GCEM

Gopalan College of Engineering and Management

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NOTES

SUBJECT: BASIC SIGNAL PROCESSING

SUBJECT CODE: 21EC33

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Go Green
Save Mother Earth

Module - 5.

Z-Transforms

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* Z-Transform is an Extension of Discrete time Fourier Transform
* This Extension can be applied to a wider class of signals than DTFT because there are many signals for which DTFT can't converges. where as
* Z-Transforms Easily converges.
* ZT is the techniques using this we can analyse signal [x(n)] better
* signal system better
* design of digital filter Efficiently → usefull in Real Time projects.
* ZT → It is discrete time counterpart of Laplace transform.
* Extension of DTFT

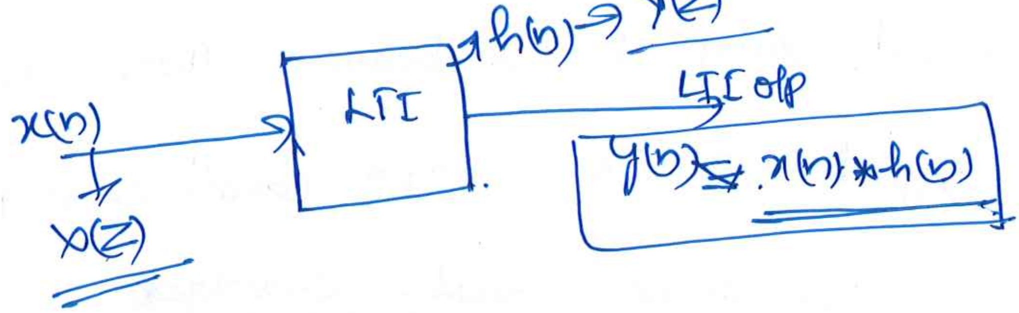
* Those signals we can't process using DTFT can be processed using Z-Transforms.

∴ Z-Transform of ~~an~~ discrete time signal $x(n)$ (2).

is given by.

$$\boxed{X(Z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}} \rightarrow \text{std.}$$

Ex: consider LTI system with i/p $x(n)$, impulse response $h(n)$ & o/p $y(n)$, ~~then~~ ie



Problems

6M

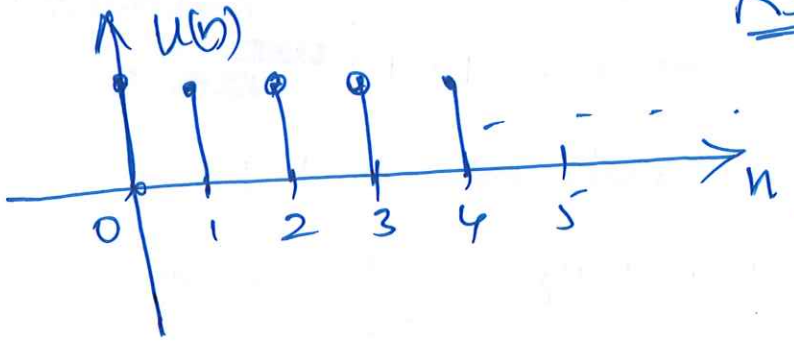
1) Find the Z-Transform of $x(n) = \alpha^n u(n)$. RSS

Soln. $x(n) = \alpha^n u(n)$ (given)

WKT. $u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & n < 0 \end{cases}$

RSS

ie.



$X(Z)$ of $x(n)$ is

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \rightarrow \text{std}$$

$$X(Z) = \sum_{n=0}^{\infty} \left[\alpha^n \underline{u(n)}\right] z^{-n}$$

$$X(Z) = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

$$X(Z) = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

WKT Using Infinite sum formula we can write

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\underline{\underline{\alpha}}} \quad ; \quad |\alpha| < 1$$

∴ By Applying above formula,

$$X(Z) = \frac{1}{1-\alpha z^{-1}} \quad ; \quad \underline{\underline{|\alpha z^{-1}| < 1}}$$

i.e condition to get $X(Z)$ converging or $\underline{\underline{ROC}}$

is

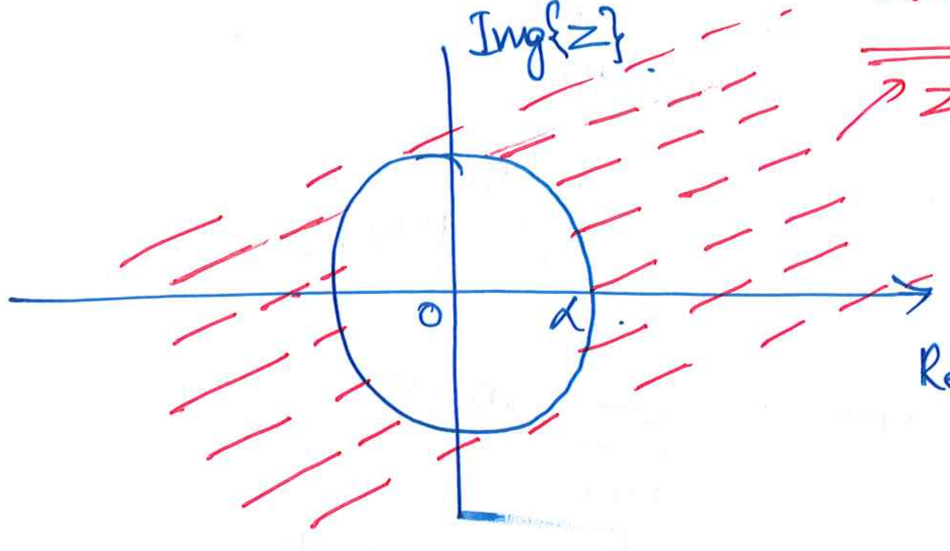
$$|\alpha z^{-1}| < 1$$

$$\left|\frac{\alpha}{z}\right| < 1, \quad \text{i.e } |\alpha| < |z|$$

ROC : Also $|z| > |\alpha|$.

ROC plot in z plane.

ROC : $|z| > |\alpha|$



∴ If Right-sided sequence u(n), then ROC is always outside the circle of radius α.

Repeat 2). $x(n) = \beta^n u(n)$

③. Find $X(z)$ of $x(n) = -\alpha^n \underline{u(-n-1)}$ & find ROC. → LSS

Soln. $x(n) = -\alpha^n u(-n-1) \longrightarrow$ [Given]

WKT $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \longrightarrow$ std

$$X(z) = \sum_{n=-\infty}^{-1} \left[-\alpha^n u(-n-1)\right] . z^{-n}$$

$$X(z) = -\sum_{n=-\infty}^{-1} \alpha^n (1) z^{-n}$$

$$\therefore X(z) = -\sum_{n=-\infty}^{-1} (\alpha z^{-1})^n$$

$$\text{wkt.}\quad \sum_{n=-\infty}^{-1} (\alpha)^n = \sum_{n=1}^{\infty} (\alpha^{-1})^n$$

Applying above formula for DTS. $x(n)$ we get—

$$X(z) = -\sum_{n=1}^{\infty} (\alpha^{-1} z)^n$$

Apply ISF ie. starting from seq 1;

wkt $\sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha}$; condition $|\alpha| < 1$.

$$\therefore \boxed{X(z) = \frac{-(\alpha^{-1} z)}{1 - \alpha^{-1} z}}$$

The condition to get $X(z)$ exist is

ROC $\equiv$ $|\alpha^{-1} z| < 1$.

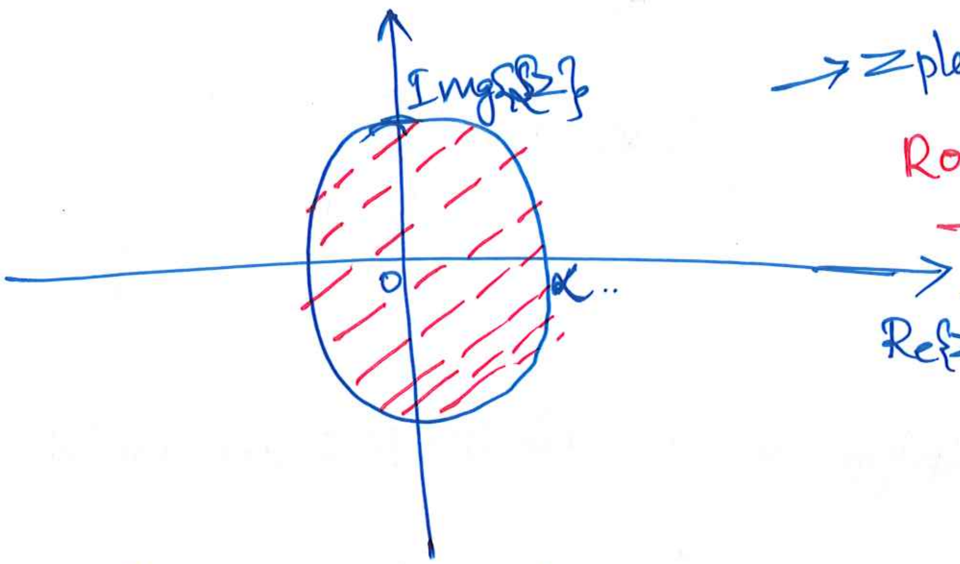
ie

$$\frac{|z|}{|\alpha|} < 1$$

ROC ; $\therefore |z| < |\alpha|$.

cont to 1

Roc plot in z-plane is



→ z plane

ROC : $|z| < |\alpha|$.

* So sequence given is left-sided seq, ROC comes inside a circle.

Repeat

$$x(n) = -\alpha^{n} u(-n-2)$$

Z-Transform – 2nd session. [Module-5]. – 18EC45

Signals and systems

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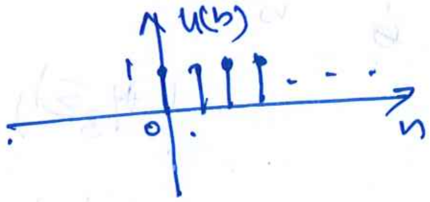
B) For the signal $x(n) = 7(1/3)^n u(n) - 6(1/2)^n u(n)$

Find Z-T & ROC.

Soln) Given. $x(n) = 7(1/3)^n u(n) - 6(1/2)^n u(n)$

$$\left[\text{WKT. } u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}\right.$$



WKT $X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$ — (std).

$$X(Z) = 7 \sum_{n=0}^{\infty} (1/3)^n \cancel{Z} Z^{-n} - 6 \sum_{n=0}^{\infty} (1/2)^n (1) Z^{-n}$$

$$\therefore X(Z) = 7 \sum_{n=0}^{\infty} (1/3 Z^{-1})^n - 6 \sum_{n=0}^{\infty} (1/2 Z^{-1})^n.$$

$$\therefore X(Z) = 7 \frac{1}{(1 - 1/3 Z^{-1})} - 6 \frac{1}{1 - 1/2 Z^{-1}} \quad \text{— (1)}.$$

$$\left[\text{Apply ISF } \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}\right.$$

$$\therefore X(Z) = \frac{7(1 - 1/2 Z^{-1}) - 6(1 - 1/3 Z^{-1})}{(1 - 1/3 Z^{-1})(1 - 1/2 Z^{-1})}.$$

$$X(Z) = \frac{7 - 7/2 Z^{-1} - 6 + \cancel{2} Z^{-1}}{(1 - 1/3 Z^{-1})(1 - 1/2 Z^{-1})} = \frac{1 - 3/2 Z^{-1}}{(1 - 1/3 Z^{-1})(1 - 1/2 Z^{-1})}$$

[See max power of z^{-1} in Denominator] ②

* Multiply & Divide N & D by z^2.

$$\therefore X(z) = \frac{z^2 \left[1 - \frac{3}{2} z^{-1}\right]}{z^2 (1 - \frac{1}{3} z^{-1})(1 - \frac{1}{2} z^{-1})}$$

$$X(z) = \frac{z (z - \frac{3}{2})}{(z - \frac{1}{3}) \left[z - \frac{1}{2}\right]}$$

$\therefore X(z)$ Exists only when both terms converges. The condn is.

$$\left|\frac{1}{3} z^{-1}\right| < 1 \quad \& \quad \left|\frac{1}{2} z^{-1}\right| < 1$$

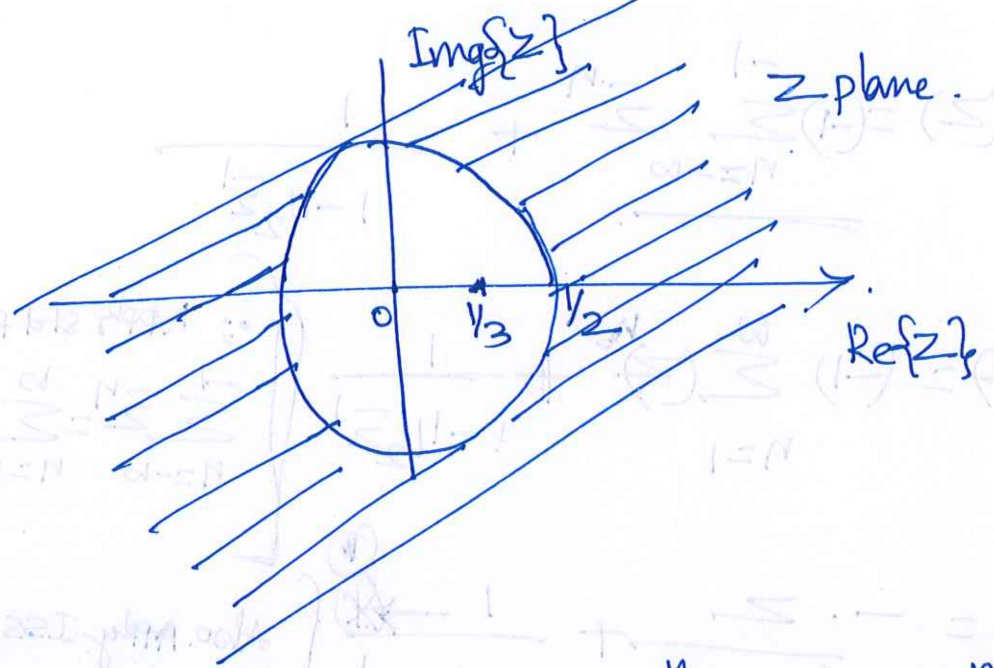
$\left|\frac{1}{3}\right| < z$ ||ly $\left|\frac{1}{2}\right| < |z|$

or $|z| > \frac{1}{3}$. $\therefore |z| > \frac{1}{2}$

$\therefore$ ROC is : $|z| > \frac{1}{2}$

$\therefore$ plot ROC in z plane. we get

$\rightarrow$.



