GCEM

Gopalan College of Engineering and Management

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NOTES

SUBJECT: BASIC SIGNAL PROCESSING

SUBJECT CODE: 21EC33

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Go Green

Save Mother Earth.

Module - 3

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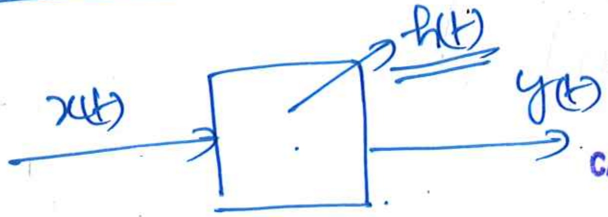
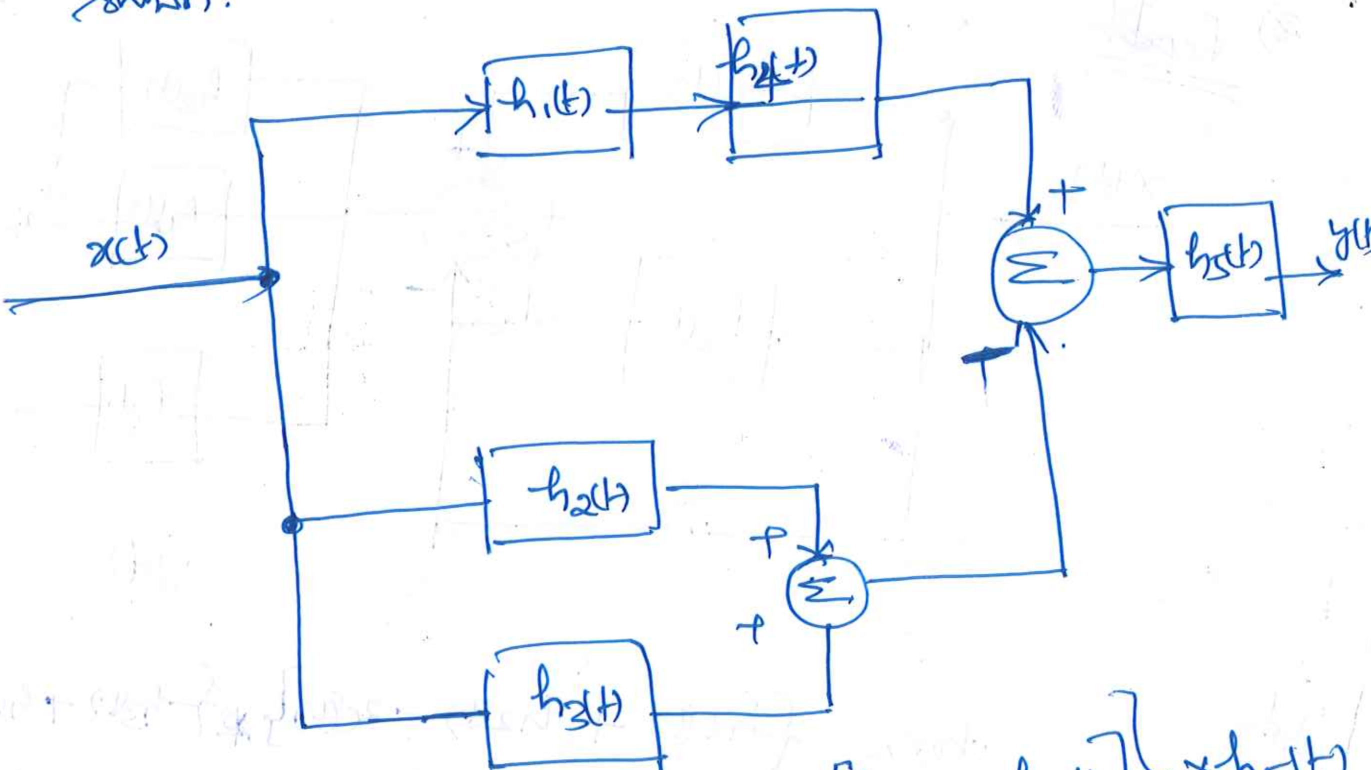
LTI System properties in terms of Impulse response

System Interconnection

1) ~~Find~~



1) Find the overall impulse response h(t) in terms of the Impulse response of each subsystems for the system shown.



Soln $h(t) = \left\{ h_1(t) * h_4(t) - \left[h_2(t) + h_3(t) \right] \right\} * h_5(t)$

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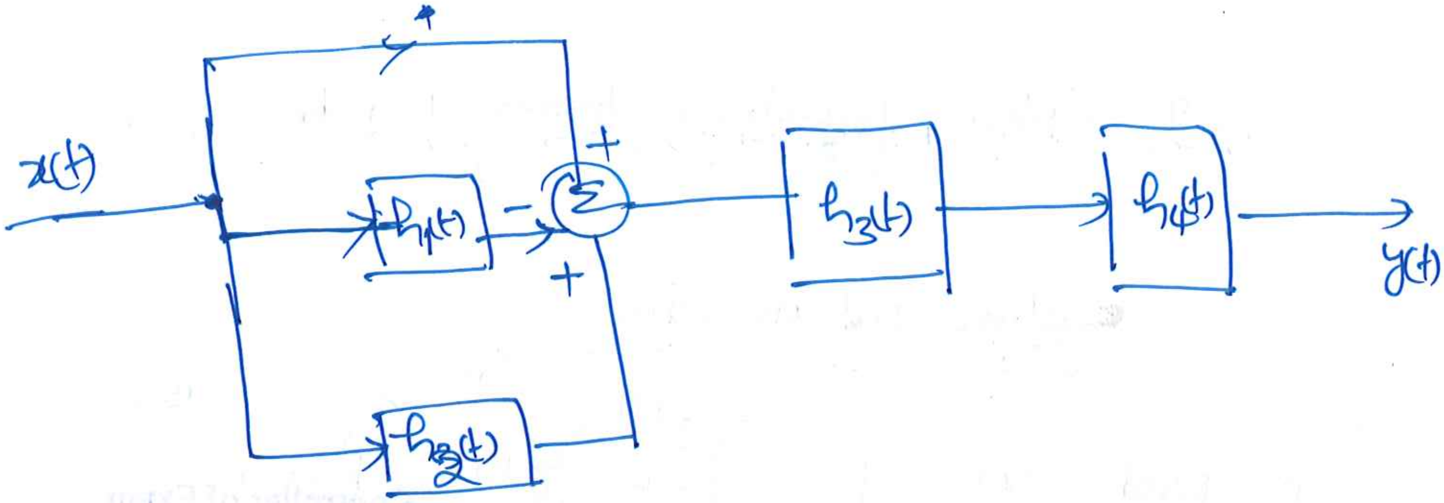
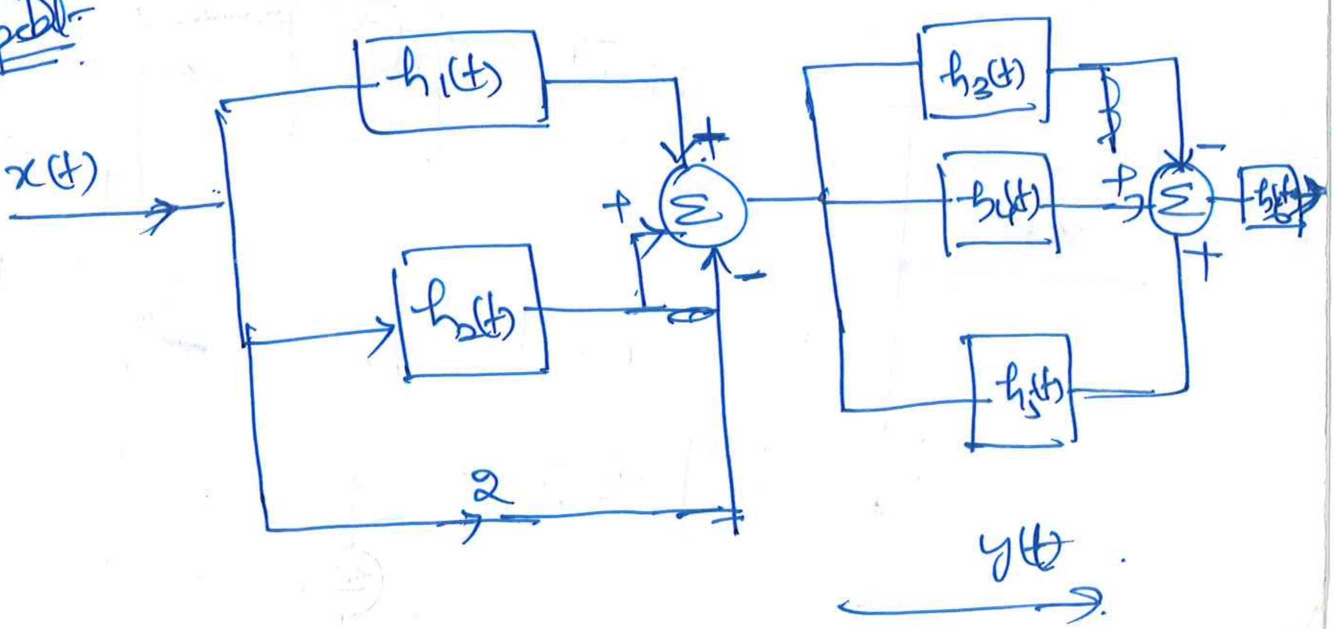
2) Find overall impulse response $h(t)$ in terms of impulse response of each subsystem for the system shown.



Soln $h(t) = \{\delta(t) - h_1(t) + h_2(t)\} * h_3(t) * h_4(t)$.

3) Repeat.

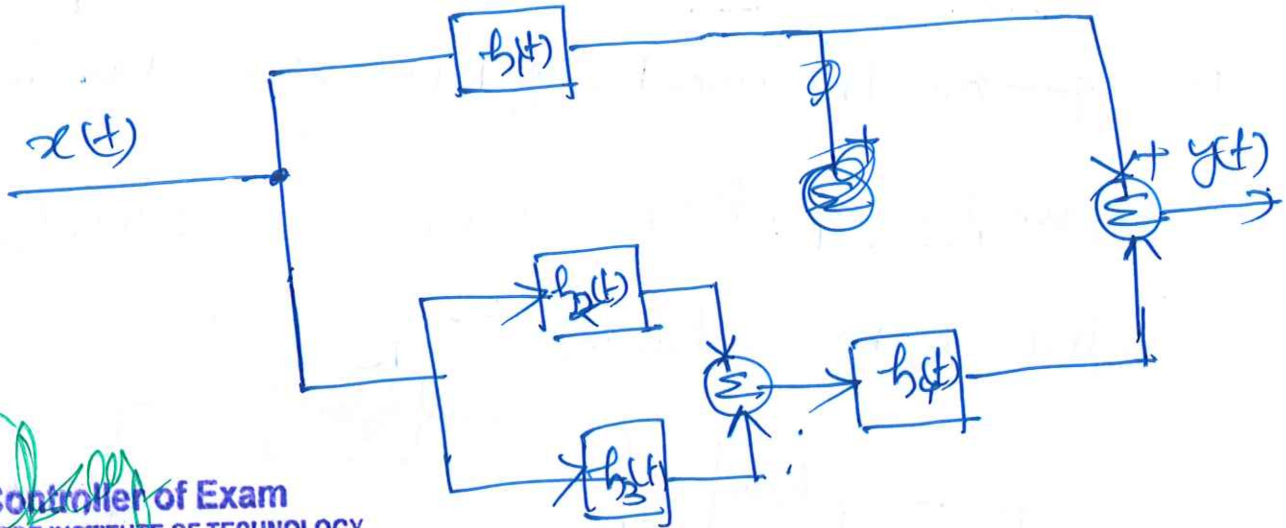


Soln → $h(t) = \{h_1(t) + h_2(t) - 2\delta(t)\} * \{-h_3(t) + h_4(t) + h_5(t)\} * h_6(t)$.

