an expression for the free longitudinal vibration of a uniform bar of length “L’ which is free-free.

10. Find the frequency and normal modes of transverse vibration of a simply supported beam of length L.

11. A bar of length l fixed at one end is pulled at the other end with a force P. The force is suddenly released. Investigate the vibration of the bar.

12. Derive 1D wave equation for torsional vibrations of a uniform shaft