(4) Repeat for $x(n) = 6(1/3)^n u(n) - 5(1/2)^n u(n)$

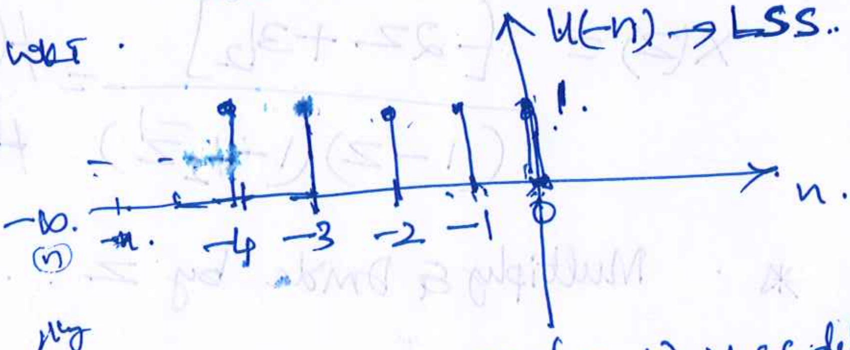
(5). Determine Z-Transform of

$$x(n) = -u(-n-1) + (1/2)^n u(n)$$

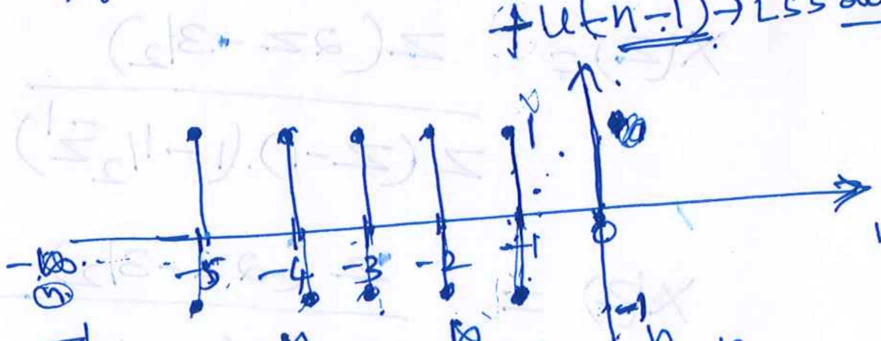
Find ROC & poles, zeros location of $X(Z)$ in the z-plane.

Soln. WKT $X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \rightarrow (std)$

WKT.



lly



$$\therefore X(Z) = \sum_{n=-\infty}^{-1} (-1) Z^{-n} + \sum_{n=0}^{\infty} (1/2)^n Z^{-n}.$$

$$X(z) = (-1)\sum_{n=-\infty}^{-1} z^{-n} + \frac{1}{1-1/2\,z^{-1}}$$

$$X(z) = (-1)\sum_{n=1}^{\infty} (z)^{n} + \frac{1}{1-1/2\,z^{-1}}$$

∴ Apply std formula

$$\sum_{n=-\infty}^{-1} z^{-n} = \sum_{n=1}^{\infty} z^{n}$$

$$X(z) = -\cdot\frac{z}{(1-z)} + \frac{1}{1-1/2\,z^{-1}} \rightarrow (*)$$

Also. ~~Apply~~ ISS formula

$$\sum_{n=1}^{\infty} z^{n} = \frac{z}{1-z}$$

$$X(z) = \frac{-z(1-1/2\,z^{-1}) + 1.(1-z)}{(1-z)(1-1/2\,z^{-1})}$$

$$X(z) = \frac{-z+1/2+1-z}{(1-z)(1-1/2\,z^{-1})}$$

$$X(z) = \frac{[-2z+3/2]}{(1-z)(1-1/2\,z^{-1})} = \frac{+(2z-3/2)}{+(z-1)(1-1/2\,z^{-1})}$$

* Multiply & Divide by z ∴

$$X(z) = \frac{z(2z-3/2)}{z(z-1).(1-1/2\,z^{-1})}$$

$$X(z) = \frac{z(2z-3/2)}{(z-1)(z-1/2)} \rightarrow (2)$$

IB

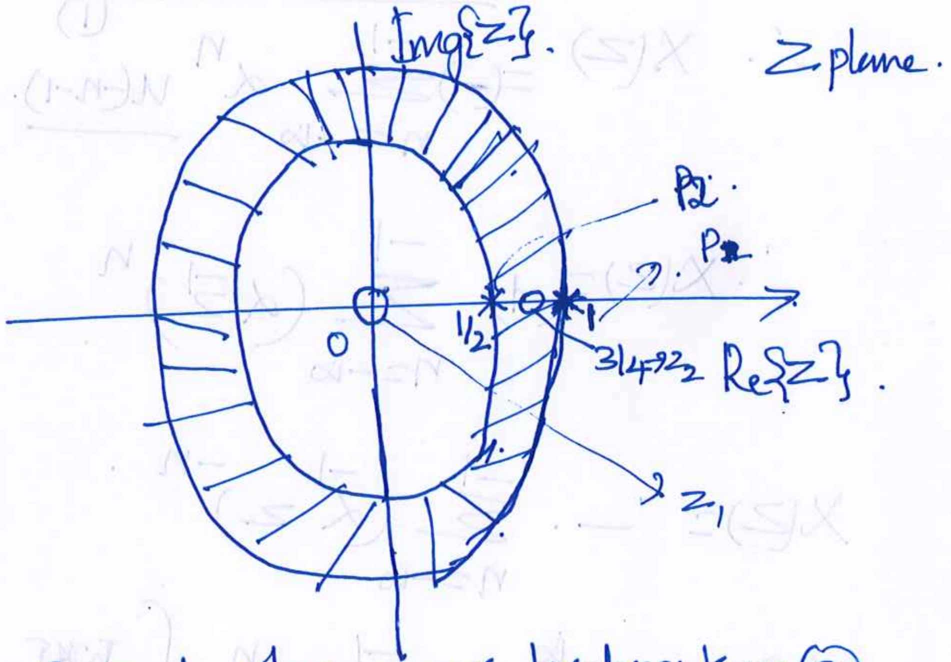
* To get $X(Z)$ exist, both the terms of eq ① must converge.

i.e $|Z| < 1$ & $|\frac{1}{2} Z^{-1}| < 1$.

$|Z| < 1$ & $|\frac{1}{2}| < |Z|$

$|Z| < 1$ & $|Z| > 1/2$

$\therefore$ ROC : $\frac{1}{2} < |Z| < 1$.



To find poles & zeros locations from ②.

Nr → zeros → $Z = 0 = Z_1 \rightarrow 0$

$2Z - 3/2 = 0$

$2Z = 3/2$

$\therefore Z = 3/4 \rightarrow Z_2 \rightarrow 0$

Dr → poles → $Z - 1 = 0$

$\therefore Z = 1 \rightarrow$ poles $P_1 \rightarrow *$.

$Z - \frac{1}{2} = 0$

$\therefore Z = \frac{1}{2} \rightarrow P_2 \rightarrow *$.

6 Repeat $x(n) = \underbrace{-u(-n-2)}_{Rss.} + \underbrace{(1/3)^n u(n)}_{Lss}$

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7 Find $X(z)$, if $x(n) = -\alpha^n u(-n-1)$. Also find ROC.

** 6M.

Soln. WKT $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \rightarrow (std)$

~~Here x(n)~~ * $\underline{Lss \rightarrow u(-n-1)}$.

$\therefore X(z) = (-1)\sum_{n=-\infty}^{-1} \alpha^n \underline{u(-n-1)} . z^{-n}$ (1)

$\therefore X(z) = (-1) \sum_{n=-\infty}^{-1} (\alpha z^{-1})^n$

* $X(z) = - \sum_{n=-\infty}^{-1} (\alpha^{-1} z)^{-n}$.

$X(z) = (-1)\sum_{n=1}^{\infty} \underline{(\alpha^{-1} z)^n}$ $\left[\text{WKT} \sum_{n=-\infty}^{-1} \alpha^{-n} = \sum_{n=1}^{\infty} \alpha^n \right]$

$$\boxed{X(z) = \frac{-(\alpha^{-1} z)}{1-\alpha^{-1} z}} \rightarrow (1) \left[\because \sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha} \right]$$

$X(z) = \frac{-\left(\frac{z}{\alpha}\right)}{1-z/\alpha} = \frac{-z/\alpha}{\frac{\alpha - z}{\alpha}} = \frac{-z}{\alpha - z}$.

$$\boxed{X(z) = \frac{\cancel{+}z}{\cancel{+}(z-\alpha)} = \frac{z}{z-\alpha}} \rightarrow (2)$$

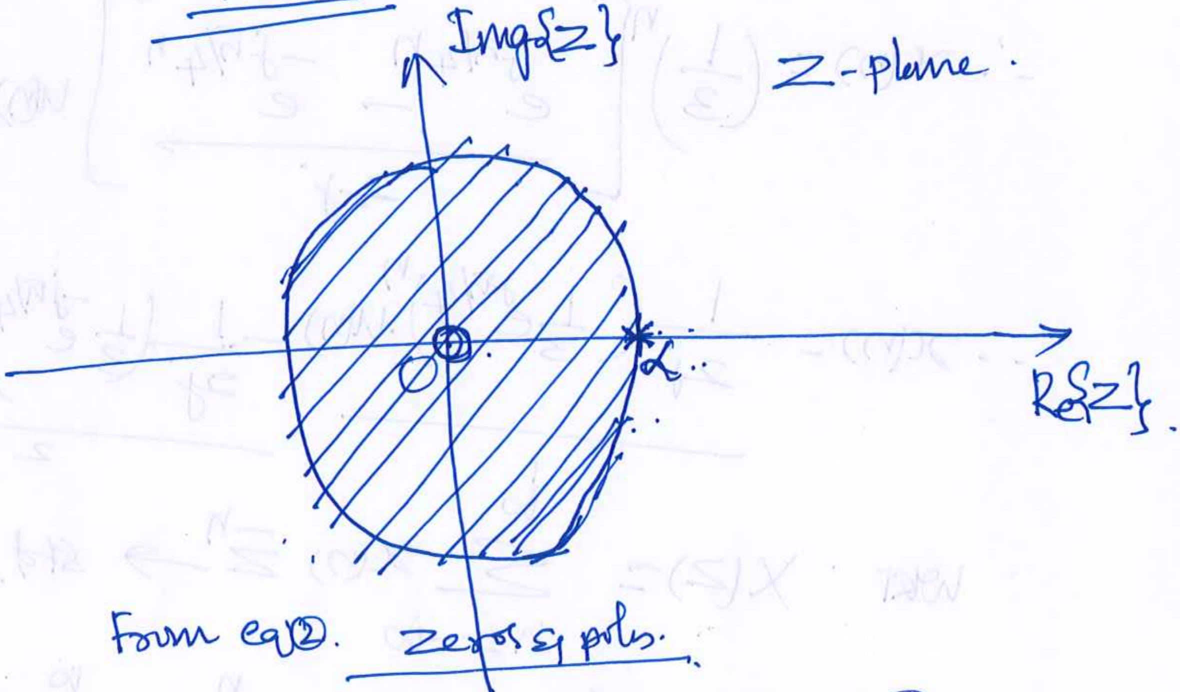
From (1) $X(z)$ Exists only when.

$$|\alpha^{-1}z| < 1$$

ie $\left|\frac{z}{\alpha}\right| < 1$

$\therefore$ ROC : $|z| < |\alpha|$

$\therefore$ ROC



From eq(2). Zeros & poles.

Nr $\rightarrow$ $z = 0, \rightarrow z_1 \rightarrow 0$

Dr $\rightarrow$ $z - \alpha = 0$, $z = \alpha \rightarrow p_1 \rightarrow *$

(8) Repeat for $x(n) = -\beta^n u(-n-1)$.