(4) Draw the Interconnection of the systems required to obtain the overall impulse response.

$$h(t) = h_1(t) + \{h_2(t) + h_3(t)\} * h_4(t)$$

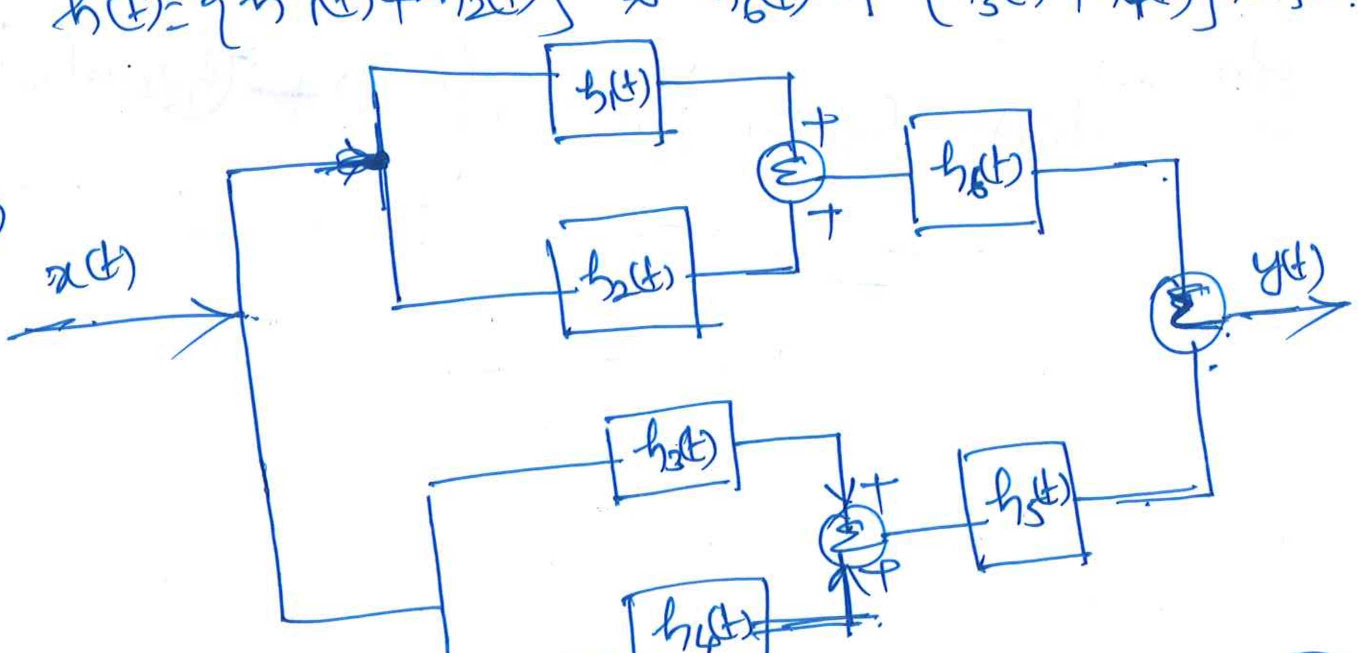
Soln



(5). Draw the Interconnection of the systems required to obtain the overall impulse response.

$$h(t) = \{h_1(t) + h_2(t)\} * h_6(t) + \{h_3(t) + h_4(t)\} * h_5(t).$$

Soln



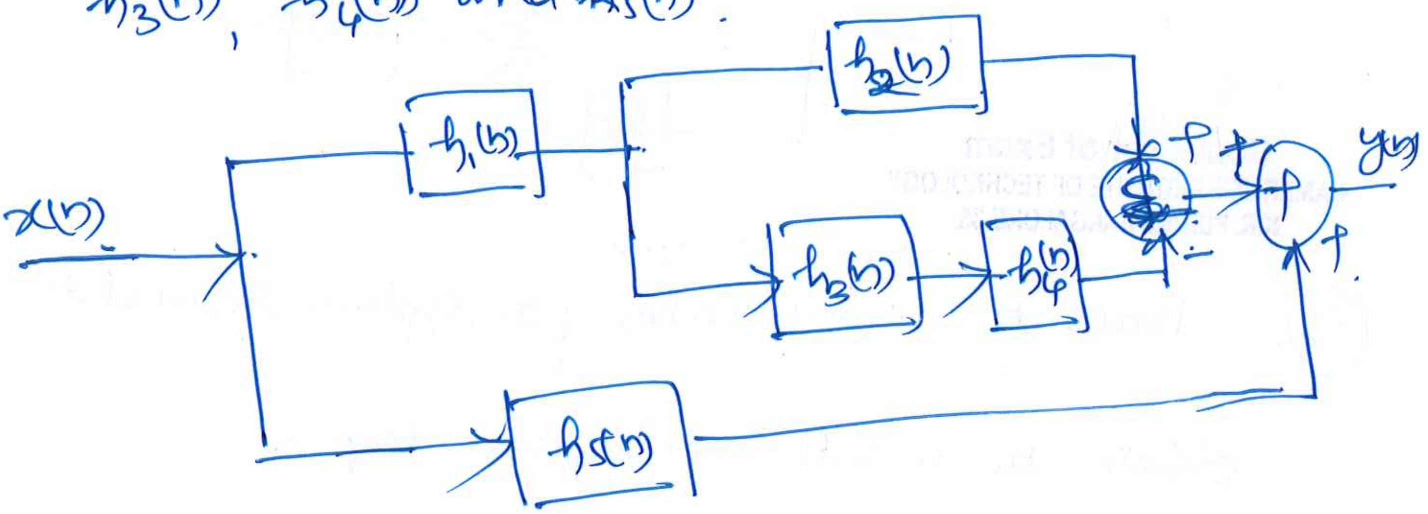
6). Repeat

$x(t)$

$$h(t) = \{h_1(t) + \delta(t) + h_2(t)\} * h_3(t) + \{h_4(t) + h_5(t) - 2\delta(t)\} * h_6(t)\} + \{h_7(t) + h_8(t)\}$$

DTS [Interconnecting systems].

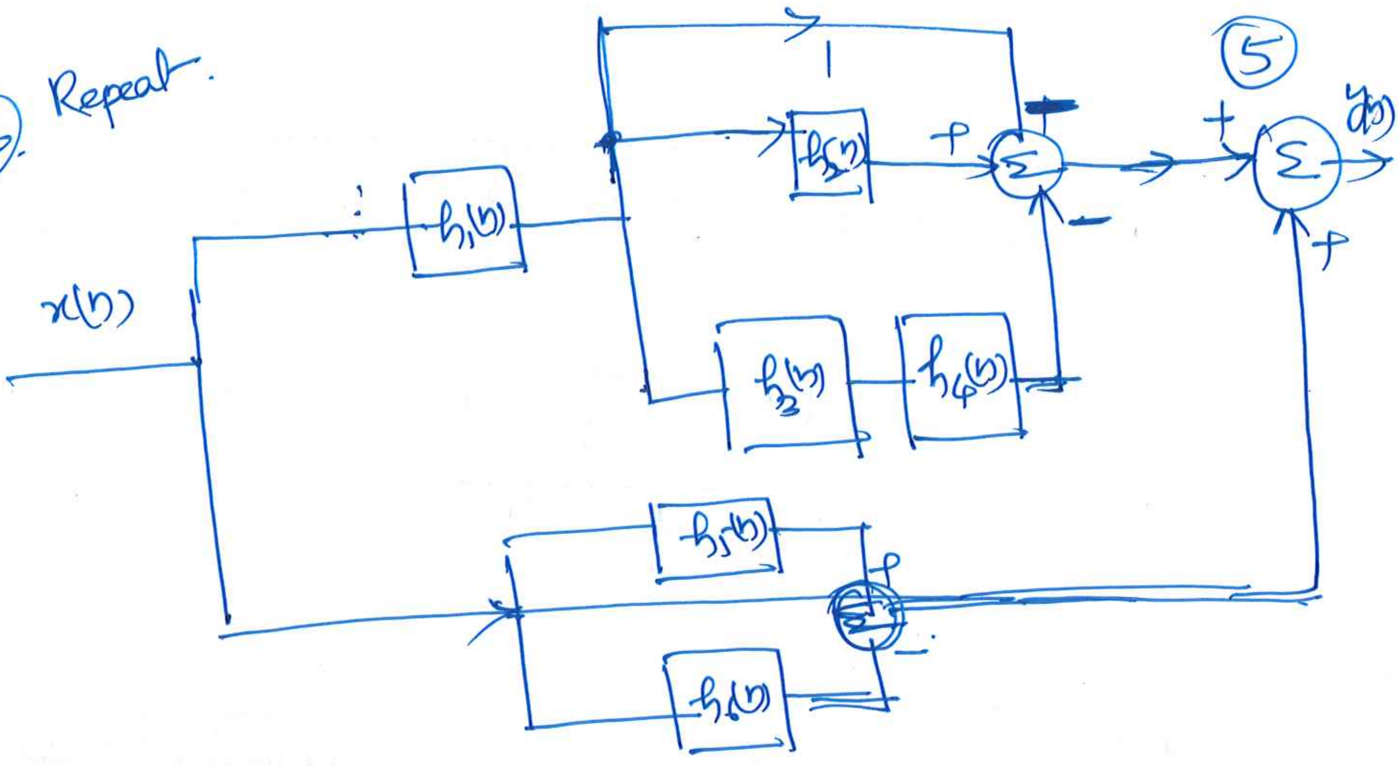
7). For the Interconnections of LTI Systems shown below. Express the overall IR $h(n)$ in terms of $h_1(n)$, $h_2(n)$, $h_3(n)$, $h_4(n)$ and $h_5(n)$.



soln

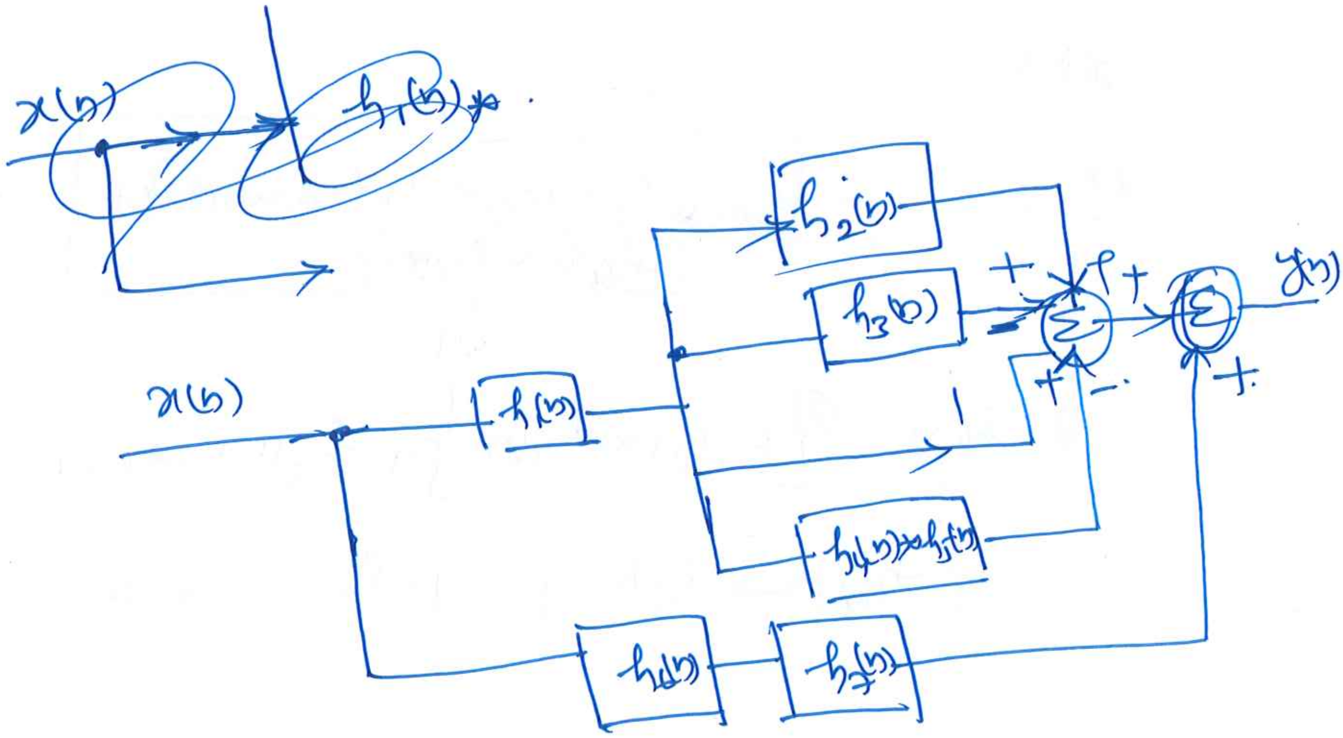
$$h(n) = h_5(n) + \{h_1(n) * [h_2(n) - (h_3(n) * h_4(n))]\}.$$

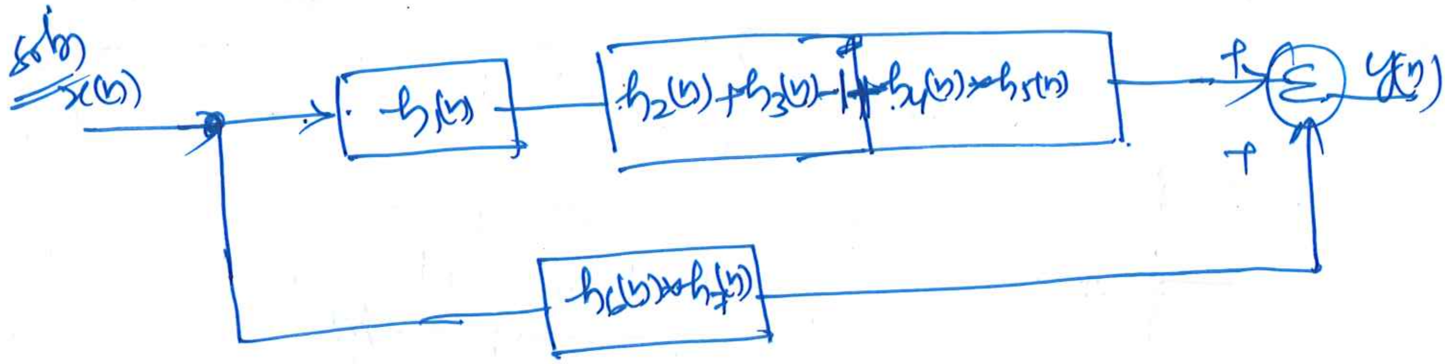
6. Repeat.

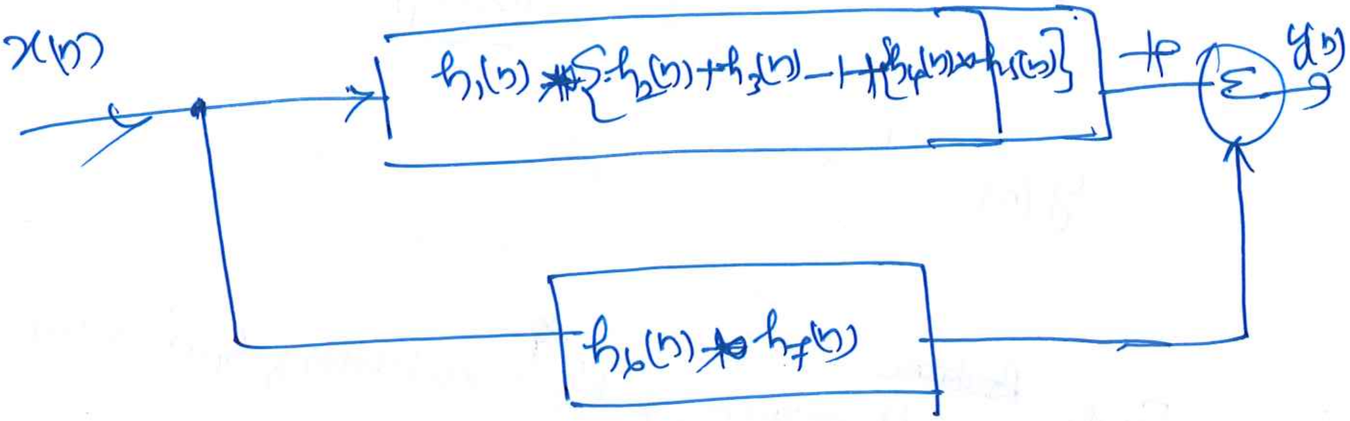


$y(n) = \ ?$

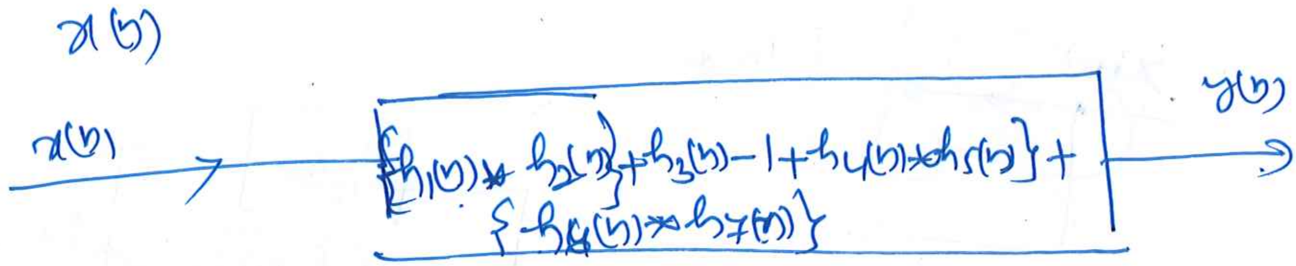
7. Find ~~Reduce~~ overall response of $h(n)$ in terms of $h_1(n)$ $h_2(n)$, etc.







∴

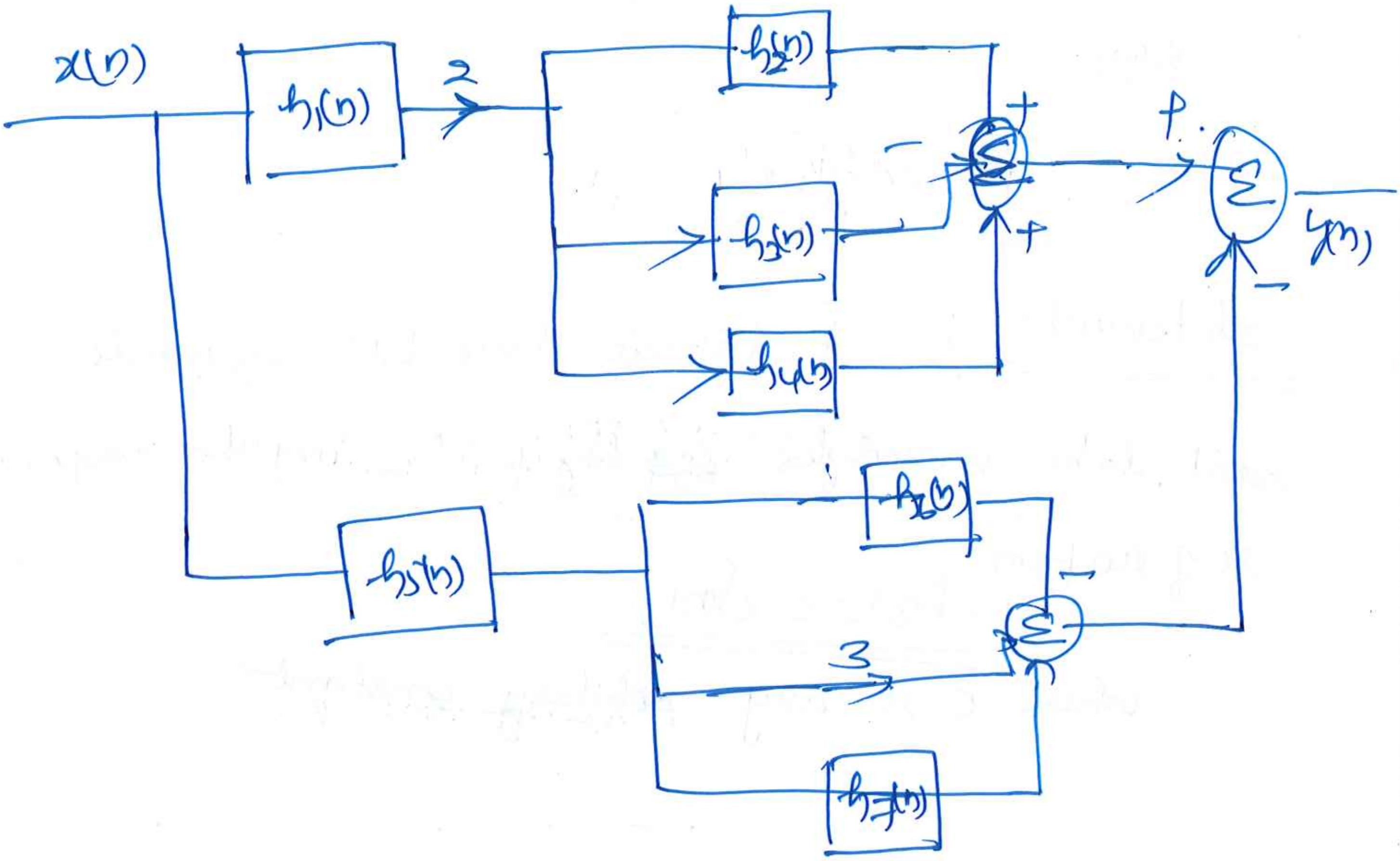


$$h(n) = \{h_1(n) * h_2(n)\} + h_3(n) - 1 + \{h_4(n) * h_5(n)\} + h_6(n) * h_7(n)$$

(8). Repeat

~~Draw~~ Reduce the Interconnection ~~&~~ & overall response of the h(n) (7)

~~x(n)~~



Soln

(9). Draw the Interconnected system for the the Express:

$$h(n) = h_1(n) + \{h_2(n) * h_3(n)\}3 + \{h_4(n) * h_5(n) * h_6(n)\}5 - h_7(n) * h_8(n)$$

System properties in terms of Impulse response $h(n)$

* State & prove memoryless, causal, stable etc in terms of $h(n)$

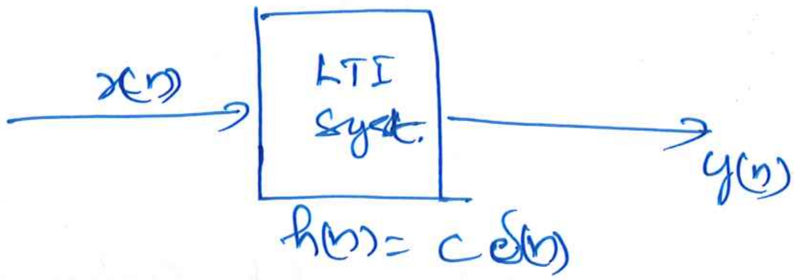
Memoryless Systems

Statement: A Discrete time LTI system is said to be memoryless, iff only if its impulse response is of the form.

$$h(n) = c\,\delta(n).$$

where c is any arbitary constant

proof



* Consider LTI system represented by its impulse response $h(n)$, WKT the response of LTI system is obtained by performing convolution of i/p sequence $x(n)$ with impulse $h(n)$ i.e

$$y(n) = x(n) * h(n)$$

if $h(n) = c\delta(n)$

then $y(n) = x(n) * c\delta(n)$

$$\left[\begin{array}{l} \because \text{WKT} \\ x(n) * \delta(n) = x(n) \end{array}\right]$$

$y(n) = c\,[x(n) * \delta(n)]$

$$\boxed{y(n) = c\,x(n)}$$

The o/p at n^{th} value of $y(n)$ depends only on n^{th} value of i/p.

Hence it is Memoryless system

2) Causality

* A LTI system is said to be causal, iff only if its impulse response $h(n)$ is zero for $n<0$.

Proof.

* O/p response of an LTI system with an arbitrary i/p $x(n)$ & impulse response $h(n)$ is given by

$y(n) = x(n) * h(n)$

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \longrightarrow (1)$$

* If $h(n) = 0$ for $n < 0$ then $h(n-k) = 0$ for $n-k < 0$

which is eqt to $h(n-k) = 0$, means $n-k < 0$

ie $n < k$

(or) $k > n$.

* Using this fact, the terms in the infinite summation of eq (1) for $k > n$ are zeros.

∴ we can write

$$y(n) = \sum_{k=-\infty}^{n} x(k)\, h(n-k). \longrightarrow (2)$$

* From the above eqn it is clear that $y(n)$ at any time 'n' is a weighted sum of the values of i/p $x(k)$ for k less than or equal to n.

∴ o/p depends on present & past values of i/p.

∴ Hence the system is causal

Stability

Stat → A LTI system is BIBO stable, if & only if its impulse response h(n) is absolutely summable

i.e. $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$ for BIBO stability

Proof

The o/p response of LTI system which is having x(n) as i/p & h(n) as impulse response is given by

$$y(n) = x(n) * h(n)$$

$$y(n) = h(n) * x(n) \quad \text{(using commutative property)}$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \longrightarrow (1)$$

* Taking magnitude on both sides of the eqn (1) we get

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \right| \longrightarrow (2)$$

* Using Triangular Inequality property, we can say that the magnitude of the sum is less than

or equal to sum of the magnitudes.

ie $$|y(n)| = \left|\sum_{k=-\infty}^{\infty} h(k)\, x(n-k)\right| \leq \sum_{k=-\infty}^{\infty} \left|h(k)\, x(n-k)\right|$$

Illy we can write the magnitude of the product is (also) less than or equal to the product of magnitudes.

ie $$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)| \longrightarrow (3)$$

* since $x(n)$ is bounded, we can write

$$|x(n)| \leq M < \infty.$$

ie $$|x(n-k)| \leq M < \infty.$$

Substitute this into eqn (3), we get

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)|\,.M.$$

* From statement WKT $\sum_{k=-\infty}^{\infty} |h(k)|$ is finite (given).