(8)

9) For the following signal shown, find ROC, Z-T & sketch pole-zeros locations.

$$x(n) = \left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u(n)$$

soln. given. $x(n) = (1/3)^n \sin\left(\frac{\pi}{4}n\right) \cdot u(n)$

Apply Formula.

i.e. $\sin\left(\frac{\pi}{4}n\right) = \frac{e^{j\pi/4 n} - e^{-j\pi/4 n}}{2j}$

$$\therefore x(n) = \left(\frac{1}{3}\right)^n \left[\frac{e^{j\pi/4 n} - e^{-j\pi/4 n}}{2j}\right] u(n).$$

$$\therefore x(n) = \underline{\frac{1}{2j}\left(\frac{1}{3}e^{j\pi/4}\right)^n \cdot u(n)} - \underline{\frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u(n)}.$$

WKT $X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \rightarrow$ std.

$$X(Z) = \frac{1}{2j}\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{j\pi/4}z^{-1}\right)^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{-j\pi/4}z^{-1}\right)^n$$

Apply I.S.S. i.e. $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$

$$\boxed{X(Z) = \frac{1}{2j}\frac{1}{\left(1-1/3\, e^{j\pi/4}z^{-1}\right)} - \frac{1}{2j}\cdot\frac{1}{\left(1-1/3\, e^{-j\pi/4}z^{-1}\right)}} \quad (1).$$

$$\therefore \boxed{X(z) = \frac{\frac{1}{3\sqrt{2}} z}{\left(1 - \frac{1}{3} e^{j\pi/4} z^{-1}\right)\left(1 - \frac{1}{3} e^{-j\pi/4} z^{-1}\right)}} \rightarrow (2).$$

* this we get by solving eq (1).

ie
$$X(z) = \frac{1}{2j} \frac{\left[\cancel{1} - \frac{1}{3} e^{-j\pi/4} z^{-1} - \cancel{1} + \frac{1}{3} e^{j\pi/4} z^{-1}\right]}{\left(1 - \frac{1}{3} e^{j\pi/4} z^{-1}\right)\left(1 - \frac{1}{3} e^{-j\pi/4} z^{-1}\right)}$$

$$X(z) = \frac{1}{2j} \frac{\left[\left(\frac{1}{3}\right) z^{-1} \left(-e^{-j\pi/4} + e^{j\pi/4}\right)\right]}{\left(1 - \frac{1}{3} e^{j\pi/4} z^{-1}\right)\left(1 - \frac{1}{3} e^{-j\pi/4} z^{-1}\right)}$$

$$X(z) = \frac{\left(\frac{1}{3}\right) z^{-1} \left[\frac{e^{j\pi/4} - e^{-j\pi/4}}{2j}\right]}{\left(1 - \frac{1}{3} e^{j\pi/4} z^{-1}\right)\left(1 - \frac{1}{3} e^{-j\pi/4} z^{-1}\right)}$$

$$X(z) = \frac{\frac{1}{3} z^{-1} \sin(\pi/4)}{\left(1 - \frac{1}{3} e^{j\pi/4} z^{-1}\right)\left(1 - \frac{1}{3} e^{-j\pi/4} z^{-1}\right)}$$

$$\therefore \sin\frac{\pi}{4} = \frac{e^{j\pi/4} - e^{-j\pi/4}}{2j}$$

$$\cos\frac{\pi}{4} = \frac{e^{j\pi/4} + e^{-j\pi/4}}{2}$$

$$\therefore X(z) = \frac{\frac{1}{3}\left(z^{-1}\right)\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{z - \frac{1}{3} e^{j\pi/4}}{z}\right)\left(\frac{z - \frac{1}{3} e^{-j\pi/4}}{z}\right)}$$

$$X(z) = \frac{\frac{1}{3\sqrt{2}} z^{-1} \cdot z^{2}}{\left(z - \frac{1}{3} e^{j\pi/4}\right)\left(z - 1/3\, e^{-j\pi/4}\right)}$$

$$\boxed{X(z) = \frac{\frac{1}{3\sqrt{2}} z}{\left(z - 1/3\, e^{j\pi/4}\right)\left(z - 1/3\, e^{-j\pi/4}\right)}} \rightarrow 2$$

* Using the above eqn. we can find zeros-poles.

Zeros, $z = 0, \rightarrow z_1 \rightarrow 0$

poles. $z - 1/3\, e^{j\pi/4}$

$\therefore z = 1/3\, e^{j\pi/4} \longrightarrow P_1 \rightarrow *$

$z - \frac{1}{3} e^{-j\pi/4}$

$\therefore z = 1/3\, e^{-j\pi/4} \longrightarrow P_2 \rightarrow *$

* Also $X(z)$ exists, only when eq (1) converges ie.

ie. $\left|\frac{1}{3} e^{j\pi/4} z^{-1}\right| < 1$ & $\left|\frac{1}{3} e^{-j\pi/4} z^{-1}\right| < 1$

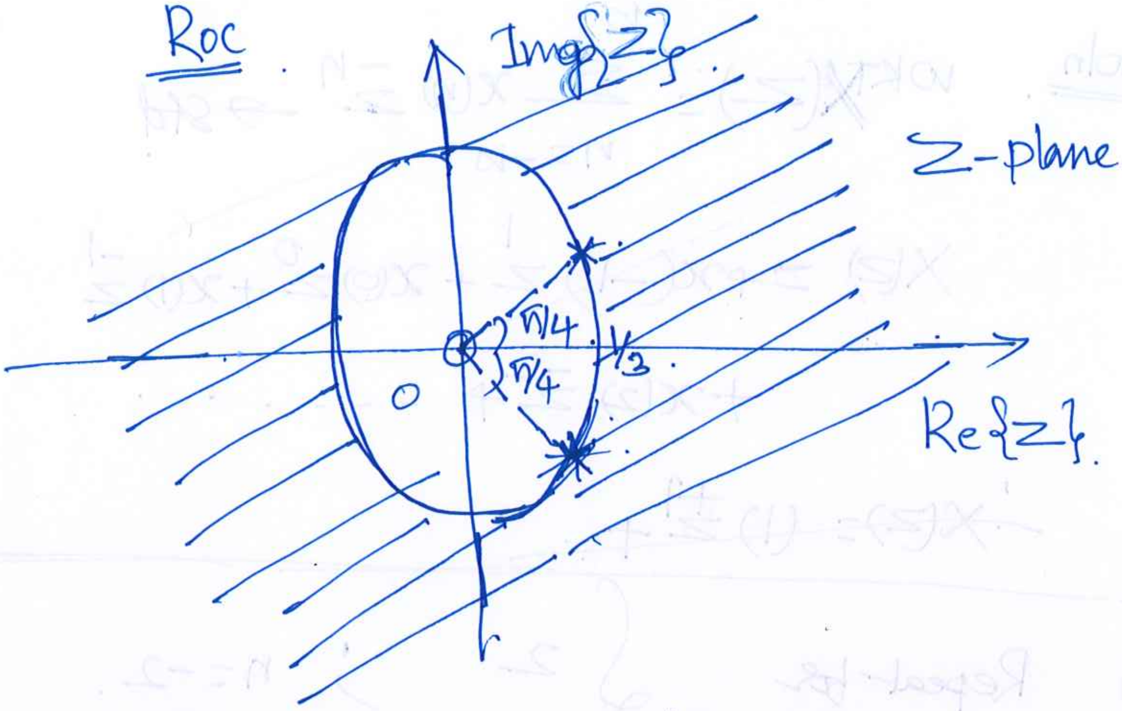
ie $\left|\frac{1}{3} e^{j\pi/4}\right| < |z|$ & $\left|\frac{1}{3} e^{-j\pi/4}\right| < |z|$

ie $|z| > \frac{1}{3} e^{j\pi/4}$ & $|z| > \frac{1}{3} e^{-j\pi/4}$

$\therefore |z| > 1/3$. & $|z| > 1/3$

$\therefore$ ROC: $|z| > 1/3$.

Roc



(10) Repeat for $x(n) = \left(\frac{1}{3}\right)^n \cos\left(\frac{\pi}{4} n\right) u(n)$.

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(11) Determine the Z-Transform of the signal and its ROC

if

$$x(n) = \begin{cases} 1 & ; n = -1 \\ 2 & ; n = 0 \\ -1 & ; n = 1 \\ 1 & ; n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Soln. WKT $X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \rightarrow (8(d))$

$$X(Z) = x(-1) Z^{1} + x(0) Z^{0} + x(1) Z^{-1} + x(2) Z^{-2} + \cdots$$

$$X(Z) = (1) Z^{1} + 2(1) + (-1) Z^{-1} + (1) Z^{-2} + \cdots$$

$$X(Z) = Z + 2 - \frac{1}{Z} + \frac{1}{Z^2}$$

∴ ROC → Exists on entire Z-plane except $Z = 0$ & $Z = \infty$

soln. WKT $X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \rightarrow$ std

$X(z) = + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1}$
$+ x(2) z^{-2} + \ldots\ldots$

~~$\therefore X(z) = (1) z^{-1} + .$~~

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(12) Repeat for

$$x(n) = \begin{cases} 2 & ; n=-2 \\ 1 & ; n=-1 \\ -2 & ; n=0 \\ 1 & n=1 \\ 2 & n=2 \\ 0 & \text{Otherwise} \end{cases}$$

---

(13) Determine Z-Transform of the following sequences.

a) $x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$

b) $x_2(n) = \{1, 2, 3, \underset{\uparrow}{4}, 5, 0, 7\}$

soln. $x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5, 0, 7\}$

WKT $X_1(z) = \sum_{n=0}^{6} x_1(n)\, z^{-n} \rightarrow$ (std)

$X_1(z) = 1z^{0} + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}$
$+ 0z^{-5} + 7z^{-6}.$

$\therefore X_1(z) = 1 + 2/z + \frac{3}{z^2} + \frac{4}{z^3} + \frac{5}{z^4} + 0 + \frac{7}{z^6}$.

$\underline{z=0}$

$\therefore X_1(z) = \infty$.

$z = \infty$.

$X_1(z) = 1+0+0+0+0+0 \ldots = 1$

$\therefore$ Roc ; Entire Z-plane except $\underline{z=0}$.

Illy. ~~Repeat~~ $x_2(n) = \{1, 2, 3, \underset{\uparrow}{4}, 5, 0, 7\}$

Properties of ROC [Region of Convergence]

** 6M

1) Roc of $X(z)$ consists of a ring in the z-plane centered about the origin.

2) ROC does n't contain any poles.

3) If $x(n)$ is of finite duration, then ROC is the entire z-plane except ① $z=0$ or $z=\infty$ &
② $z=0$ And $z=\infty$.

4) If $x(n)$ is right sided sequence (ie $u(n)$), then Roc is the region in the z-plane outside the circle.

5) If $x(n)$ is left sided sequence [ie $u(-n)$], then Roc is the region inside the circle.

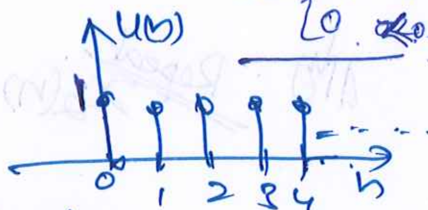
6) If $x(n)$ is 2 sided sequence [both $u(n)$ & $u(-n)$], Then ROC exist in the centre ring in the z-plane.

Problems

1) Find z-transform of unit step sequence & plot its ROC & poles & zeros.

Soln

$$u(n) = \begin{cases} 1 \; ; \; n \geq 0 \\ 0 \; ; \; n < 0 \end{cases}$$



given $x(n) = u(n)$

WKT $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ → (std)

$$X(z) = \sum_{n=0}^{\infty} u(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} (1) z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^{n}$$

$$X(z) = \frac{1}{(1 - z^{-1})} \rightarrow ①$$

$$\left[\text{GSF..} \quad \sum_{n=0}^{\infty} \alpha^{n} = \frac{1}{1-\alpha} \right].$$

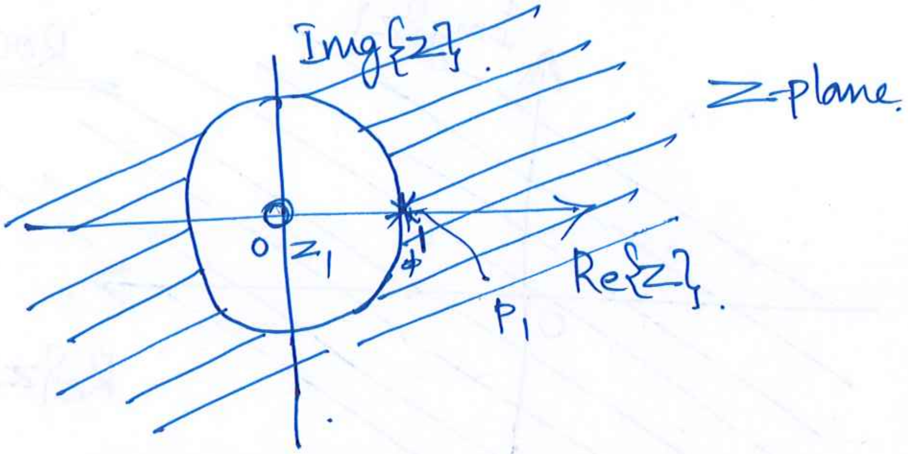
$$X(z) = \frac{z}{z-1} \rightarrow ②$$

Apply condn to eq①

* $X(z)$ Exists if $|z^{-1}| < 1$

$$\left| \frac{1}{z} \right| < 1$$

ie. ROC : $|z| > 1$



* poles & zeros location.

Zeros → $z=0$, → z_1

poles → $z-1=0$

$z=1$ → p_1

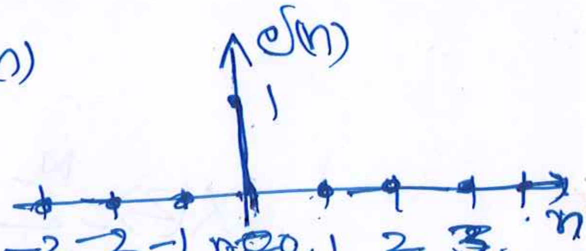
---

(2) Repeat Z-Transform of $x(n) = 2u(n)$

(3). Determine the Z-Transform & ROC for the following signal.

$$x(n) = \delta(n)$$

Soln. given $x(n) = \delta(n)$



WKT

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \rightarrow \text{(1)}$$

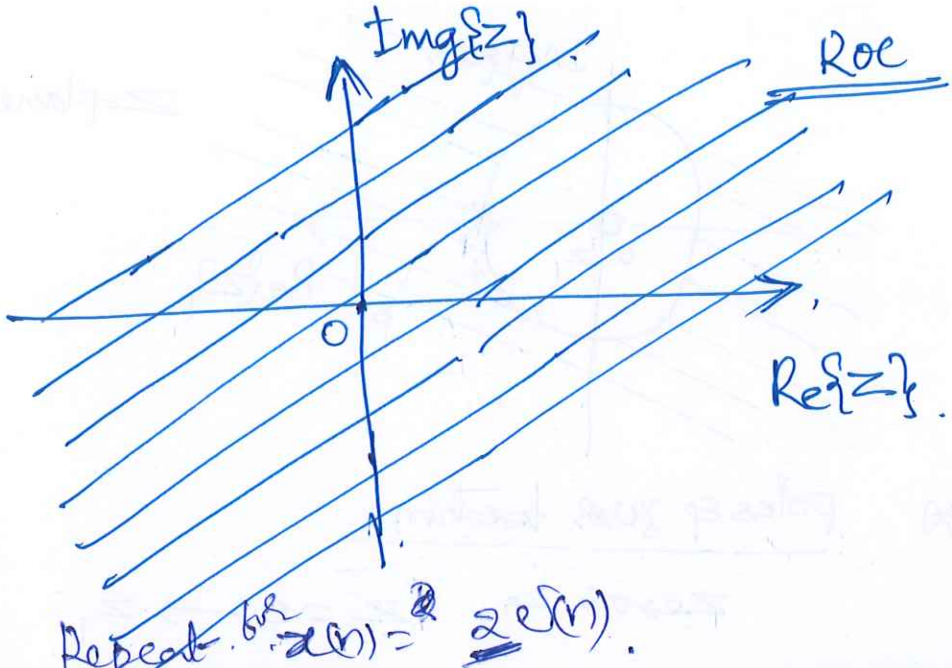
$$\delta(n) = \begin{cases} 1 ; & n=0 \\ 0 ; & n \neq 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$$X(z) = \sum_{n=0} \delta(n) z^{-n} = \delta(0) \cdot z^{0} = 1$$

∴ $\delta(n)$ exist only at $n=0$. [from graph].