∴ then $|y(n)|$ for all 'n' gives bounded. o/p eq is given by

$$|y(n)| \leq M_y < \infty.$$

∴ System is BIBO. stable

$||^{ly}$ ~~for~~ a continuous system.

Stat. A LTI system is BIBO stable, iff only if its impulse response h(t) is absolutely Integrable. i.e

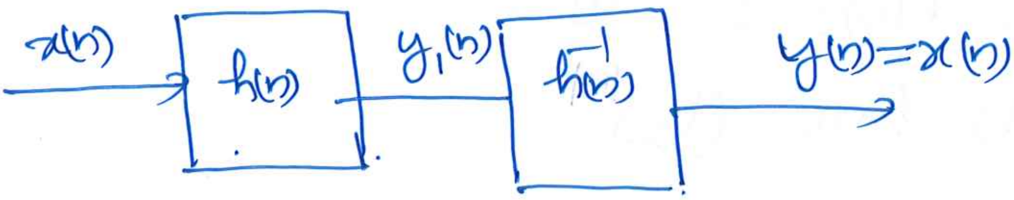
i.e
$$\int_{\tau=-\infty}^{\infty} |h(\tau)|\, d\tau < \infty .$$

Investibility

* A Discrete LTI system with impulse response $h(n)$ is said to be investible, iff only if $h(n) * h^{-1}(n) = \delta(n)$

Here $h^{-1}(n)$ is the impulse response of the Inverse system.

Proof.



cascade of a system & its inverse.

* Let us consider 2 LTI system, Interconnected as shown above. Then overall response of the system is given by

$$y(n) = \underline{y_1(n)} * h^{-1}(n)$$
$$= \left[x(n) * h(n)\right] * h^{-1}(n)$$

Using Associative property, we can write

$$y(n) = x(n) * [h(n) * h^{-1}(n)]$$

So $[h(n) * h^{-1}(n)] = \delta(n)$ (given in start) then we get

$$y(n) = x(n) * \delta(n)$$

$$\boxed{\therefore y(n) = x(n)}$$

[$\because$ WKT. $x(n) * \delta(n) = \delta(n) * x(n) = x(n)$]

$\therefore$ we can recover i/p from o/p.. Hence it is Invertible system..

Problems

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1) check the following for causality & stability.

a) $h(n) = 2^n u(n-1)$.

b) $h(n) = (1/2)^{|n|}$.

soln a) $h(n) = 2^n u(n-1)$.

System is causal $\because h(n) = 0$ for $n < 0$.

Ex $n = -1$

$$h(-1) = 2^{-1} u(-1-1)$$

$$\boxed{h(-1) = \frac{1}{2} u(-2) = 1/2\,(0) = 0}$$

$\therefore h(n) = 0$ for $n < 0$.

2) let $S = \sum_{n=-\infty}^{\infty} |h(n)|$.

$$S = \sum_{n=1}^{\infty} 2^n \cancel{u(n-1)}^{1}$$

$$S = \sum_{n=1}^{\infty} 2^n (1) = 2+4+8+16+ \cdots \infty.$$

$$S = \infty.$$

Hence it is Unstable system.

2) Repeat. $h(n) = 3^n u(n-1)$.

3) Check $h(n) = (0.5)^{|n|}$.

soln

a). causality $h(n) = (0.5)^{-n} u(-n-1) + 0.5^{n} \cdot u(n)$

$h(n) \neq 0$, for $n<0$. is the condn.

when $n<0$, $n=-1$.

$$h(-1) = (0.5)^{+1} u(1-1) + (0.5)^{-1} u(-1).$$

$$h(-1) = 0.5 \cdot u(0) + (0.5)^{-1} \cancel{(0)}^{0}.$$

$$h(-1) = 0.5 \cdot (1) = 0.5.$$

∴ it is Noncausal.

2) stability

let $S = \sum_{n=-\infty}^{\infty} h(n) \rightarrow (1)$

$= \sum_{n=-\infty}^{-1} (0.5)^{-n} + \sum_{n=0}^{\infty} (0.5)^{n}$

$S = \underline{\underline{\sum_{n=-\infty}^{-1} (0.5)^{-n}}} + \frac{1}{1-0.5}$ $\left[\because \sum_{m=0}^{\infty} \alpha^m = \frac{1}{1-\alpha} \right]$

let $n = -m$

$\left[\therefore -n = m \quad \text{||ly} \quad n = -1 \right.$
$m = -(-1) = 1$
when $n = -\infty$
$\left. m = -(-\infty) = \infty \quad \therefore m = 1 \right]$

$\therefore S = \sum_{m=\infty}^{1} (0.5)^{m} + \frac{1}{1-0.5}$

$\underline{\underline{S = \sum_{m=1}^{\infty} (0.5)^{m} + \frac{1}{0.5}}}$

$\underline{\underline{S = -1 + \sum_{m=0}^{\infty} (0.5)^{m} + 2}}$

$\left[\because \text{WKT} \sum_{m=1}^{\infty} (0.5)^{m} = 0.5 + 0.5^2 + 0.5^3 + \cdots \right.$

$\sum_{m=0}^{\infty} (0.5)^{m} = 1 + 0.5^1 + 0.5^2 + 0.5^3 + \cdots$

$\left. \therefore -1 + \sum_{m=0}^{\infty} (0.5)^{m} = \sum_{m=1}^{\infty} (0.5)^{m} \right.$

$$\therefore \; S = -1 + \sum_{m=0}^{\infty} (0.5)^{m} + 2.$$

$$S = -1 + \frac{1}{1-0.5} + 2$$

$$\left[\sum_{m=0}^{\infty} \alpha^{m} = \frac{1}{1-\alpha} \right]$$

$$\therefore S = -1 + \frac{1}{0.5} + 2.$$

$$\therefore S = -1 + 2 + 2 = 3$$

$\boxed{\therefore S = 3 < \infty.}$ $\therefore$ it is stable.

(4) Repeat $h(n) = (0.4)^{|n|}$.

(5). Check $h(n) = \alpha^{n} u(n)$ as causal & stable.

Soln Causal. $h(n) = 0$ for $n < 0$.

$n = -1$

$$h(-1) = \alpha^{-1} \cancel{u(-1)}^{0} = 0.$$

$\therefore$ it is causal.

Stable. let $S = \sum_{n=0}^{\infty} \alpha^{n} u(n) = \frac{1}{1-\alpha}.$

it is stable provided $|\alpha| < 1$

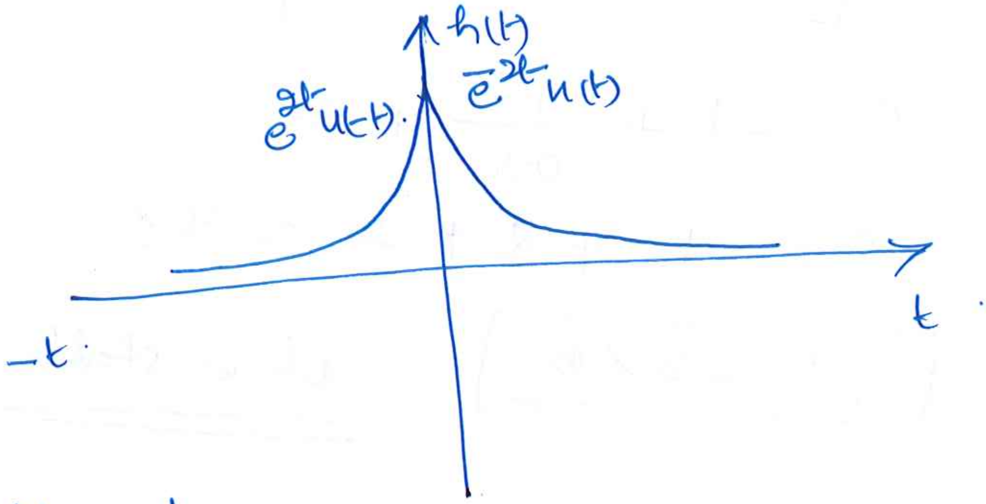
Repeat $h(n) = \alpha^{n} 2 . u(n)$

(9). check the following system for

(a) Memoryless. (b) causal (c) stable.

$h(t) = e^{-2|t|}$.

Soln



* Memoryless

system is not memoryless $\because$ h(t) is not of the form.

$h(t) = c\,\delta(t)$..

* Causality

$h(t) \neq 0$ for $t < 0$, $\therefore$ system is Noncausal.

i.e

wkt $h(t) = e^{-2(-t)} u(-t-1) + e^{-2(t)} u(t)$

ie $h(t) = e^{2t} u(-t-1) + e^{-2t} u(t)$

$t = -1$

$h(-1) = e^{2(-1)} u(+1-1) + e^{+2} \cancel{u(-1)}^{\,0}$

$$h(-1) = e^{-2}\, u(0) + 0.$$

$$\therefore\ h(-1) = \frac{1}{e^2}(1) = \frac{1}{e^2} \neq 0.$$

$\therefore$ Noncausal system.

2) Stability.

$$S = \int_{\tau=-\infty}^{\infty} |h(\tau)|\, d\tau.$$

$$S = \int_{-\infty}^{0} e^{2\tau}(1)\, d\tau + \int_{0}^{\infty} e^{-2\tau}(1)\, d\tau.$$

$$S = \left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{0} + \left[\frac{e^{-2\tau}}{-2}\right]_{0}^{\infty}$$

$$S = \frac{e^{0} - e^{-\infty}}{2} + \frac{e^{-\infty} - 1}{-2}$$

$$S = \frac{1 - \frac{1}{e^{\infty}}}{2} + \frac{\frac{1}{e^{\infty}} - 1}{-2}$$

$$S = \frac{1 - \frac{1}{\infty}}{2} + \frac{\frac{1}{\infty} - 1}{-2}$$

$\therefore$ Stable system

$$\therefore\ S = \frac{1-0}{2} - \frac{0-1}{2} = \frac{1}{2} + \frac{1}{2} = \boxed{1 < \infty}$$

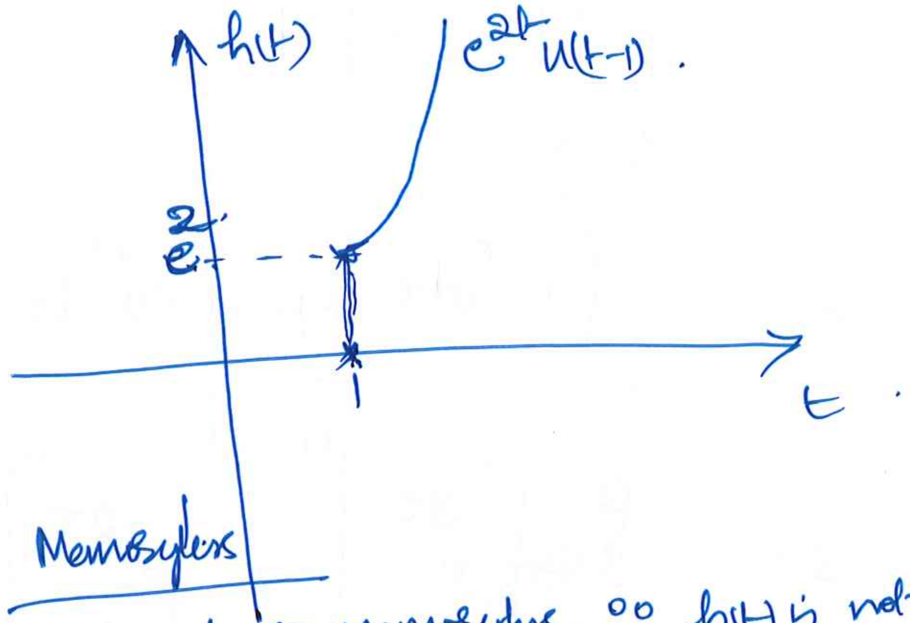
Repeat ~~for~~ 8) $h(t) = e^{-3|(t)|}$

9) $h(t) = e^{-4|t|}$ $\longrightarrow$ Ans: $S = 1/4 + 1/4 = 1/2 < \infty$

stable

10) $h(t) = e^{2t} u(t-1)$

Soln.



1) Memoryless

System is not memoryless. ∵ $h(t)$ is not of the form $h(t) = c\,\delta(t)$

2) Causality

System is causal ∵ $h(t) = 0$ for $t < 0$.

ie $t = -1$

$h(-1) = e^{2(-1)} u(-1-1)$

$= e^{-2} u(-2)$

$\boxed{h(-1) = e^{-2}(0) = 0}$

∴ causal system.

(3) Stability

$$S = \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

$$S = \cancel{\int_{-\infty}^{1} e^{2\tau} u(\tau-1)\, d\tau} + \int_{1}^{\infty} e^{2\tau} u(\tau-1)\, d\tau$$

$$S = \cancel{\int_{-\infty}^{1} e^{2\tau}(0)\, d\tau} + \int_{1}^{\infty} e^{2\tau}(1)\, d\tau$$

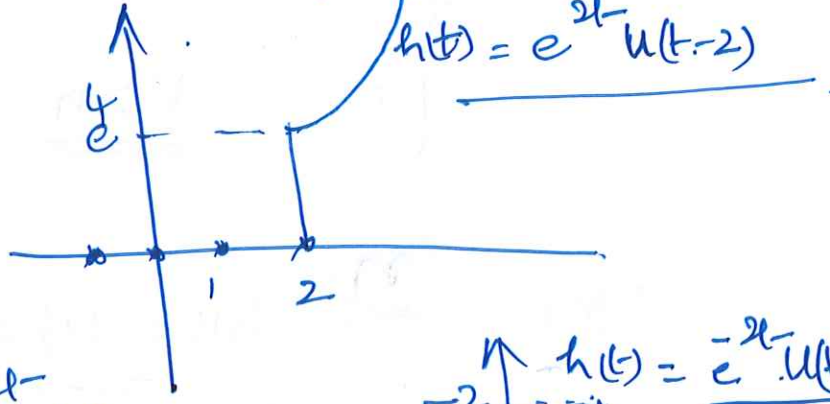
$$S = 0 + \int_{1}^{\infty} e^{2\tau}\, d\tau$$

$$S = \left[\frac{e^{2\tau}}{2}\right]_{1}^{\infty} = \frac{e^{2\infty} - e^{2}}{2} = \frac{\infty - 1}{2} = \infty$$

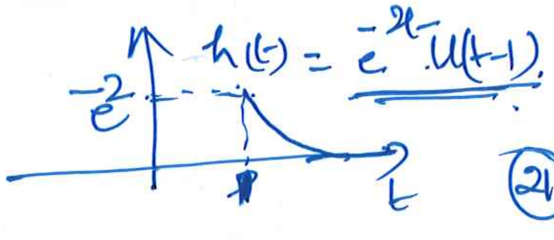
$$\boxed{S = \infty}$$

∴ System is Unstable.

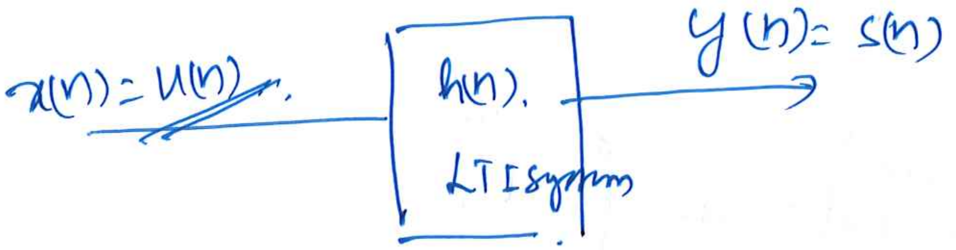
(11) Repeat $h(t) = e^{2t} u(t-2)$.



(12) $h(t) = e^{-2t} u(t-1)$



Step response of a LTI system.



* Consider an LTI system characterized by its I R $h(n)$ & is shown in fig.
* then Step response of the system is given by

$$s(n) = u(n) * h(n)$$

$$s(n) = h(n) * u(n) \qquad \left[\text{using commutative property}\right]$$

$$s(n) = \sum_{k=-\infty}^{\infty} h(k)\, u(n-k)$$

From the defn of the step fn, we can write.

$$u(n-k) = \begin{cases} 1 & (n-k) \geq 0, \text{ i.e } n \geq k, \text{ or } k < n \\ 0 & k > n \end{cases}$$

$$\therefore\ s(n) = \sum_{k=-\infty}^{n} h(k)\, \cancel{u(n-k)} \rightarrow 1.$$

$$\therefore\ \boxed{s(n) = \sum_{k=-\infty}^{n} h(k)} \rightarrow (1)$$

∴ Step response $s(n)$ in case of DTS is nothing but running sum of Impulse response.

$\|^{ly}$ the step response $s(t)$ in case of CTS is given by running Integral of the impulse response $h(t)$

ie
$$\boxed{s(t) = \int_{\tau=-\infty}^{t} h(\tau)\, d\tau}$$

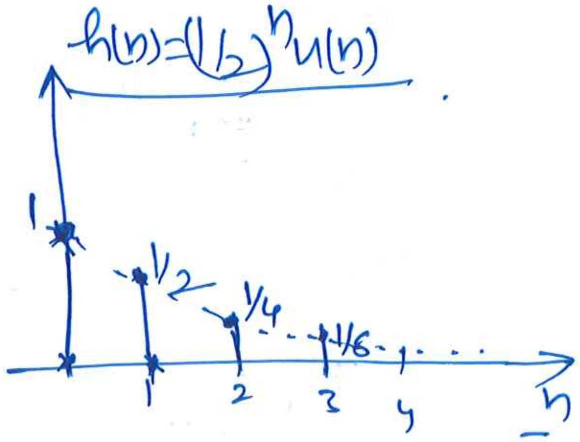
Problems

1) Evaluate the step response of an LTI system represented by the Impulse response

$$h(n) = (1/2)^n u(n)$$

Soln
$$s(n) = \sum_{k=-\infty}^{\infty} h(k) \longrightarrow (1)$$

given $h(n) =$



For $n<0$, $s(n)=0$.

$n\geq 0$, we can write.

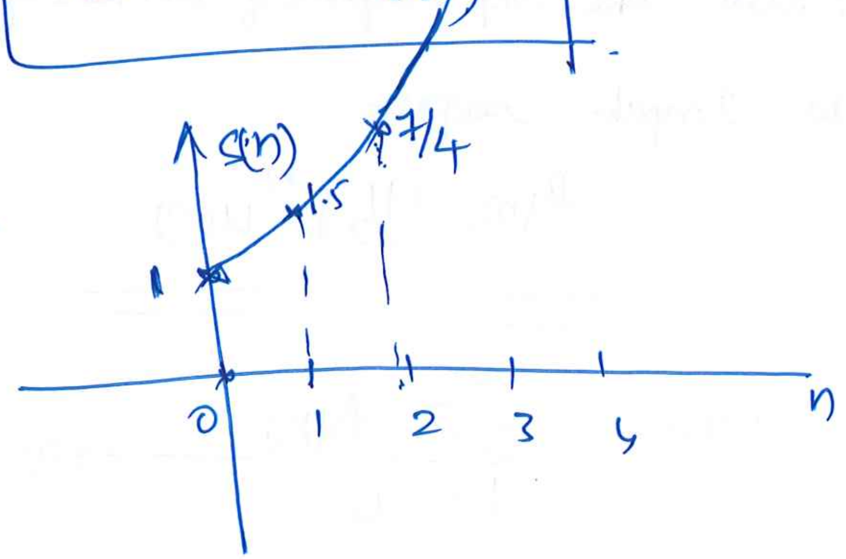
$$s(n)=\sum_{k=0}^{n} h(k) = \sum_{k=0}^{n} (1/2)^k$$

$$\therefore s(n)=\frac{1-(1/2)^{n+1}}{1-1/2} = \frac{1-(1/2)^n\cdot(1/2)}{1/2}$$

$$\therefore s(n)= 2\left[1-(1/2)^n\cdot(1/2)\right]$$

$$\therefore \boxed{s(n)= 2-(1/2)^n}$$

$\therefore$



$n=0$

$s(0)= 2-(1/2)^0 = 2-1=1$

$n=1$

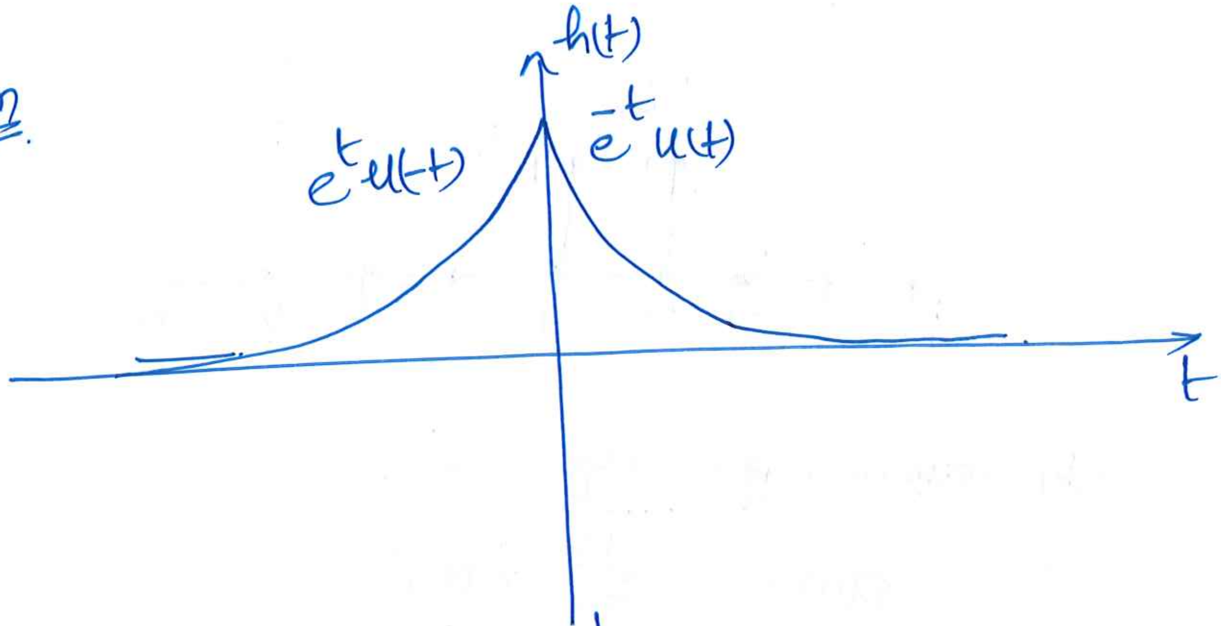
$s(1)= 2-1/2 = 3/2$.

Etc.

(2) Repeat $h(n)=(1/3)^n u(n)$

(5) Find the step response of an LTI system whose IR is given by $h(t) = e^{-|t|}$ (26)

soln.



* step response is given by

$$s(t) = \int_{\tau=-\infty}^{t} h(\tau)\, d\tau.$$

GF is given by

$$s(t) = \int_{\tau=-\infty}^{t} e^{-|\tau|}\, d\tau.$$

$$s(t) = \int_{-\infty}^{0} e^{\tau\,\cancel{u(-\tau)}^{1}}(1)\, d\tau + \int_{0}^{t} e^{-\tau\,\cancel{u(\tau)}^{1}}(1)\, d\tau.$$

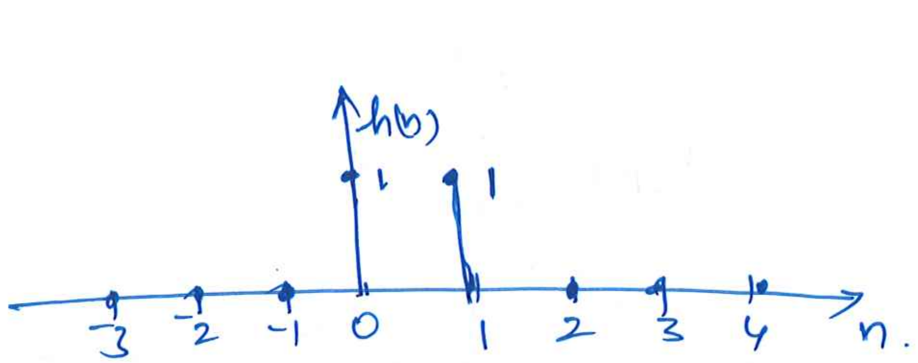
$$s(t) = \left[\frac{e^{\tau}}{1}\right]_{-\infty}^{0} + \left[\frac{e^{-\tau}}{-1}\right]_{0}^{t}$$

$$s(t) = \left[e^{0} - e^{-\infty}\right] + -e^{-t} + e^{0} = 1 - \cancel{e^{-\infty}} - e^{-t} + 1$$

$$= 2 - \frac{1}{\infty} + e^{-t}.$$

(3). Find the step response of the system, whose IR is given by $h(n) = \delta(n) + \delta(n-1)$.