∴ ROC ; for All z,, No zeros & poles.



(*) Repeat for $x(n) = 2\delta(n)$.

(**) 6M

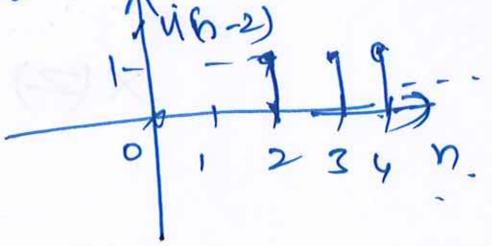
(4) Find Z-Transform of signal $x(n) = \left(\frac{1}{2}\right)^n u(n-2)$ and also its Roc, zeros & poles location.

Soln

$$x(n) = \left(\frac{1}{2}\right)^n u(n-2) \text{ (given)} \rightarrow \text{(Std)}$$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \rightarrow \text{(Std)}$$

$$u(n-2) = \begin{cases} 1 ; & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$



$$X(Z) = \sum_{n=2}^{\infty} \left[\left(\frac{1}{2}\right)^n Z^{-n}\right]$$

$$X(Z) = \sum_{n=2}^{\infty} \left(\frac{1}{2} Z^{-1}\right)^n$$

$$X(Z) = \sum_{n=2}^{\infty} \left(\frac{1}{2} Z^{-1}\right)^n$$

∞ GSF standard formula

$$\sum_{n=k=2}^{\infty} \alpha^k = \frac{\alpha^k}{1-\alpha}$$

$$\boxed{X(Z) = \frac{\left(\frac{1}{2} Z^{-1}\right)^2}{1 - \frac{1}{2} Z^{-1}}} \rightarrow (1)$$

$$\therefore X(z) = \frac{1}{\dfrac{4z^2}{(1-\frac{1}{2}z^{-1})}} = \frac{1}{4z(z-\frac{1}{2})}.$$

$$\boxed{X(z) = \frac{1}{4z(z-\frac{1}{2})}} \rightarrow (2)$$

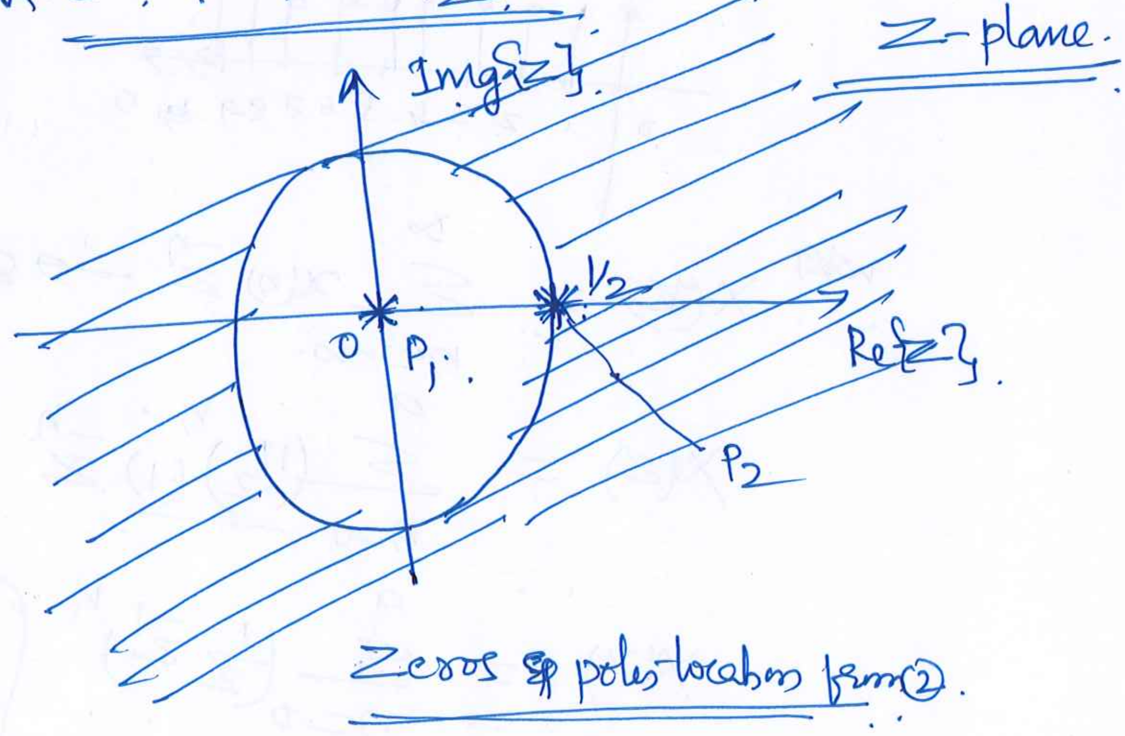
* From eq(1)

$X(z)$ Exists (converges) only when

$$\left|\frac{1}{2}z^{-1}\right| < 1$$

ie $\left|\frac{1}{2}\right| < |z|$

ie ROC : $|z| > 1/2$



Zeros & poles locations from (2).

* → No zeros.

* poles → $z=0$ → P_1 → *

$z - 1/2 = 0$

$z = 1/2$ → P_2 → *

(5) Repeat for $x(n) = \left(\frac{1}{2}\right)^n u(n-3)$

b) Find 'ZT, ROC & plot zeros-poles locations.

for the ~~given~~ signal

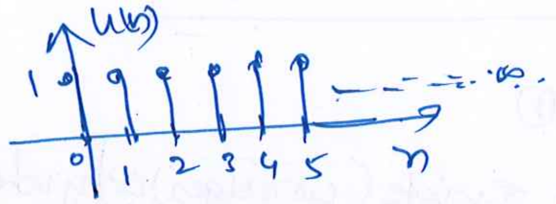
$$x(n) = \left(\frac{1}{2}\right)^n \{u(n) - u(n-10)\}.$$

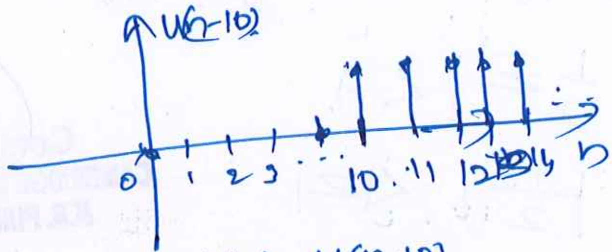
Soln

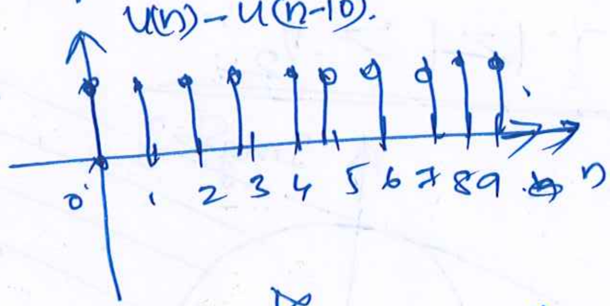
Given $x(n) = \left(\frac{1}{2}\right)^n \{u(n) - u(n-10)\}$

$$\left[u(n) - u(n-10) = \{\underset{\uparrow\, n=0}{1}, 1, 1, 1, 1, 1, 1, 1, 1, 1\}. \right]$$

WKT







WKT $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \longrightarrow$ std

$$X(z) = \sum_{n=0}^{9} (1/2)^n (1) z^{-n}.$$

$$X(z) = \sum_{n=0}^{9} \left(\frac{1}{2} z^{-1}\right)^n$$

$$\left[\therefore \text{FS} \quad \sum_{n=0}^{N-1} \alpha^n = \frac{1 - \alpha^N}{1 - \alpha}. \right]$$

* . Apply Finite sum formula (FSF.)

$$\left[\sum_{n=0}^{N-1} \alpha^n = \begin{cases} \dfrac{1 - \alpha^N}{1 - \alpha}, & \text{if } \alpha \neq 1 \\ N, & \text{if } \alpha = 1 \end{cases} \right]$$

N = 10

$1^0 + 1^1 + 1^2 + 1^3 + 1^4 + \cdots 1^9 = 10 = N.$

$$X(z) = \frac{1-\left(\frac{1}{2}z^{-1}\right)^{10}}{1-\frac{1}{2}z^{-1}} \rightarrow ①$$

ie
$$X(z) = \frac{1-\left(\frac{1}{2}\right)^{10}\cdot\frac{1}{z^{10}}}{\left(1-\frac{1}{2}z^{-1}\right)}$$

$$\therefore X(z) = \frac{\frac{1}{z^{10}}\left[z^{10}-\left(\frac{1}{2}\right)^{10}\right]}{\left(1-\frac{1}{2}z^{-1}\right)}$$

[see Dr power of z^{-1} $\rightarrow z^{1}$]

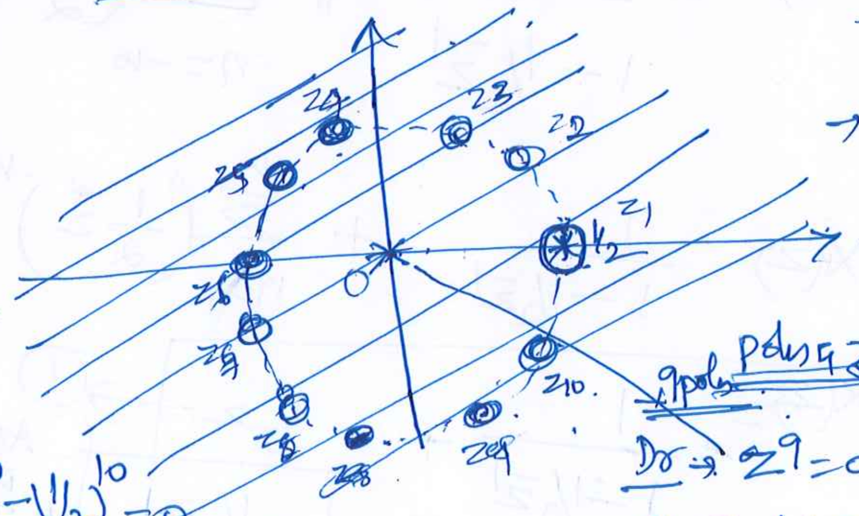
[Multiply & Divide by z]

$$\boxed{X(z) = \frac{1\left[z^{10}-\left(\frac{1}{2}\right)^{10}\right]}{z^{9}\cdot\left(z-\frac{1}{2}\right)}} \rightarrow ②$$

From eq ①, we get.

ROC: All z plane except $z=0$,

[even $z=\infty$. $X(z)$ converges.]



Total, 10 zeros, 9 poles + 1 pole

Poles & zeros.

9 poles

Nr $\rightarrow z^{10}-\left(\frac{1}{2}\right)^{10}=0$

$z^{10}=\left(\frac{1}{2}\right)^{10} \rightarrow$ 10 zeros at a range $\frac{1}{2}$ on the circle.

Dr $\rightarrow z^{9}=0$, $z=0 \rightarrow * \rightarrow$ 9 poles

$z-\frac{1}{2}=0$.

$z=\frac{1}{2} \rightarrow P_2 \rightarrow *$

*7) Repeat for $x(n) = \left(\frac{1}{2}\right)^n \left[u(n) - u(n-8)\right]$

**8) Find Z-Transform of $x(n) = \left(\frac{1}{2}\right)^{|n|}$ find its ROC.

soln Given: $x(n) = \left(\frac{1}{2}\right)^{|n|}$. [whenever $|n|$ comes]

** it is written as +ve & -ve n with unit step.

$$\therefore x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{2}\right)^{-n} u(-n-1).$$

WKT $X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$ — (std.)

$$X(Z) = \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{2}\right)^{-n} u(-n-1)\right] Z^{-n}.$$

$$X(Z) = \sum_{n=0}^{\infty} \left(\frac{1}{2} Z^{-1}\right)^n + \sum_{n=-\infty}^{-1} \left[\left(\frac{1}{2}\right)^{-1} Z^{-1}\right]^n.$$

$$X(Z) = \frac{1}{1 - \frac{1}{2} Z^{-1}} + \sum_{n=-\infty}^{-1} \left(\frac{1}{2} Z\right)^{-n}.$$

$$X(Z) = \frac{1}{1 - \frac{1}{2} Z^{-1}} + \sum_{n=1}^{\infty} \left(\frac{1}{2} Z\right)^n$$

Formula
$$\left[\therefore \sum_{n=-\infty}^{-1} \alpha^{-n} = \sum_{n=1}^{\infty} \alpha^n\right]$$

$$\boxed{X(Z) = \frac{1}{1 - \frac{1}{2} Z^{-1}} + \frac{\frac{1}{2} Z}{1 - \frac{1}{2} Z}} \rightarrow (1)$$

$$\left[\therefore \text{Apply ISF from 1. } \sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha}\right]$$

$$\therefore X(Z) = \frac{1 - \frac{1}{2}z + \frac{1}{2}z\left(1 - \frac{1}{2}z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{2}z\right)}$$

$$X(Z) = \frac{1 - \cancel{\frac{1}{2}z} + \cancel{\frac{1}{2}z} - \frac{1}{4}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{2}z\right)}$$

$$X(Z) = \frac{3/4}{\left(1 - \frac{1}{2}z^{-1}\right)\left(-\frac{1}{2}\right)\left[z - 2\right]}$$

* Mult & Divide by z.