Soln



Step response is given by

$$s(n) = \sum_{k=-\infty}^{n} h(k).$$

For $n<0$, $h(n)=0$

$n=0$, $s(0)=1$

for $n \geq 1$ $\quad s(n) = 1+1+0+0 = \underline{\underline{2}}$

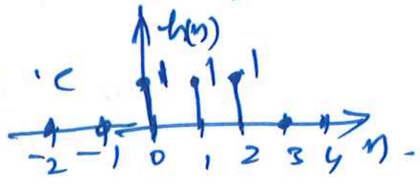
ie

$$s(n) = \sum_{k=1}^{n} h(k) = 1+1+0\cdots = \underline{\underline{2}}$$

$$\boxed{\therefore s(n) = \begin{cases} 0 & ; n<0 \\ 1 & n=0 \\ 2. & n \geq 1. \end{cases}}$$

(4) Repeat $h(n) = \delta(n) + \delta(n-1) + \delta(n-2)$.

or $h(n) = \{1, 1, 1\}$, ie



$s(t) = 1 + \frac{1}{\infty} - e^{-t} + 1$

$$\boxed{s(t) = 2 + 0 - e^{-t} = 2 - e^{-t}}$$

$\therefore$

$$\boxed{s(t) = \begin{cases} e^{t} & t < 0 \\ 2 - e^{-t} & t \geq 0. \end{cases}}$$

$\therefore$ generally for $t<0$, $s(t) = \int_{\tau=-\infty}^{t} e^{\tau} d\tau.$

$t<0$, $s(t) = \left[e^{\tau}\right]_{-\infty}^{t} = e^{t} - e^{-\infty} = e^{t} - 0 = e^{t}$

---

(6) Repeat $h(t) = e^{+|t|}$

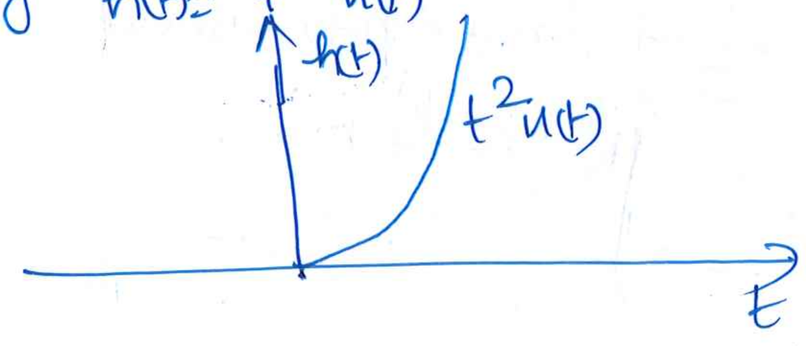
(7) $h(t) = e^{-|2t|}.$

(8) $h(t) = e^{+|2t-1|}.$

(**) 6M

(9) Find the step response of an LTI system whose IR is given by $h(t) = t^2 u(t)$

Soln



General step response is given by

$$s(t) = \int_{\tau=-\infty}^{t} h(\tau)\, d\tau.$$

For $t<0$, $s(t)=0$,

$t \geq 0$, $s(t) = \int_0^t \tau^2 d\tau = \left[\frac{\tau^3}{3}\right]_0^t$

$$\boxed{s(t) = \frac{t^3 - 0}{3} = \frac{t^3}{3}}$$

$$\therefore \boxed{s(t) = \begin{cases} 0 & t<0 \\ \frac{t^3}{3} & t \geq 0 \end{cases}}$$

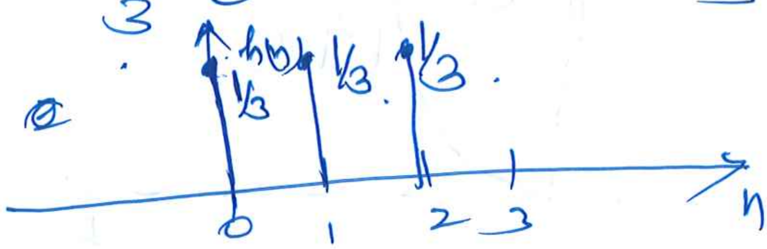
(10) Repeat for $h(t) = t^3 u(t)$

(11) Find the step response $s(n)$ of an LTI system whose IR is given by

$$h(n) = \frac{1}{3}\sum_{k=0}^{2} \delta(n-k).$$

Soln

$$h(n) = \frac{1}{3}\left[\delta n + \delta(n-1) + \delta(n-2)\right]$$



Step response is given by

$$s(n) = \sum_{k=-\infty}^{n} h(k).$$

For $n<0$, $s(n)=0$

$n=0$ $\quad s(0) = 1/3$

$n=1$ $\quad s(1) = 1/3 + 1/3$

$n \geq 2$ $\quad s(2) = 1/3 + 1/3 + 1/3 = \underline{\underline{1}}$

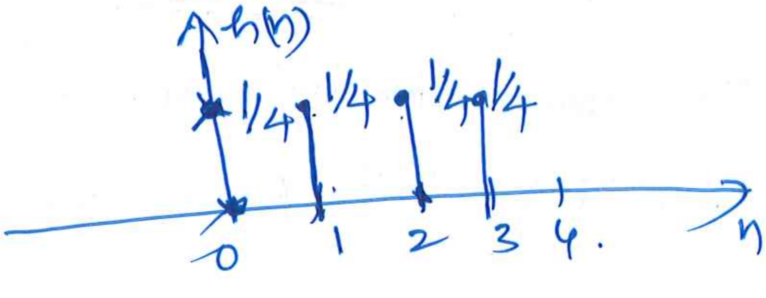
Hence
$$s(n) = \begin{cases} 0 & n<0 \\ 1/3 & n=0 \\ 2/3 & n=1 \\ 1 & n \geq 2 \end{cases}$$

---

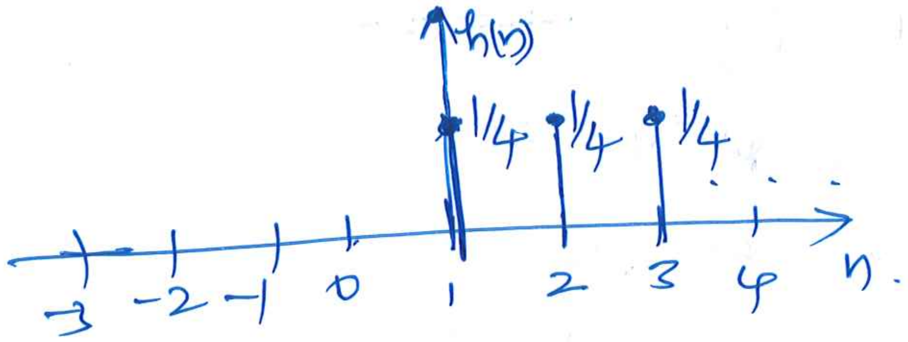
(12) Repeat $h(n) = 1/4 \sum_{k=0}^{3} \delta(n-k)$.

Repeat $h(n) = 1/4 \sum_{k=1}^{3} \delta(n-k)$.

Hint $h(n) = 1/4 \sum_{k=0}^{3} \delta(n-k)$



Ans $h(n) = \frac{1}{4} \sum_{k=1}^{3} \delta(n-k)$.



Fourier Representation of signals

Introduction. The convolution sum & Integral we studied so far provides us a convenient way to find the response of an LTI system if its IR is known.

* Also we know that the LTI system can be completely characterized by its IR.
* In this Topic [Fourier], we will study different alternative representation for signals & systems.
* We represent signals as a weighted superposition of complex sinusoids. If such a signal is applied to an LTI system, then the system o/p obtained also a weighted superposition of complex sinusoidal o/ps.

Conclusion. The representation of signals & systems using complex sinusoids is called Fourier representation of signals.

Fourier representations for signal classes

* Depending on the periodic nature of a signal, there are 4 distinct Fourier representation.

* Periodic signals have Fourier series representations [Module 3]

* whereas Nonperiodic signals have Fourier Transform representations. [Module-4]

ie

1) Continuous time periodic signals are represented as Fourier Series [FS]

2) Discrete Time periodic signals are represented as DTFS.

3) Non periodic continuous time signals are represented as Fourier Transforms (FT)

4) Non periodic Discrete time signals are represented by [DTFT]

Discrete time Fourier Series. [DTFS]

A periodic discrete time signal can be mathematically expressed by using complex sinusoids. given by

$$x(n) = \sum_{K=\langle N\rangle} X(K).e^{jK\Omega_0 n} \longrightarrow (1)$$

[where $e^{j\omega t} = \cos\omega t + j\sin\omega t$ ↓ complex sinusoids.]

$$X(K) = \frac{1}{N} \sum_{n=\langle N\rangle} x(n) e^{-jK\Omega_0 n} \longrightarrow (2).$$

where $x(n)$ is original discrete time signal. & it has fundamental ~~frequency~~ period 'N'. &

* fundamental freq $\Omega_0 = \frac{2\pi}{N}$. [radians]

$X(K) \longrightarrow$ is known as Discrete time Fourier Series co-efficients of $x(n)$. The co-efficients specifies a decomposition of $x(n)$ into a sum of 'N' harmonically related complex exponentials.

Eq(1) is known as Std synthesis eqn.

eq 2 is ~~known~~ known as Analysis eqn.

* Always $x(n)$ & $X(K)$ forms a DTFS pair & is denoted by

$$x(n) \xleftrightarrow{DTFS:\ \Omega_0} X(K)$$

where $x(n)$ is time domain representation of a signal & $X(K)$ is freq domain representation of a signal.

$|X(K)|$ is known as magnitude spectrum of $x(n)$

$\angle X(k)$ is known as phase spectrum of $x(n)$

* $|X(k)|$ is given by

$$\boxed{|X(k)| = \sqrt{A^2(k) + B^2(k)}}$$

phase is given by

$$\boxed{\angle X(k) = \tan^{-1}\left\{\frac{B(k)}{A(k)}\right\}}$$

Properties of DTFS.

helps in analysis signals better in freq domain. & solving problems.
No proofs required now, we need it in 5^{th} sem DSP.

1) Linearity property
2) Time shift.
3) ~~scaling~~ Freq shift
4) scaling
5) convolution.
6) Modulation.
7) parseval's Theorem
8) Duality 9) symmetry.

Linearity property.

Stmt: if $x(n) \xleftrightarrow{DTFS:\ \Omega_0} X(K)$ $\quad \Omega_0 = \frac{2\pi}{N}$.

& $y(n) \xleftrightarrow{DTFS:\ \Omega_0} Y(K)$ then PT.

$$z(n) = a x(n) + b y(n) \xleftrightarrow{DTFS:\ \Omega_0} Z(K) = a X(K) + b Y(K)$$

Proof :- WKT $\quad X(K) = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jK\Omega_0 n}$.

$$Y(K) = \frac{1}{N} \sum_{n=\langle N \rangle} y(n)\, e^{-jK\Omega_0 n}.$$

$$Z(K) = \frac{1}{N} \sum_{n=\langle N \rangle} z(n)\, e^{-jK\Omega_0 n}.$$

$$Z(K) = \frac{1}{N} \sum_{n=\langle N \rangle} \left[a x(n) + b y(n)\right] e^{-jK\Omega_0 n}.$$

$$Z(K) = \frac{1}{N} \sum_{n=\langle N \rangle} a\, x(n)\, e^{-jK\Omega_0 n} + b y(n)\, e^{-jK\Omega_0 n}.$$

$$Z(K) = \frac{1}{N} . a . \sum_{n=\langle N \rangle} x(n)\, e^{-jK\Omega_0 n} + \frac{1}{N} b . \sum_{n=\langle N \rangle} y(n)\, e^{-jK\Omega_0 n}$$

$$\therefore \boxed{Z(K) = a X(K) + b Y(K)} = \text{RHS}.$$

Hence the proof

2) Time shift property.

Stat. If $x(n) \xleftrightarrow{DTFS:\Omega_0} X(k)$ $\Omega_0 = \frac{2\pi}{N}$.

then PT $w(n) = x(n-n_0) \xleftrightarrow{DTFS:\Omega_0} W(k) = e^{-jk\Omega_0 n_0} X(k)$

ie shift in time domain gives multiplication of exponential with Fourier-coefficients.

③ Freq-shift property.

Stat. If $x(n) \xleftrightarrow{DTFS:\Omega_0} X(k)$. $\Omega_0 = \frac{2\pi}{N}$.

then $g(n) = e^{jk_0\Omega_0 n} x(n) \xleftrightarrow{DTFS:\Omega_0} G(k) = X(k-k_0)$.

ie shift in frequency domain gives the multiplication of exponential with time domain signals. x(n).

④ Modulation

Stat. If $x(n) \xleftrightarrow{DTFS:\Omega_0} X(k)$

& $y(n) \xleftrightarrow{DTFS:\Omega_0} Y(k)$ & $\Omega_0 = \frac{2\pi}{N}$.

then PT $z(n) = x(n).y(n) \xleftrightarrow{DTFS:\Omega_0} Z(k) = X(k) ⊛ Y(k)$

ie. Multiplication in time domain is eqt convolution of DTFS-coefficients.

Scaling property

Stat If $x(n) \xleftrightarrow{DTFS:\Omega_0} X(K)$ & $\Omega_0 = \frac{2\pi}{N}$.

then PT $z(n) = x(pn) \xleftrightarrow{DTFS:\Omega_0} Z(K) = \underline{p \cdot X(K)}$.

where $p > 0$.

ie Scaling operations if it scales the time domain Discrete signal by p then at o/p it amplifies the DTFS co-efficients by p.

Convolution property

If $x(n) \xleftrightarrow{DTFS:\Omega_0} X(K)$ $\Omega_0 = \frac{2\pi}{N}$.

& $y(n) \xleftrightarrow{DTFS:\Omega_0} Y(K)$.

then PT. $z(n) = x(n) \circledast y(n) \xleftrightarrow{DTFS:\Omega_0} N \cdot X(K) \cdot Y(K)$

ie convolution in time domain is transformed into multiplication of DTFS-coefficients along with period 'N'.

Parseval's theorem

Stat. If $x(n) \xleftrightarrow{DTFS:\Omega_0} X(K)$ & $\Omega_0 = \frac{2\pi}{N}$. then

PT $\frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 = \sum_{K=\langle N \rangle} |X(K)|^2$.

the sequence $|X(k)|^2$ for $k = 0, 1, 2 \cdots N-1$ is nothing but distribution of power as a function of frequency & it is called power Density spectrum of the signal $x(n)$..

Duality

stat: If $x(n) \xleftrightarrow{DTFS : \Omega_0} X(k)$ & $\Omega_0 = \frac{2\pi}{N}$.

then PT $X(n) \xleftrightarrow{DTFS : \Omega_0} \frac{1}{N} x(-k)$.

Symmetry

stmt If $x(n) \xleftrightarrow{DTFS : \Omega_0} X(k)$ $\Omega_0 = \frac{2\pi}{N}$.

then PT

1) If $x(n)$ is real. then $X^*(k) = X(-k)$.

2) If $x(n)$ is real & even then

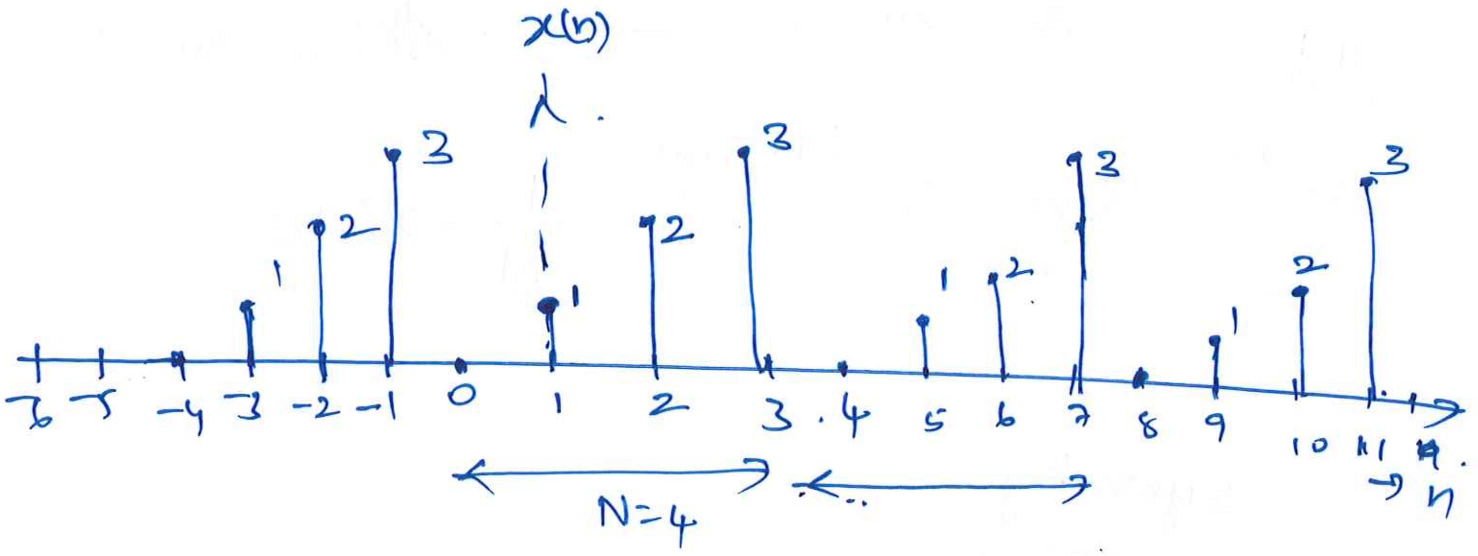
$$\text{Img}\{X(k)\} = 0$$

3) If $x(n)$ is real & odd then

$$\text{Re}\{X(k)\} = 0$$

(***) 10M. **Problems**

1) Evaluate the DTFS representation for the signal $x(n)$ shown in fig. Ep. Sketch the spectrum. Also the verify. parseval's Identity.



Soln

Fundamental period $N = 4$

$$\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{N} = \frac{2\pi}{4} = \pi/2$$

w.k.t $$X(k) = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-jk(\Omega_0)n}.$$

X(k) in terms of k.
$$X(k) = 1/4 \sum_{n=0}^{3} x(n) e^{-jk\left(\frac{\pi}{2}\right)n}$$

$$\therefore X(k) = \frac{1}{4}\left[x(0) + x(1) e^{-jk\frac{\pi}{2}(1)} + x(2) e^{-jk\left(\frac{\pi}{2}\right)(2)} + x(3) e^{-jk(\pi/2)(3)}\right].$$

$$X(k) = \frac{1}{4}\left[0 + (1)\, e^{-j\pi/2 k} + 2\, e^{-j\frac{\pi}{2}(k)} + 3e^{-j3(\frac{\pi}{2})k}\right].$$

$\longrightarrow k = 0,1,2,3.$

$\therefore$ ~~$X(k) = \frac{1}{4}[$~~ $X(k)$ in terms of k

when $k=0$

$$X(0) = \frac{1}{4}\left[0 + 1 + 2 + 3\right] = \frac{6}{4} = 3/2 = \underline{\underline{1.5}}$$

$$X(1) = \frac{1}{4}\left[0 + e^{-j(1)\cdot\frac{\pi}{2}(1)} + 2\cdot e^{-j(1)\frac{\pi}{2}(2)} + 3e^{-j(1)\frac{\pi}{2}(3)}\right]$$

$$X(1) = \frac{1}{4}\left[0 + e^{-j\pi/2} + 2e^{-j\pi} + 3e^{-j3\frac{\pi}{2}}\right].$$

$$X(1) = \frac{1}{4}\left[0 - \underline{\underline{j}} + 2(-1) + 3(\underline{\quad -\quad})\right]$$

$\because\ e^{-j\pi/2} = \cos\frac{\pi}{2} - j\sin\pi/2$

$= 0 - j = \underline{\underline{-j}}$

$e^{-j\pi} = \cos\pi - j\sin\pi.$

$= -1 - 0 = \underline{\underline{-1}}$

lly $e^{-j3\pi/2} = \underline{\underline{\cos 3\pi/2 - j\sin 3\pi/2 = (\qquad)}}$

$$X(1) = \underline{\underline{-0.5 + j0.5}}$$

$$X(2) = \frac{1}{4}\left[0 + x(1)\, e^{-j\pi} + 2e^{-j\pi(2)} + 3\, e^{-j\frac{3\pi}{2}(3)}\right]$$

$$X(2) = \frac{1}{4}\left[x(1) e^{-j\pi} + 2e^{-j2\pi} + e^{-j3\pi}\right]$$

$$X(2) = \underline{-0.5}$$

$$X(3) = 1/4\left[0 + (1)\, e^{-j\pi/2(3)} + 2e^{-j\pi(3)} + 3e^{-j\frac{3\pi}{2}(3)}\right]$$

$$X(3) = 1/4\left[0 - e^{-j3/2\pi} + 2e^{-j3\pi} + 3e^{-j9/2\pi}\right]$$

$$X(3) = \underline{-0.5 - j0.5}$$

$$\therefore\ X(K) = \{\underset{\substack{\uparrow \\ K=0}}{1.5}, -0.5 + j0.5, -0.5, -0.5 - j0.5\}$$

Magnitude spectra

$$\therefore\ |X(K)| = \sqrt{A(K)^2 + B(K)^2}$$

$$|X(0)| = \sqrt{1.5^2} = 1.5$$

$$|X(1)| = \sqrt{(0.5)^2 + (0.5)^2} = \sqrt{0.25 + 0.25} = \sqrt{0.50} = \underline{\underline{0.707}}$$

lly $|X(2)| = 0.5$, $\underline{\underline{|X(3)| = 0.707}}$

phase angle of $X(k)$

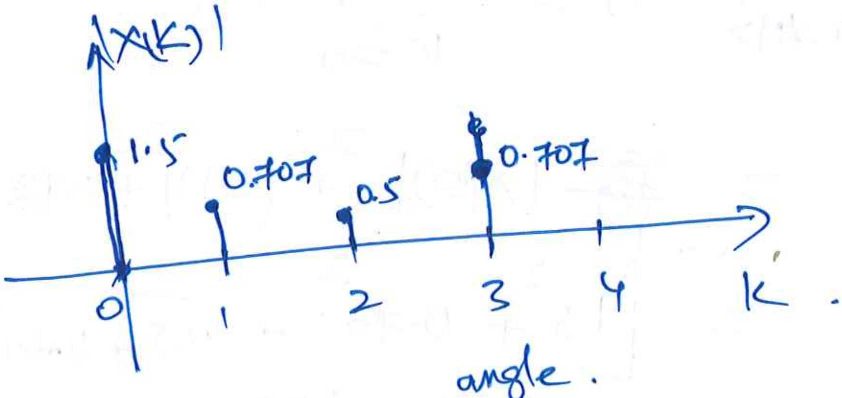
$$\angle X(k) = \tan^{-1}\left\{\frac{B(k)}{A(k)}\right\}$$

$$\therefore \angle X(0) = \tan^{-1}\left\{\frac{B(k)}{A(k)}\right\} = \tan^{-1}\left\{\frac{(0)}{(1.5)}\right\} = \tan^{-1}(0) = \underline{\underline{0}}$$

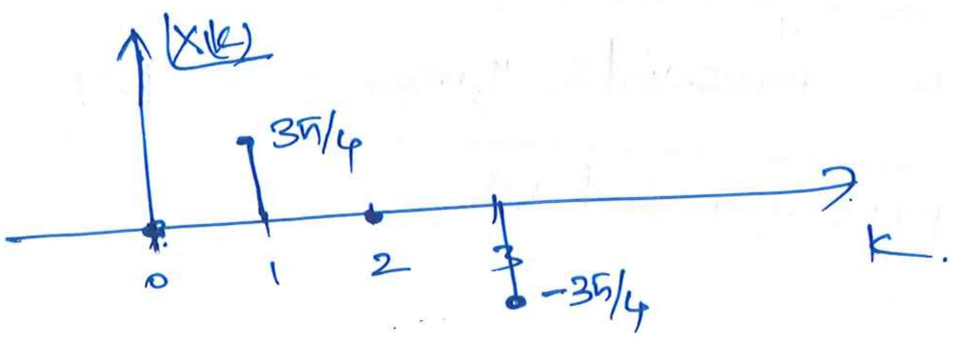
$$\angle X(1) = \tan^{-1}\left[\frac{(0.5)}{-0.5}\right] = \tan^{-1}(-1) = \underline{\frac{3\pi}{4}}$$

$$\angle X(2) = 0, \quad \angle X(3) = \underline{-3\pi/4}$$

$\therefore$ Magnitude spectrum



angle.
Phase Spectrum



Verification of parseval's theorem

WKT PT is.