$\cancel{X(Z)}$

$$X(Z) = -\frac{\frac{3}{\cancel{4}_2} z \cdot \cancel{2}}{\left(z - \frac{1}{2}\right)\left(z - 2\right)}$$

$$\boxed{X(Z) = \frac{-\frac{3}{2} z}{\left(z - \frac{1}{2}\right)\left(z - 2\right)}} \rightarrow (2)$$

$\therefore$ From eq(1). Apply condn

$X(Z)$ exist. if

$\left|\frac{1}{2}z^{-1}\right| < 1$ & $\left|\left(\frac{1}{2}z\right)\right| < 1$

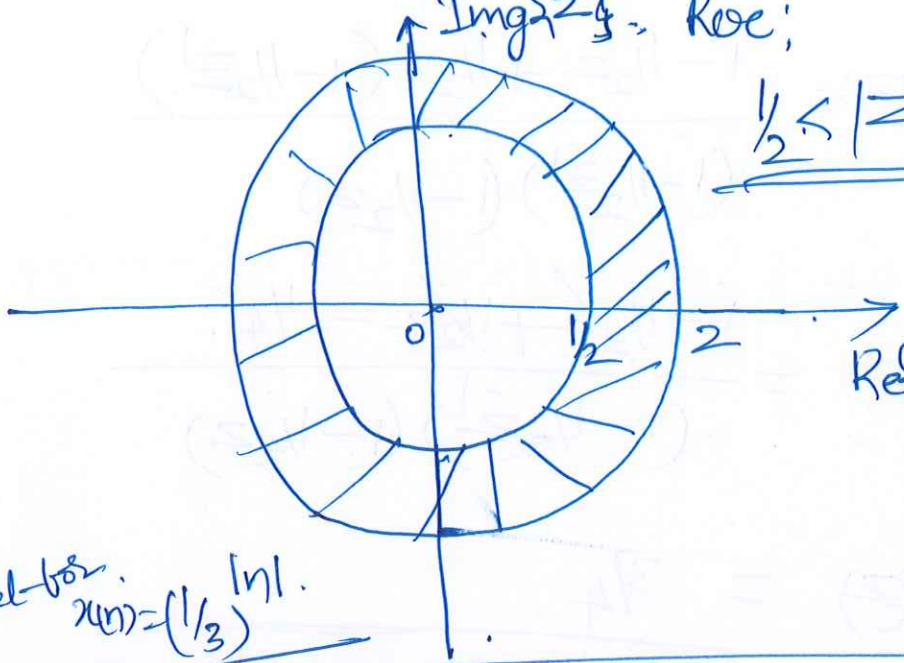
$\left|\frac{1}{2}\frac{1}{z}\right| < 1$ $\quad$ 1ly $\quad |z| < 2$

$\therefore \frac{1}{2} < |z|$

$\therefore |z| > \frac{1}{2}$ $\quad \therefore$ ROC; $\frac{1}{2} < |z| < 2$

Roc;

$\frac{1}{2} < |Z| < 2$



9) Repeat for $x(n) = \left(\frac{1}{3}\right)^{|n|}$.

Properties of Z-Transform.

* Z Transform has many properties, which are useful to study discrete-time signals & systems Effectively.

1) Linearity.

2) Time shifting.

3) Scaling in Z Domain.

4) Time Reversal.

5) convolution.

6) Differentiation in Z domain.. Etc.

8M ** 1) Linearity

~~statement~~ If $x_1(n) \xleftrightarrow{Z} X_1(Z)$ with $ROC = R_1$

& $x_2(n) \xleftrightarrow{Z} X_2(Z)$ with $ROC = R_2$

then $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{Z} a_1 X_1(Z) + a_2 X_2(Z)$

with ROC atleast $R_1 \cap R_2$

Proof:

WKT $Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \rightarrow$ (def)

$$Z\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] z^{-n}.$$

$$Z\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) z^{-n}.$$

$$\therefore Z\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}.$$

$$= a_1 X_1(Z) + a_2 X_2(Z) = \text{RHS}.$$

Hence the proof

② Time shifting

If $x(n) \xleftrightarrow{Z} X(Z)$ with $ROC = R$

then $x(n-n_0) \xleftrightarrow{Z} Z^{-n_0} \cdot X(Z)$ with $ROC = R$

Proof $Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ → x(std).

$$Z\{x(n-n_0)\} = \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n} \rightarrow ①.$$

* sub $n-n_0 = l$. then limits changes from $n \to l$.

i.e.

$$Z\{x(n-n_0)\} = \sum_{l=-\infty}^{\infty} x(l) z^{-(l+n_0)}$$

i.e. go to ①.

When $n = -\infty$. then $l = \cancel{\infty}\, \infty - n_0$, $l = -\infty$.

$n = \infty$. $l = \infty - n_0$, $l = \infty$.

∴ new limits are $\sum_{l=-\infty}^{\infty}$

$$\therefore Z\{x(n-n_0)\} = \sum_{l=-\infty}^{\infty} x(l) z^{-l} \cdot [z^{-n_0}].$$

$$\therefore Z\{x(n-n_0)\} = z^{-n_0} \cdot X(z) \quad \left[\because X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}\right]$$

Hence the proof

3) Scaling in the Z-Domain prop.

Stat. If $x(n) \xleftrightarrow{Z} X(z)$ with ROC $= R$

then $\alpha^n x(n) \xleftrightarrow{Z} X\left(\frac{z}{\alpha}\right)$ with ROC $= |\alpha| R$

where α is a complex no.

Proof. WKT $Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$ → (std)

$$Z\{\alpha^n x(n)\} = \sum_{n=-\infty}^{\infty} [\alpha^n x(n)] . Z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} x(n) \alpha^n Z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} x(n) (\alpha^{-1} . Z)^{-n}$$

$$Z\{\alpha^n x(n)\} = X(\alpha^{-1} Z)$$

$\left[\because \text{ WKT } X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}\right]$

$$\therefore = X\left(\frac{Z}{\alpha}\right)$$

i.e. $(e^{j\Omega})^n x(n) \xleftrightarrow{Z} X(e^{-j\Omega} . Z).$

(4) Time Reversal Property.

Stat. If $x(n) \xleftrightarrow{Z} X(Z)$ with ROC $= R$.

& $x(-n) \xleftrightarrow{Z} X\left(\frac{1}{Z}\right)$ with ROC $= \frac{1}{R}$

Proof. $Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$ → (std)

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) Z^{-n}$$

sub $l = -n$.

$\therefore \left[\text{when } n = -\infty \quad \& \quad n = \infty \right.$
$\left. l = -(-\infty) = +\infty \quad l = -\infty \right]$

$$\therefore Z\{x(-n)\} = \sum_{l=\infty}^{-\infty} x(l)\, z^{l}$$

$$= \sum_{l=-\infty}^{\infty} x(l)\, z^{l}$$

$\left[\because \sum_{l=\infty}^{-\infty} x(l) z^{l} = \sum_{l=-\infty}^{\infty} x(l) z^{l} = \right]$

$$Z\{x(-n)\} = \sum_{l=-\infty}^{\infty} x(l) (z^{-1})^{-l}$$

$\left[\because \sum_{l=-\infty}^{\infty} x(l) (z^{-1})^{-l} = X(z^{-1})\right]$

$$\therefore \boxed{Z\{x(-n)\} = X(z^{-1}) = X\left(\frac{1}{z}\right)} \rightarrow \text{Hence the proof}$$

6M:

(5) Convolution property.

stat: if $x(n) \xleftrightarrow{Z} X(z)$ with ROC = R_1

& $y(n) \xleftrightarrow{Z} Y(z)$ with ROC = R_2

then $x(n) * y(n) \xleftrightarrow{Z} X(z) \cdot Y(z)$ with ROC atleast $R_1 \cap R_2$.

Proof: WKT ~~...~~ (LC).

$$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(n-k) \rightarrow \text{[Std Formula] [2nd Module]}$$

$$\therefore Z[x(n) * y(n)] = \sum_{n=-\infty}^{\infty} [x(n) * y(n)]\, z^{-n}.$$

ie

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)\, z^{-n}$$

Interchange the order of summation,

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k) z^{-n}$$

sub $n-k=l$;

$$\left[\text{we get when } n=-\infty \text{ \& } n=\infty \atop l=-\infty . \quad l=\infty . \right]$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{l=-\infty}^{\infty} y(l) z^{-(l+k)}$$

$$\therefore Z\{x(n) * y(n)\} = \underline{\sum_{k=-\infty}^{\infty} x(k) z^{-k}} \cdot \underline{\sum_{l=-\infty}^{\infty} y(l) z^{-l}}$$

$$Z[x(n) * y(n)] = X(Z) \cdot Y(Z) = RHS.$$

* 6) **Accumulation property**

Stat: If $x(n) \xleftrightarrow{Z} X(Z)$ with ROC $= R$

then PT $\sum_{k=-\infty}^{n} x(k) \xleftrightarrow{Z} \dfrac{X(Z)}{1-z^{-1}}$ with ROC at least $= R \cap [|z|>1]$

Proof: WKT convolution of signal $x(n)$ with unit step fn $u(n)$ is given by [Module 2]

$$x(n) * u(n) = \sum_{k=-\infty}^{\infty} x(k) . u(n-k) \longrightarrow (1)$$

Also WKT $u(n-k) = \begin{cases} 1 & ; n-k \geq 0 \text{ ie } n \geq k \text{ or } \underline{k \leq n} \\ 0 & ; n-k < 0 \text{ ie } n < k \text{ or } \underline{k > n} \end{cases}$

Sub These in eq ① we get-

$$x(n) * u(n) = \sum_{k=-\infty}^{n} x(k)(1) \rightarrow ②$$

Take Z-Transform on both sides.

$$Z\{x(n) * u(n)\} = Z\left\{\sum_{k=-\infty}^{n} x(k)\right\}$$

$$\therefore Z\left[\sum_{k=-\infty}^{n} x(k)\right] = X(Z) \cdot U(Z)$$

[∵ WKT. conv-prop $x(n) * u(n) \overset{Z}{\longleftrightarrow} X(Z) \cdot U(Z)$]

$$= X(Z) \cdot \frac{1}{1-Z^{-1}}$$

[$Z\{u(n)\} = U(Z) = \frac{1}{1-Z^{-1}}$]

$$\therefore \sum_{k=-\infty}^{n} x(k) \overset{Z}{\longleftrightarrow} \frac{X(Z)}{1-Z^{-1}} = RHS.$$

Hence the proof.

⑦ Differentiation in the Z-Domain.

6M.

Stat. If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$ with ROC = R

then $n\,x(n) \overset{Z}{\longleftrightarrow} -Z\dfrac{d[X(Z)]}{dZ}$ with ROC = R

Proof: WKT $Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \rightarrow$ std

Differentiating both side wrt Z, we get-

$$\frac{d[X(Z)]}{dZ} = \sum_{n=-\infty}^{\infty} x(n)(-n) Z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} n\,x(n)\, z^{-n} \cdot z^{-1}$$

$$= -z^{-1} \sum_{n=-\infty}^{\infty} n\,x(n)\, z^{-n}.$$

$$= \frac{-1}{z} \sum_{n=-\infty}^{\infty} [n\,x(n)]\, z^{-n}$$

$$\therefore\ -z\frac{d[X(z)]}{dz} = \sum_{n=-\infty}^{\infty} (n\,x(n))\, z^{-n}.$$

$$= Z\{n\,x(n)\}.$$

[$\because$ WKT. $X[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$]

$$\boxed{\therefore\ n\,x(n) \xleftrightarrow{Z} -z\frac{d[X(z)]}{dz}.}$$

Hence the proof.

Z-Transform of Basic signals.

Sl. No	Signal.	Z-Transforms	Roc
1.	$\delta(n)$	1.	All z. points.
2	$\delta(n-k)$	z^{-k}.	All z. Except 0.
3	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\|>1$.
4.	$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\|z\|<1$.

Sl No	Signal	Z-Transforms	ROC
5	$n\,u(n)$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$.
6	$-n\,u(-n-1)$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert < 1$
7	$\alpha^n u(n)$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
8	$-\alpha^n u(-n-1)$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
9.	$\alpha^{\lvert n\rvert}$; $\lvert\alpha\rvert<1$	$\frac{1-\alpha^2}{(1-\alpha z^{-1})(1-\alpha z)}$	$\lvert\alpha\rvert < \lvert z\rvert < \frac{1}{\lvert\alpha\rvert}$
10.	$n\alpha^n u(n)$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
11.	$-n\alpha^n u(-n-1)$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
12.	$\cos(\Omega_0 n)\,u(n)$	$\frac{1-(\cos\Omega_0)z^{-1}}{1-2\cos\Omega_0 z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$.
~~13~~	~~$\sin(\Omega_0 n)\,u(n)$~~	$\cancel{\frac{\sin\Omega_0 z^{-1}}{1-2\cos\Omega_0 z^{-1}+z^{-2}}}$	$\cancel{\lvert z\rvert}$
13	$\sin(\Omega_0 n)\,u(n)$	$\frac{(\sin\Omega_0)z^{-1}}{1-2\cos\Omega_0 z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$.

Problems [using prop^s, Eq. Table, Z-T formula]

1) Find Z-Transform of the signal $x(n) = u(-n)$.

Soln. given. $x(n) = u(-n)$.

WKT $u(n) \xleftrightarrow{Z} \dfrac{1}{1-z^{-1}}$ $|z|>1$

using Time reversal property, [ie sub. $z = \frac{1}{z}$]

$$u(-n) \xleftrightarrow{Z} \frac{1}{1-(1/z)^{-1}}$$

ie $$u(-n) \xleftrightarrow{Z} \frac{1}{1-(z^{-1})^{-1}}$$

$$\boxed{u(-n) \xleftrightarrow{Z} \frac{1}{1-z}}$$ ROC. $|z|<1$.

2). Find Z-Transform of the signal $x(n) = 3(2)^n u(-n)$ using the appropriate properties.

Soln Given $x(n) = 3(2)^n u(-n)$.

WKT. $\left(\frac{1}{2}\right)^n$ this can be written as

$$x(n) = 3\left(\frac{1}{2}\right)^{-n} u(-n)$$

WKT. $\left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{Z} \dfrac{1}{1-\frac{1}{2}z^{-1}}$; $|z|>1/2$

* Using time reversal property.

$$\left(\frac{1}{2}\right)^{-n} u(-n) \xleftrightarrow{Z} \frac{1}{1-\frac{1}{2}z} \quad |z|<2$$

ie. $\left.\frac{1}{1-\frac{1}{2}z^{-1}}\right|_{z=1/z}^{\text{sub}} = \frac{1}{1-\frac{1}{2}\left(\frac{1}{z}\right)^{-1}}$

$$= \frac{1}{1-\frac{1}{2}\left(z^{-1}\right)^{-1}} = \frac{1}{1-\frac{1}{2}z}$$

ie $\frac{1}{2}z < 1$, $\therefore$ $z < 2$ $\therefore$ ROC

* Using linearity property.

$$3\left(\frac{1}{2}\right)^{-n} u(-n) \xleftrightarrow{Z} \frac{3}{1-\frac{1}{2}z} \; ; |z|<2.$$

---

* (3) Repeat. $x(n) = 4(3)^n u(-n)$..

(A*) 6M (4) Find Z-Transform of $x(n) = n\alpha^n u(n)$ using appropriate properties.

Soln

wkt $u(n) \xleftrightarrow{Z} \frac{z}{z-1} \quad |z|>1$

Using scaling in Z-Domain prop.

$$\alpha^n u(n) \xleftrightarrow{Z} \frac{\frac{z}{\alpha}}{\left(\frac{z}{\alpha}-1\right)} = \frac{\frac{z}{\cancel{\alpha}}}{\frac{z-\alpha}{\cancel{\alpha}}}$$

$$\xleftrightarrow{Z} \frac{z}{z-\alpha} \; ; |z|>|\alpha|$$

* Using Differentiation in the z-domain prop.

$$n\alpha^n u(n) \xleftrightarrow{Z} -z \frac{d\left(\frac{z}{z-\alpha}\right)}{dz}$$

$$\xleftrightarrow{Z} \frac{z\alpha}{(z-\alpha)^2}$$

$$\cancel{\text{i.e.} \left[\because \frac{z(1)-(z-\alpha)(1)}{(z-\alpha)^2} = \frac{z-z+\alpha}{(z-\alpha)^2} = \frac{\alpha}{(z-\alpha)^2} \right]}$$

$$\therefore n\alpha^n u(n) \xleftrightarrow{Z} -z \left[\frac{-\alpha}{(z-\alpha)^2} \right]$$

$$\left[\because \frac{(z-\alpha)(1) - z(1)}{(z-\alpha)^2} = \frac{z-\alpha-z}{(z-\alpha)^2} = \frac{-\alpha}{(z-\alpha)^2} \right]$$

$$\therefore n\alpha^n u(n) \xleftrightarrow{Z} \frac{z\alpha}{(z-\alpha)^2} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}, \quad |z|>|\alpha|$$

Repeat for $x(n) = \alpha \beta^n u(n)$ using appropriate prop^s

4) 6M (5) Using appropriate properties, find Z-Transform of

$$x(n) = n^2 \left(\frac{1}{2}\right)^n u(n-3).$$

Soln given.

$$x(n) = n^2 \left(\frac{1}{2}\right)^n u(n-3)$$

this can be written as

$$x(n) = n^2 \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{n-3} u(n-3)$$

$$x(n) = \frac{1}{8} \cdot n^2 \left(\frac{1}{2}\right)^{n-3} \cdot u(n-3).$$

WKT. [From Table]

$$\left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{z} \frac{1}{1-\frac{1}{2}z^{-1}} ; |z| > \frac{1}{2}$$

* Using Time shift prop.

$$\left(\frac{1}{2}\right)^{n-3} u(n-3) \xleftrightarrow{z} \frac{z^{-3}}{1-\frac{1}{2}z^{-1}} ; |z| > \frac{1}{2} \text{ except } z=0$$

$$\xleftrightarrow{z} \frac{1}{z^3(1-\frac{1}{2}z^{-1})}$$

$$\xleftrightarrow{z} \frac{1}{z^3 - \frac{1}{2}z^2}.$$

→ using Differentiation in z-domain prop.

$$\boxed{d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}}$$

$$n \cdot \left(\frac{1}{2}\right)^{n-3} \cdot u(n-3) \xleftrightarrow{z} -z \frac{d}{dz}\left[\frac{1}{z^3 - \frac{1}{2}z^2}\right] \text{(is missing)}$$

$$\xleftrightarrow{z} \frac{3z^3 - z^2}{\left(z^3 - \frac{1}{2}z^2\right)^2}.$$

* Using differentiation again in z-domain

$$n^2 \left(\frac{1}{2}\right)^{n-3} u(n-3) \xleftrightarrow{z} -z \frac{d}{dz}\left[\frac{3z^3 - z^2}{(z^3 - \frac{1}{2}z^2)^2}\right]$$

$$\xleftrightarrow{Z} \frac{z^4(9z^2 - 5.5z + 1)}{(z^3 - 1/2\, z^2)^4}$$

*∴ Using linearity property.

$$x(n) = \frac{1}{8} n^2 \left(\frac{1}{2}\right)^{n-3} u(n-3) \xleftrightarrow{Z} \frac{1}{8} \frac{z^4(9z^2 - 5.5z + 1)}{(z^3 - 1/2\, z^2)^4}$$

5* Repeat. $x(n) = n^2 \left(\frac{1}{3}\right)^n u(n-3)$

---

6) Find Z-Transform of the signal.

$$x(n) = n \sin\left(\frac{\pi}{2} n\right) u(-n).$$

Soln

Given $x(n) = n \sin(\pi/2\, n)\, u(-n)$.

9606263919 (up)

* this can be written as.

$$x(n) = -n \sin(-\pi/2\, n)\, u(-n).$$

[From Table]

* WKT $\sin(\pi/2\, n).u(n) \xleftrightarrow{Z} \frac{\sin(\pi/2) z^{-1}}{1 - 2\cos(\pi/2) z^{-1} + z^{-2}}, |z| > 1.$

$$\xleftrightarrow{Z} \frac{(1) z^{-1}}{1 - 2(0) z^{-1} + z^{-2}}; |z| > 1$$

$$\xleftrightarrow{Z} \frac{z^{-1}}{1 + z^{-2}}$$

$$\xleftrightarrow{Z} \frac{z}{z^2 + 1}.$$

* Using Differentiation in Z Domain prop.

$$n \sin(\pi/2\, n)\, u(n) \xleftrightarrow{Z} -z \frac{d}{dz}\left[\frac{z}{z^2+1}\right]$$

$$\xleftrightarrow{Z} \frac{\cancel{z^2}\cdot z}{(z^2+1)^2}$$

$$\left[-z\,\frac{(z^2+1)(1) - z(2z)}{(z^2+1)^2} \right.$$

$$= \frac{-z(z^2+1-2z^2)}{(z^2+1)^2} = \frac{z^3-z}{(z^2+1)^2}$$

*: Using Time reversal property.

$$-n\sin(-\pi/2\,n)\,u(-n) \xleftrightarrow{Z} \left.\frac{z^3-z}{(z^2+1)^2}\right|_{z=1/z} ; |z|<1$$

$$\longleftrightarrow \frac{z(1-z^2)}{(z^2+1)^2}$$

$$\frac{\left(\frac{1}{z}\right)^3 - \frac{1}{z}}{\left[\left(\frac{1}{z}\right)^2+1\right]^2} = \frac{\frac{1}{z^3}-\frac{1}{z}}{\left[\frac{1}{z^2}+1\right]^2}$$

$$= \frac{\frac{1-z^2}{z^3}}{\left[\frac{1+z^2}{z^2}\right]^2}$$

$$= \frac{1-z^2}{z^3} \Big/ \frac{(1+z^2)^2}{z^4}$$

$$= \frac{z(1-z^2)}{(1+z^2)^2}$$

7) Using appropriate prop, find Z-Transform of

$$x(n) = \alpha^n \sin(\Omega_0 n)\, u(n)$$

Soln WKT

$$\sin(\Omega_0 n) \cdot u(n) \xleftrightarrow{Z} \frac{(\sin \Omega_0) z^{-1}}{1 - (2\cos\Omega_0) z^{-1} + z^{-2}} ; |z| > 1$$

* Using Scaling in Z domain property

$$\xleftrightarrow{Z} \frac{(\alpha \sin\Omega_0) z^{-1}}{1 - (2\alpha\cos\Omega_0) z^{-1} + \alpha^2 z^{-2}} ; |z| > |\alpha|$$

	Signal	Z-Transform	ROC
14)	$\alpha^n \cos(\Omega_0 n)\, u(n)$	$\frac{1 - (\alpha\cos\Omega_0) z^{-1}}{1 - (2\alpha\cos\Omega_0) z^{-1} + \alpha^2 z^{-2}}$	$\lvert z \rvert > \lvert\alpha\rvert$
15)	$\alpha^n \sin(\Omega_0 n)\, u(n)$	$\frac{(\alpha \sin\Omega_0) z^{-1}}{1 - (2\alpha\cos\Omega_0) z^{-1} + \alpha^2 z^{-2}}$	$\lvert z \rvert > \lvert\alpha\rvert$

*** 6M

8) Find Z-Transform of the signal.

$$x(n) = \left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)$$

Soln Let $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$ $\therefore X_1(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$, $|z| > 1/2$

Illy $x_2(n) = \left(\frac{1}{3}\right)^n u(n) \therefore X_2(z) = \frac{1}{1-\frac{1}{3}z^{-1}}, |z|>\frac{1}{3}$

$\therefore x(n) = x_1(n) * x_2(n)$.

using convolution property.

$x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(z) \cdot X_2(z)$.

$\therefore \left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n) \xleftrightarrow{Z} \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}, |z|>\frac{1}{2}$

---

9) Repeat for $x(n) = \left(\frac{1}{3}\right)^n u(n) * \left(\frac{1}{4}\right)^n u(n)$.

Asg.

10.) Find Z-T of the signal..

$x(n) = n\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{2}\right)^n u(n)\right]$

Soln * using conv^n property

$\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{Z} \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1})} \quad |z|>\frac{1}{2}$

$\xleftrightarrow{Z} \frac{1}{(1-\frac{1}{2}z^{-1})^2}$

* using Differentiation in Z-Domain prop.

$n\left[(1/2)^n u(n) * \left(\frac{1}{2}\right)^n u(n)\right] \xleftrightarrow{Z} -z\frac{d}{dz}\left[\frac{1}{(1-\frac{1}{2}z^{-1})^2}\right]$

$$\xleftrightarrow{Z} \frac{z^{-1}}{(1-\frac{1}{2}z^{-1})^{2}} \quad |z|>\frac{1}{2}.$$

Inverse Z - Transform

* We can recover time domain signal $x(n)$ from its $X(Z)$ [Z-transform] by using given eqn below.

i.e. $$x(n) = \frac{1}{2\pi j}\oint X(Z)\, Z^{n-1}\, dz \rightarrow (Std)$$

» But it is very difficult to carry out this integration.

∴ The 2 alternative techniques to get Inverse-Z Transform are.

1) ~~power~~ partial fraction expansion method
2) power series Expansion method.

partial fraction expansion method

» To get IZT, we use basic Z-Transform pairs & different properties of Z-Transform.

i.e let us write $X(Z) = \frac{B(Z)}{A(Z)}$

$$X(Z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_N z^{-N}}.$$

* If $M < N$, we do partial fraction expansion directly by factorizing the denominator polynomial.

* If $M \geq N$, then we do PFE by long division method.

ie $$X(z) = \sum_{k=0}^{M-N} c_k z^{-k} + \frac{\bar{B}(z)}{A(z)}$$

where Numerator polynomial $\bar{B}(z)$ has order one less than that of denominator polynomial $A(z)$.

problem.

8M

1) Find discrete-time sequence $x(n)$ which has Z-Transform

$$X(z) = \frac{-1 + 5z^{-1}}{1 - 3/2\, z^{-1} + 1/2\, z^{-2}} \quad \text{with } |z| > 1.$$

Soln. Given: $$X(z) = \frac{-1 + 5z^{-1}}{1 - 3/2\, z^{-1} + 1/2\, z^{-2}} \quad |z| > 1$$

* Factorize the Denominator it is of the form $ax^2 + bx + c$.

$$\therefore X(z) = \frac{-1 + 5z^{-1}}{(1 - 1/2\, z^{-1})(1 - z^{-1})} \quad \left[\because \text{Maths}\right]$$

* Here $M = 1$, & $N = 2$. $M < N$, $\Rightarrow$ Expand $X(z)$ by using DPFE method.

ie

$$\therefore X(z) = \frac{K_1}{(1-1/2 z^{-1})} + \frac{K_2}{1-z^{-1}}$$

$$\therefore K_1 = (1-1/2 z^{-1}) X(z) \Big|_{z=1/2}$$

$$K_1 = \frac{\cancel{(1-1/2 z^{-1})}(-1+5z^{-1})}{\cancel{(1-1/2 z^{-1})}(1-z^{-1})}\Big|_{z=1/2}$$

$$\therefore K_1 = \frac{-1+5z^{-1}}{1-z^{-1}}\Big|_{z=1/2}$$

$$\therefore K_1 = \frac{-1+5(1/2)^{-1}}{1-(1/2)^{-1}} = \frac{-1+5(2)^{1}}{1-2}$$

$$\therefore K_1 = \frac{9}{-1} = \underline{-9}$$

IIly

$$K_2 = (1-z^{-1}) X(z) \Big|_{z=1}$$

$$K_2 = \frac{\cancel{(1-z^{-1})}(-1+5z^{-1})}{(1-1/2 z^{-1})\cancel{(1-z^{-1})}}\Big|_{z=1}$$

$$K_2 = \frac{-1+5z^{-1}}{1-1/2 z^{-1}}\Big|_{z=1}$$

$$K_2 = \frac{-1+5(1)^{-1}}{1-1/2(1)^{-1}} = \frac{-1+5(1)}{1-1/2(1)} = \frac{4}{1/2}$$

$$\therefore K_2 = \underline{8}$$

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$\therefore \quad X(z) = \dfrac{-9}{1-\frac{1}{2}z^{-1}} + \dfrac{8}{1-z^{-1}}$

× * From given problem, wkt ~~ROC~~ $X(z)$ has ROC $|z|>1$, ROC exists outside the circle of radius 1.