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2 = \sum_{k=\langle N\rangle}|X(k)|^2$$

$$LHS = \frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2 = \frac{1}{4}\sum_{n=0}^{3}|x(n)|^2$$

$$= 1/4\left[x(0)^2 + x(1)^2 + x(2)^2 + x(3)^2\right]$$

$$= 1/4\left[0 + 1^2 + 2^2 + 3^2\right]$$

$$= 1/4\left[1+4+9\right] = \frac{14}{4} = \underline{\underline{3.5}}.$$

$$RHS = \sum_{n=\langle N\rangle}|X(k)|^2 = \sum_{k=0}^{3}|X(k)|^2$$

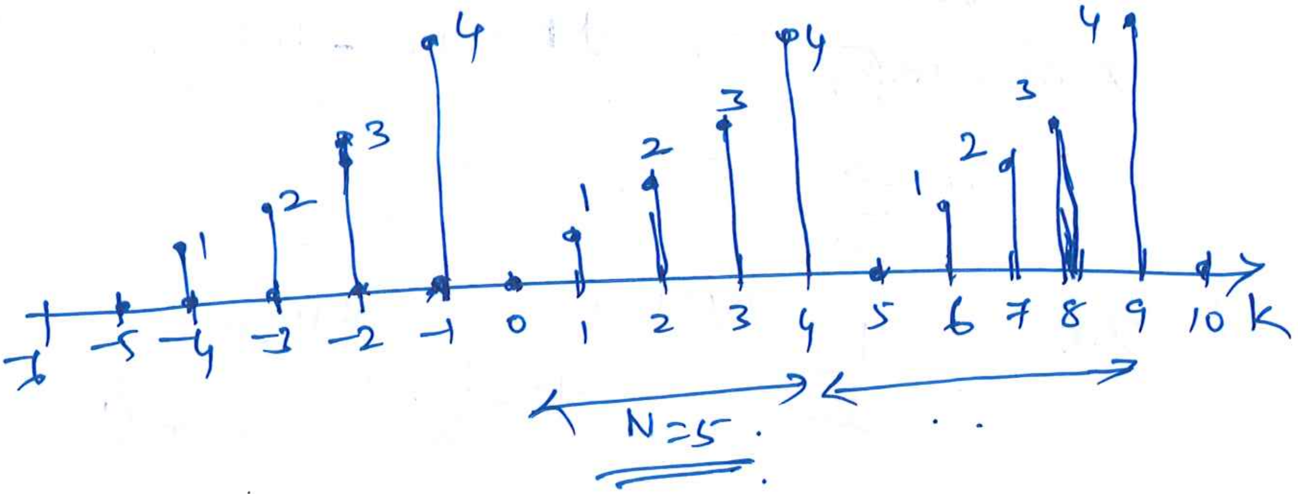
$$= |X(0)|^2 + |X(1)|^2 + |X(2)|^2 + |X(3)|^2$$

$$= \left[1.5^2 + 0.707^2 + 0.5^2 + 0.707^2\right]$$

$$RHS = 3.5$$

Hence parseval's theorem is verified.

2) Repeat DTFS of a signal for



(13) Determine the DTFS of the signal.

$$x(n) = \cos(\pi/3\, n).$$

& Draw the spectrum.

Soln. WKT $x(n) = \cos(\Omega_0 n)$ is periodic with Ω_0 as fund-amental freq.

* Compare the std eqn $x(n) = \cos(\Omega_0 n)$ with given eqn $x(n) = \cos\left(\frac{\pi}{3} n\right)$.

$$\therefore \Omega_0 = \frac{2\pi}{N} = \pi/3.$$

$\therefore$ $N = 6$ → Fundamental period.

$\therefore$ we can write std as

$$X(k) = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jk\Omega_0 n}.$$

$$X(k) = \frac{1}{6} \sum_{n=\langle 6 \rangle} x(n)\, e^{-jk\left(\frac{\pi}{3}\right) n}.$$

$k = 0, 1, 2, 3, 4, 5$

$\therefore$ other $X(k)$'s are equal to 0.

∴ we can write given as

$$x(n) = \cos\left(\frac{\pi}{3}n\right) = \frac{e^{j\left(\frac{\pi}{3}\right)n} + e^{-j\frac{\pi}{3}n}}{2}.$$

* ie $x(n) = \frac{1}{2} e^{j(1)\frac{\pi}{3}n} + \frac{1}{2} e^{j(-1)\frac{\pi}{3}n} \longrightarrow (1)$

From std we can write

$$x(n) = \sum_{k=\langle N\rangle} X(k) e^{jk\Omega_0 n} = \sum_{k=\langle 6\rangle} X(k) e^{jk\left(\frac{\pi}{3}\right)n} \rightarrow (2)$$

* Compare (1) & (2), we get

$$\frac{1}{2} e^{j(1)\pi/3 n} + \frac{1}{2} e^{j(-1)\pi/3 n} = \sum_{k=\langle 6\rangle} X(k) e^{jk(\pi/3)n}.$$

* From above eqn.

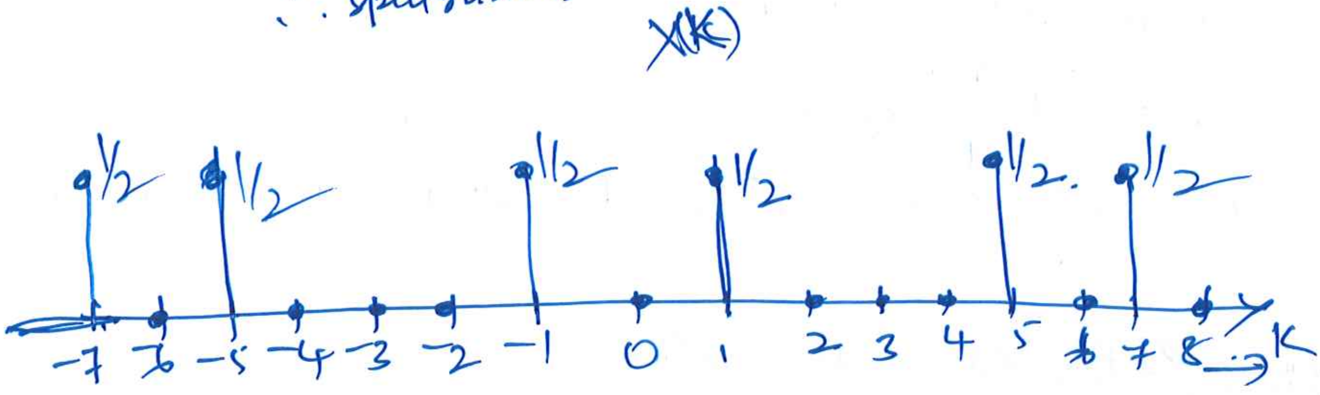
$$X(1) = 1/2, \quad X(-1) = 1/2$$

∴ Since DTFS $X(k)$ is periodic with period $N=6$, we can write

$$\ldots = X(-11) = X(-5) = X(1) = X(7) = X(13) = \ldots \ldots \; 1/2$$

$$\ldots = X(-13) = X(-7) = X(-1) = X(5) = X(11) = \ldots \ldots \; 1/2$$

$\therefore$ spectrum is



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(4) Repeat DTFS of the signal $x(n) = \cos(\pi/4)n$.

Soln compare the given $x(n) = \cos(\pi/4\, n)$ with std eqn

$$x(n) = \cos(\Omega_0 n).$$

ie $$\Omega_0 = \frac{2\pi}{N} = \pi/4$$

$\therefore N = 8.$ Fundamental period.

std $$\therefore X(K) = \frac{1}{8} \sum_{n=\langle 8 \rangle} x(n)\, e^{-jk\left(\frac{\pi}{4}\right)n}.$$

where $k = 0, 1, 2, 3, 4, 5, 6, 7.$

given eqn interms of exponentials is

$$x(n) = \cos(\pi/4\, n) = \frac{e^{j\frac{\pi}{4}n} + e^{-j\pi/4\, n}}{2}.$$

ie $$x(n) = \cos(\pi/4\, n) = \frac{1}{2} e^{j(1)\pi/4\, n} + \frac{1}{2} e^{j(-1)\pi/4\, n} \longrightarrow (1)$$

From std eqn

$$x(n) = \sum_{K=\langle 8 \rangle} X(K)\, e^{jK\Omega_0 n} = \sum_{K=\langle 8 \rangle} X(K)\, e^{jK(\pi/4)n} \longrightarrow (2)$$

$k = 0$ to 7

Compare eq ① & ②, we get

$X(1) = 1/2 \quad X(-1) = 1/2$

Since DTFS of $X(k)$ is periodic with period $N=8$, we get

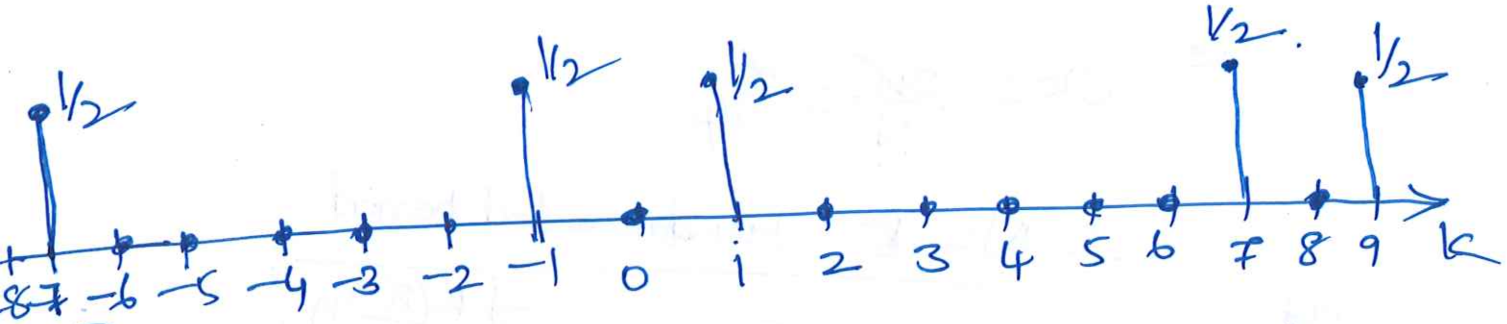
$\ldots = X(-15) = X(-7) = X(1) = X(9) = X(17) = \ldots \ldots 1/2$

$\ldots = X(-17) = X(-9) = X(-1) = X(7) = X(15) = \ldots \ldots 1/2$

other $X(k)^s = 0$.

Spectrum plot is

$X(k)$



5. Evaluate the DTFS representation for the signal.

10M

$$x(n) = \sin\left(\frac{4\pi}{21}n\right) + \cos\left(\frac{10\pi}{21}n\right) + 1$$

Sketch the magnitude & phase spectrum.

Soln given $x(n) = \sin\left(\frac{4\pi}{21}n\right) + \cos\left(\frac{10\pi}{21}n\right) + 1$

∴ Effective angular freq of summation is given by

$\Omega_0 = \text{GCD of } \frac{4\pi}{21} \text{ \& } \frac{10\pi}{21}$

$\frac{2\pi}{21} \Big| \; \frac{4\pi}{21}, \frac{10\pi}{21} \qquad \therefore \Omega_0 = \frac{2\pi}{21}$

$\qquad 2 \qquad 5$

$\Omega_0 = \frac{2\pi}{\cancel{N}} = \frac{2\pi}{21}$

$\therefore N = 21 \rightarrow$ Fundamental period.

$\therefore$ we can write given as

$$x(n) = \frac{e^{j\frac{4\pi}{21}n} - e^{-j\frac{4\pi}{21}n}}{2j} + \frac{e^{j\frac{10\pi}{21}n} + e^{-j\left(\frac{10}{21}\right)n}}{2} + 1 \rightarrow (2)$$

compare eq (1) & (2)

$$\therefore x(n) = \frac{1}{2j} e^{j(2)\left(\frac{2\pi}{21}\right)n} - \frac{1}{2j} e^{j(-2)\frac{2\pi}{21}n} + \frac{1}{2} e^{j(5)\frac{2\pi}{21}n} + \frac{1}{2} e^{j(-5)\frac{2\pi}{21}n} + 1$$

$$\cancel{x}n = \sum_{k=\langle 21 \rangle} X(k)\, e^{jk\left(\frac{2\pi}{21}\right)n}$$

$\therefore X(0) = 1,$

$X(2) = \frac{1}{2j}, \quad X(-2) = -\frac{1}{2j};$

$X(5) = \frac{1}{2} \quad X(-5) = \frac{1}{2}$

other value of $X(k) = 0$

$\therefore$ Magnitude spectra for 1 cycle is