$\therefore$ both the terms are RSS. *

~~From~~ IZT of $X(z)$ from table 5. *

* . $x(n) = -9\left(\frac{1}{2}\right)^n u(n) + 8(1)^n u(n)$

~~$x(n) = \dots$~~

(2) Repeat for $X(z) = \dfrac{-1+4z^{-1}}{1-\frac{5}{2}z^{-1}+\frac{3}{2}z^{-2}}$.

$\left[\text{Hint } X(z) = \dfrac{-1+4z^{-1}}{(1-\frac{3}{2}z^{-1})(1-z^{-1})}\right]$

** 8M.
(3) Using partial fraction expansion method, Determine $x(n)$ corresponds to ~~x~~ z-Transforms.

a) $X(z) = \dfrac{\frac{1}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}, \quad |z|>\frac{1}{2}$.

b) $X(z) = \dfrac{\frac{1}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} \quad |z|<\frac{1}{4}$

c) $X(z) = \dfrac{\frac{1}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} \quad \frac{1}{4}<|z|<\frac{1}{2}$

Soln. Here $M=1$, & $N=2$, $[M<N \rightarrow \text{D.PFEM}]$.

ie $X(z) = \dfrac{k_1}{(1-1/2 z^{-1})} + \dfrac{k_2}{(1-1/4 z^{-1})} \rightarrow ①$

$\therefore k_1 = (1-1/2 z^{-1}) X(z) \Big|_{z=1/2}$

$k_1 = \dfrac{\cancel{(1-1/2 z^{-1})} (1/4 z^{-1})}{\cancel{(1-1/2 z^{-1})}(1-1/4 z^{-1})} \Bigg|_{z=1/2}$

$\therefore k_1 = \dfrac{1/4 (\frac{1}{2})^{-1}}{1-1/4(1/2)^{-1}} = \dfrac{\frac{1}{\cancel{4}}(\cancel{2})}{1-\frac{1}{\cancel{4}}(\cancel{2})}$

$\therefore k_1 = \dfrac{1/2}{\frac{2-1}{2}} = \cancel{1/2} / \cancel{1/2} = 1$

|||ly $k_2 = (1-1/4 z^{-1}) X(z) \Big|_{z=1/4}$

$= \dfrac{\cancel{(1-1/4 z^{-1})} \cdot (1/4 z^{-1})}{(1-1/2 z^{-1}) \cancel{(1-1/4 z^{-1})}} \Bigg|_{z=1/4}$

$k_2 = \dfrac{1/4 z^{-1}}{1-1/2 z^{-1}} \Bigg|_{z=1/4}$

$k_2 = \dfrac{1/4 (1/4)^{-1}}{1-1/2(1/4)^{-1}} = \dfrac{\frac{1}{\cancel{4}}(\cancel{4})}{1-1/\cancel{2}(\cancel{4})_2} = \dfrac{1}{-1}$

$\therefore k_2 = -1$

$\therefore X(z) = \dfrac{1}{1-1/2 z^{-1}} + \dfrac{1}{1-1/4 z^{-1}} \longrightarrow ②$

∴ Since given $|z| > 1/2$, ROC exist outside the circle of radius $1/2$.

∴ Both terms are RSS.

∴ IZT of $X(z)$ is

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)$$

b) Since $|z| < 1/4$, ROC exist inside the circle of radius $1/4$; ∴ both must be LSS (same expn.)

∴ IZT of $X(z)$ is

$$x(n) = -\left(\frac{1}{2}\right)^n u(-n-1) + \left(\frac{1}{4}\right)^n u(-n-1).$$

c) ROC: $1/4 < |z| < 1/2$, ROC exist inside the ring i.e [outside the 1st circle of radius $1/4$ & inside the 2nd circle of radius $1/2$].

∴ both LSS & RSS exist.

IZT of $X(z)$ is

$$x(n) = -\left(1/2\right)^n u(-n-1) - \left(1/4\right)^n u(n)$$

(4) Repeat for

a) $X(z) = \dfrac{1/3\, z^{-1}}{(1 - 1/2\, z^{-1})(1 - 1/3\, z^{-1})}$; $|z| > 1/2$

b) $X(z) = \dfrac{1/3\, z^{-1}}{(1 - 1/2\, z^{-1})(1 - 1/3\, z^{-1})}$; $|z| < 1/3$

c) $X(z) = \dfrac{1/3\, z^{-1}}{(1 - 1/2\, z^{-1})(1 - 1/3\, z^{-1})}$; $1/3 < |z| < 1/2$

(5) Find Z-Transform of the following using Partial fraction expansion method (4.5)

$$X(Z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 3/2 z^{-1} + 1/2 z^{-2}}$$ with ROC $|z| > 1$

Soln. given $X(Z) = \dfrac{1 + 2z^{-1} + z^{-2}}{1 - 3/2 z^{-1} + 1/2 z^{-2}}$.

$\therefore X(Z) = \dfrac{1 + 2z^{-1} + z^{-2}}{(1 - 1/2 z^{-1})(1 - 1 \cdot z^{-1})}$ [Factorize Denominator]

Here $M = 2$ & $N = 2$. [$\because M \not< N$ $\therefore$ 2nd method]

$\therefore$ Bring $X(Z)$ to the Normal form by Long Division method.

$$
\begin{array}{r|l}
 & 2 \\
\hline
\frac{1}{2} z^{-2} - 3/2 z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\
 & z^{-2} - 3z^{-1} + 2 \\
\hline
 & 5z^{-1} - 1
\end{array}
$$

$\therefore X(Z) = 2 + \dfrac{5z^{-1} - 1}{1/2 z^{-2} - 3/2 z^{-1} + 1}$

ie $X(Z) = 2 + \dfrac{5z^{-1} - 1}{(1 - 1/2 z^{-1})(1 - z^{-1})} \longrightarrow (1)$ $\uparrow X_1(Z)$

Now apply PF Expansion for the IInd term ie

$$X_1(Z) = \frac{5z^{-1} - 1}{(1 - 1/2 z^{-1})(1 - z^{-1})}.$$

(46)

$$X_1(z) = \frac{k_1}{(1-\frac{1}{2}z^{-1})} + \frac{k_2}{(1-z^{-1})}$$

$$k_1 = (1-\tfrac{1}{2}z^{-1})X_1(z)\Big|_{z=1/2}$$

$$\therefore k_1 = \frac{\cancel{(1-\frac{1}{2}z^{-1})}(5z^{-1}-1)}{\cancel{(1-\frac{1}{2}z^{-1})}(1-z^{-1})}\Bigg|_{z=1/2}$$

$$k_1 = \frac{5(\frac{1}{2})^{-1}-1}{1-(\frac{1}{2})^{-1}} = \frac{5(2)-1}{1-2} = \frac{9}{-1}$$

$$\therefore k_1 = -9$$

||ly $$k_2 = (1-z^{-1})X_1(z)\Big|_{z=1}$$

$$k_2 = \frac{\cancel{(1-z^{-1})}(5z^{-1}-1)}{\cancel{(1-z^{-1})}(1-\frac{1}{2}z^{-1})}\Bigg|_{z=1}$$

$$k_2 = \frac{5z^{-1}-1}{(1-\frac{1}{2}z^{-1})}\Bigg|_{z=1} = \frac{5(1)^{-1}-1}{1-\frac{1}{2}(1)^{-1}}$$

$$\therefore k_2 = \frac{5-1}{1-\frac{1}{2}} = \frac{4}{\frac{1}{2}} = 8.$$

$$\therefore X(z) = 2^{(1)} + \frac{(-9)}{(1-\frac{1}{2}z^{-1})} + \frac{8}{(1-z^{-1})}.$$

* Since $|z|>1$, ROC exists outside circle of radius 1.
$\therefore$ both terms are RSS only & By using Linearity prop. & table (Basic z-Transform), we get

$$x(n) = 2\delta(n) - 9(\tfrac{1}{2})^n u(n) + 8\,u(n)$$

6) Find the Inverse Z-Transform of

$$X(z) = \frac{z^2 - 3z}{z^2 + 3/2\, z - 1} \; ; \; ROC : \frac{1}{2} < |z| < 2$$

Soln Given. $X(z) = \dfrac{z^2 - 3z}{z^2 + 3/2\, z - 1}$ [∵ No. z^{-1} terms] ∴ ~~Multiply~~

∴ Multiply & Divide both Nr & Dr by z^{-2} [Maxterms power]

$$\therefore X(z) = \frac{1 - 3z^{-1}}{1 + 3/2\, z^{-1} - z^{-2}}$$

$$X(z) = \frac{1 - 3z^{-1}}{(1 + 2z^{-1})(1 - 1/2\, z^{-1})} \quad \text{(Factorization)}$$

Here $M = 1$ & $N = 2$ i.e $M < N$.

$$\therefore X(z) = \frac{K_1}{1 + 2z^{-1}} + \frac{K_2}{(1 - 1/2\, z^{-1})} \rightarrow ①$$

$$\therefore K_1 = (1 + 2z^{-1}) X(z) \Big|_{z = -2}$$

$$K_1 = \frac{\cancel{(1 + 2z^{-1})}(1 - 3z^{-1})}{\cancel{(1 + 2z^{-1})}(1 - 1/2\, z^{-1})} \Bigg|_{z = -2}$$

$$K_1 = \frac{1 - 3z^{-1}}{1 - 1/2\, z^{-1}} \Bigg|_{z=-2} = \frac{1 - 3(-2)^{-1}}{1 - 1/2(-2)^{-1}}$$

$$\therefore K_1 = \frac{1 - \frac{3}{(-2)^1}}{1 - 1/2\left(\frac{1}{-2}\right)^1} = \frac{1 + 3/2}{1 + 1/4} = \frac{5/2}{5/4} = 2$$

$\therefore K_1 = 2$

lly $k_2 = (1-\tfrac{1}{2}z^{-1})X(z)\Big|_{z=1/2}$.

$\therefore k_2 = \left.\dfrac{\cancel{(1-\tfrac{1}{2}z^{-1})}(1-3z^{-1})}{(1+2z^{-1})\cancel{(1-\tfrac{1}{2}z^{-1})}}\right|_{z=1/2}$

$k_2 = \dfrac{1-3(\tfrac{1}{2})^{-1}}{1+2(\tfrac{1}{2})^{-1}} = \dfrac{1-3(2)}{1+2\cdot(2)} = \dfrac{-5}{5}$

$\therefore \underline{k_2 = -1}$.

$\therefore \underline{X(z) = \dfrac{2}{1+2z^{-1}}} + \dfrac{(-1)}{1-\tfrac{1}{2}z^{-1}}$.

Since ROC : $\tfrac{1}{2} < |z| < 2$, corresponding $x(n)$ must be 2 sided. i.e. IZT of $X(z)$ is

suitable form

$$\left[\,X(z) = \frac{2}{1-(-2)z^{-1}} + \frac{(-1)}{1-\tfrac{1}{2}z^{-1}}\,\right]$$

$\therefore \boxed{x(n) = -2(-2)^n u(-n-1) - (\tfrac{1}{2})^n u(n)}$

From Table w.k.t

$-(\alpha)^n u(-n-1) = \dfrac{1}{1-\alpha z^{-1}}$

$|z| < |\alpha|$

~~Repeat (6)~~ ~~X(z)~~

---

(7) Find the Inverse Z-Transform of

$$X(z) = \frac{z^3 + z^2 + \tfrac{3}{2}z + \tfrac{1}{2}}{z^3 + \tfrac{3}{2}z^2 + \tfrac{1}{2}z} \; ; \; \text{ROC} : |z| < \tfrac{1}{2}$$

by partial fraction expansion method.

Soln Given. $X(z) = \dfrac{z^3 + z^2 + 3/2\, z + 1/2}{z^3 + 3/2\, z^2 + 1/2\, z}$.

* Multiply & Divide by z^{-3}.

$$X(z) = \frac{1 + z^{-1} + 3/2\, z^{-2} + \frac{1}{2} z^{-3}}{1 + 3/2\, z^{-1} + \frac{1}{2} z^{-2}}$$

* Here $M = 3$ & $N = 2$ ie $M > N$ $\therefore$ bring $X(z)$ to required form by long division method.

$$\begin{array}{r|l} & z^{-1} \\ \hline \frac{1}{2} z^{-2} + 3/2\, z^{-1} + 1 & \frac{1}{2} z^{-3} + 3/2\, z^{-2} + z^{-1} + 1 \\ & \frac{1}{2} z^{-3} + 3/2\, z^{-2} + z^{-1} \\ \hline & 0 + 0 + 0 + 1 \end{array}$$

$$X(z) = z^{-1} + \underbrace{\frac{1}{1 + 3/2\, z^{-1} + \frac{1}{2} z^{-2}}}_{X_1(z)}$$

$\therefore$ $X_1(z) = \dfrac{1}{1 + 3/2\, z^{-1} + \frac{1}{2} z^{-2}}$

$X_1(z) = \dfrac{1}{(1 + z^{-1})(1 + 1/2\, z^{-1})}$

$X_1(z) = \dfrac{K_1}{(1 + z^{-1})} + \dfrac{K_2}{1 + 1/2\, z^{-1}}$

$K_1 = (1 + z^{-1})\, X_1(z) \Big|_{z=-1}$

$= \cancel{(1 + z^{-1})} \dfrac{1}{\cancel{(1 + z^{-1})}(1 + 1/2\, z^{-1})} \Big|_{z=-1}$

$$K_1 = \frac{1}{1+\frac{1}{2}(-1)^{-1}} = \frac{1}{1+\frac{1}{2}(-1)}$$

$$K_1 = \frac{1}{\frac{2-1}{2}} = \frac{2}{1} = 2$$

$$K_2 = \left(1+\frac{1}{2}z^{-1}\right) X_1(z)\Big|_{z=-1/2}$$

$$K_2 = \cancel{\left(1+\tfrac{1}{2}z^{-1}\right)} \frac{1}{(1+z^{-1})\cancel{(1+\frac{1}{2}z^{-1})}}\Bigg|_{z=-1/2}$$

$$K_2 = \frac{1}{1+z^{-1}}\Big|_{z=-1/2} = \frac{1}{1+(-1/2)^{-1}}$$

$$K_2 = \frac{1}{1+\frac{1}{(-1/2)^{1}}} = \frac{1}{1+\frac{2}{-1}} = -1$$

$$K_2 = -1$$

$$X(z) = z^{-1} + \frac{2}{1+z^{-1}} + \frac{(-1)}{1+\frac{1}{2}z^{-1}}$$

ROC:

* Since $|z| < 1/2$, $x(n)$ must be left sided sequence.

By using LP & Table of IZT formula we get IZT as.

$$x(n) = \delta(n-1) - 2(-1)^n u(-n-1) + (-1/2)^n u(-n-1)$$

$$x(n) = \delta(n-1) - 2(-1)^n u(-n-1) + (-1/2)^n u(-n-1).$$

IInd method → Power series Expansion Method

[we use this IInd of PFE, whenever to find IZT of signal that are not a ratio of polynomials]

through example will see this using power series Expansion method [Taylor's series Maths.

Problems

8M 1) Find Inverse Z-Transform of the following using power series Expansion method [Taylor's series method]

i) $X(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}$; $|z| > 1/2$

ii) $X(z) = \dfrac{1}{1+1/2\, z^{-1}}$ $|z| < 1/2$

Soln ① Given $X(z) = \dfrac{1}{1+1/2\, z^{-1}}$; $|z| > 1/2$

* Since ROC of X(z) is outside the circle of radius 1/2
* x(n) must be RSS ~~convert~~. ~~X(z)~~
* convert X(z) into power series having only −ve powers of z like below.

i.e. [WKT $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$ → For x(n) is RSS → get −ve powers of z only]

$$
\begin{array}{r|l}
 & 1 - (1/2)z^{-1} + 1/4\, z^{-2} - 1/8\, z^{-3} \\
\hline
(1+1/2\, z^{-1}) & \cancel{1} \\
 & -\cancel{1} + 1/2\, z^{-1} \\
\hline
 & 0 - 1/2\, z^{-1} \\
 & -1/2\, z^{-1} + 1/4\, z^{-2} \\
\hline
 & 0
\end{array}
$$

$$0 + \tfrac{1}{4}z^{-2}$$

$$\tfrac{1}{4}z^{-2} + \tfrac{1}{8}z^{-3}$$

$$-\tfrac{1}{8}z^{-3} \text{ series continues.}$$

$$\therefore X(z) = 1 - \tfrac{1}{2}z^{-1} + \tfrac{1}{4}z^{-2} - \tfrac{1}{8}z^{-3} + \ldots$$

$$-\tfrac{1}{8}z^{-3} - \tfrac{1}{16}z^{-4}$$

continues.

$$\therefore X(z) = 1 - \tfrac{1}{2}z^{-1} + \tfrac{1}{4}z^{-2} - \tfrac{1}{8}z^{-3} + \ldots$$

$\therefore$ Take IZT of $X(z)$ we get $x(n)$

$$x(n) = \delta n - \tfrac{1}{2}\delta(n-1) + \tfrac{1}{4}\delta(n-2) - \tfrac{1}{8}\delta(n-3) + \ldots$$

Not Required.

$$x(n) = \left(-\tfrac{1}{2}\right)^n u(n) \quad \therefore [\text{RSS}]$$

Why ②

$$X(z) = \frac{1}{1+\tfrac{1}{2}z^{-1}} = \frac{1}{1-(-\tfrac{1}{2})z^{-1}} \quad |z| > \tfrac{1}{2}$$

by Applying formula, we can write

$$\left(-\tfrac{1}{2}\right)^n u(n)$$

Why ②. Given $X(z) = \dfrac{1}{1+\tfrac{1}{2}z^{-1}}$; ROC $|z| < \tfrac{1}{2}$

Since ROC of $X(z)$ is inside the circle of radius $\tfrac{1}{2}$

* $x(n)$ must be LSS.
* Convert $X(z)$ into power series expansion method having +ve powers of z like below.

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$n=-2$.

$x(2) = -\left(-\tfrac{1}{2}\right)^{-2} u(\cancel{-n}-1)$ ↗ 1

$x(2) = -\dfrac{1}{(-1/2)^2}(1) = -\dfrac{1}{1/4} = -4$ ②

$\|^{ly}$ $n=-3$

$x(3) = -\left(-\tfrac{1}{2}\right)^{-3} u(\cancel{-n}-1)$ ↗ 1

$x(3) = -\dfrac{1}{(-1/2)^3}(1) = -1[-8] = 8$ ③

---

③ Repeat for $X(z) = \dfrac{1}{1+1/4\, z^{-1}}$; $|z| > 1/4$

$X(z) = \dfrac{1}{1+1/4\, z^{-1}}$; $|z| < 1/4$

** 8M

④ Find IZT of

$X(z) = \dfrac{1}{1-1.5z^{-1}+0.5z^{-2}}$; $|z|>1$

using power series Expansion method.

Soln Since ROC is outside circle of radius 1,

* $x(n)$ must be RSS.

* convert $X(z)$ into power series with only $-ve$ powers of z

ie

$$\begin{array}{r|l} & 1+1.5z^{-1}+1.75z^{-2}+1.875z^{-3} \\ \hline 1-1.5z^{-1}+0.5z^{-2} & \cancel{1} \\ & -\cancel{1}-1.5z^{-1}+0.5z^{-2} \\ \hline & 0+\cancel{1.5z^{-1}}-0.5z^{-2} \\ & +\cancel{1.5z^{-1}}-2.25z^{-2}+0.75z^{-3} \\ \hline & 00 \end{array}$$

$$\left(\frac{1}{2}z^{-1}+1\right) \overline{)\;1\;}\quad 2z - 4z^2 + 8z^3 + \dots$$

$1 + 2z$

$0 - 2z$

$(-)\; -2z - 4z^2$

$0 + 4z^2$

$+4z^2 + 8z^3$

$0 - 8z^3$

[write $1+\frac{1}{2}z^{-1}$ as $\frac{1}{2}z^{-1}+1$]

using power series exp^n

$$X(z) = 2z^{1} - 4z^{2} + 8z^{3} - 16z^{4} + \dots$$

Taking I ZT

$$x(n) = 2\delta(n+1) - 4\delta(n+2) + 8\delta(n+3) + \dots$$

From Table this is eqt to

$$x(n) = -\left(-\tfrac{1}{2}\right)^n u(-n-1)$$

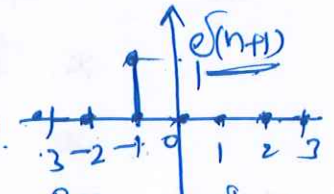
[w.r.t. $\frac{1}{1+\frac{1}{2}z^{-1}} = \frac{1}{1-\left(-\frac{1}{2}z^{-1}\right)}$ for $|z|<\frac{1}{2}$ $= -\left(-\tfrac{1}{2}\right)^n u(-n-1)$]

Expand the above we get $x(n)$ only

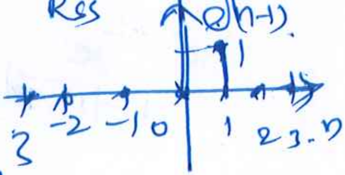
$\delta(n+1)$ means $\delta(n)$ starts 1 sec advance

i.e.

LSS.



RSS



$n=-1$ $x(n) = -\left(-\tfrac{1}{2}\right)^n u(-n-1)$

$$x(-1) = -\left(-\tfrac{1}{2}\right)^{-1}(1)$$

$$x(-1) = -\left[\frac{1}{\left(-\frac{1}{2}\right)^{1}}(1)\right] = -\left[-2\right] = 2$$

$$\cancel{1.75z^{-2}} - 0.75z^{-3}$$
$$\cancel{1.75z^{-2}} - 2.625z^{-3} + 0.875z^{-4}$$

$$0 \quad + 1.875z^{-3} - 0.875z^{-1}$$
$$1.875z^{-3}$$

$$X(z) = 1 + 1.5z^{-1} + 1.75z^{-2} + 1.875z^{-3} + \cdots \rightarrow (1)$$

Compare this eqn with std. eqn of $X(z)$ for RSS. eqn

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \longrightarrow (2) \text{ we get}$$

$x(n) = 0 \quad n < 0.$

$x(0) = 1,$

$x(1) = 1.5,$

$x(2) = 1.75, \quad x(3) = 1.875$

$$\boxed{x(n) = \{\underset{\substack{\uparrow \\ n=0}}{1}, 1.5, 1.75, \ldots\}}$$

(5) ~~Repeat~~ Find IZT of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \text{ for } |z| < 0.5$$

Soln: Since ROC is inside the circle of rad 0.5.

* So $x(n)$ must be LSS.
* convert $X(z)$ into power series having +ve powers of z only.

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$$0.5z^{-2}-1.5z^{-1}+1 \overline{\big)\; 1 \;}\quad 2z^2+6z^3+14z^4+30z^5$$

$$\cancel{1}$$
$$1-3z+2z^2$$
$$0+3z-2z^2$$
$$3z-9z^2+6z^3$$
$$0+7z^2-6z^3$$
$$7z^2-21z^3+14z^4$$
$$0+15z^3-14z^4$$

$$\therefore X(z)=2z^2+6z^3+14z^4+30z^5+\ldots \quad \text{--- (1)}$$

Compare this series with std eqn weget-

$$X(z)=\sum_{n=-2}^{\infty} x(n)z^{-n} \rightarrow (2)$$

$x(-2)=2, \quad x(-3)=6, \quad x(-4)=14, \quad x(-5)=30.$

$x(0)=0, \quad x(-1)=0.$

$$\boxed{\therefore x(n)=\{\ldots 30, 14, 6, 2, 0, \underset{\uparrow n=0}{0}\}}$$

6). Find the Inverse Z-Transform.

$$X(z)=(1+z^{-1})^4 \quad |z|>0.$$

Soln. Given $X(z)=(1+z^{-1})^4 \;;\; |z|>0$

$$=(1+z^{-1})^2(1+z^{-1})^2$$

$$X(z)=(1+z^{-2}+2z^{-1})(1+z^{-2}+2z^{-1})$$

$$\cancel{=1+z^{-2}+2z^{-1}+z^{-2}+z^{-4}+2z^{-3}+z^{-4}+z^{-6}+2z^{-5}}$$

$$X(z) = (1+z^{-2}+2z^{-1})(1+z^{-2}+2z^{-1})$$

$$X(z) = 1+z^{-2}+2z^{-1}+z^{-2}+z^{-4}+2z^{-3}+2z^{-1}+2z^{-3}+4z^{-2}$$

$$X(z) = 1+6z^{-2}+4z^{-1}+4z^{-3}+z^{-4}.$$

$$X(z) = 1+4z^{-1}+6z^{-2}+4z^{-3}+z^{-4}.$$

Taking IZT of $X(z)$ $\qquad |z|>0$

$$x(n) = \delta n+4.\delta(n-1).+6\delta(n-2)+4\delta(n-3)+\delta(n-4).$$

(7) Find the Inverse Z-Transform of

$$X(z) = z^2(1-\tfrac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})$$

$$X(z) = (z^2-\tfrac{1}{2}z)(1-z^{-2})$$

$$\therefore X(z) = z^2-\tfrac{1}{2}z-\cancel{z^{\circ}}1+\tfrac{1}{2}z^{-1}$$

$$\therefore X(z) = z^2-\tfrac{1}{2}z-1+\tfrac{1}{2}z^{-1} \rightarrow (1).$$

Compare this with std Z-Transform eqn

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \rightarrow \text{std} \rightarrow (2)$$

$$\boxed{\text{i.e } x(n) = \sum_{n=-2}^{1} x(n)z^{-n}}$$

$$x(n) = \begin{cases} 1 & ; n=-2 \\ -\tfrac{1}{2} & ; n=-1 \\ -1 & ; n=0 \\ \tfrac{1}{2} & ; n=1 \\ 0 & ; \text{otherwise} \end{cases}$$

Alternatively (OR)

$$\therefore x(n) = \delta(n+2)-\tfrac{1}{2}\delta(n+1)-\delta(n)+\tfrac{1}{2}\cancel{\delta(n+1)}\ \delta(n-1)$$

(8). A causal signal $x(n)$ has Z-Transforms

$$X(Z) = \cos(Z^{-1}) \quad \text{find } x(3) \,\&\, x(8)$$

Soln. WKT power series.

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!}$$

$$\therefore X(Z) = \cos(Z^{-1}) = 1 - \frac{Z^{-2}}{2!} + \frac{Z^{-4}}{4!} - \frac{Z^{-6}}{6!} + \longrightarrow (1)$$

Compare this given power series eqn. with std series

$$X(Z) = \sum_{n=0}^{\infty} x(n)\, Z^{-n} \longrightarrow (2). \quad \left\{ \begin{array}{l} \text{since } Z^{-n} \Rightarrow \text{i.e.} \\ \text{RSS} \end{array} \right.$$

$$x(3) = 0, \quad x(8) = \frac{1}{8!} = \frac{1}{\boxed{8\times7\times6\times5\times4\times3\times2}}$$

$$\boxed{x(8) = \frac{1}{40320}}$$

HW Repeat for $X(Z) = \sin(Z^{-1})$. find $x(3)$ & $x(1)$

{ WKT power series

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!}$$

Soln. WKT

$$X(Z) = \sin(Z^{-1}) = Z^{-1} - \frac{Z^{-3}}{3!} + \frac{Z^{-5}}{5!} - \frac{Z^{-7}}{7!} + \frac{Z^{-9}}{9!} \longrightarrow (1)$$

compare this with std

$$X(Z) = \sum_{n=0}^{\infty} x(n)\, Z^{-1} \longrightarrow (2)$$

$$x(3) = -1/3! = \frac{1}{(3\times2)} = \frac{1}{6}$$