Module 2
Gauss's Law and Divergence
Energy, Potential and Conductors

Syllabus

- **Gauss's law and Divergence**: Gauss 'law, Application of Gauss' law to point charge, line charge, Surface charge and volume charge, Point (differential) form of Gauss law, Divergence. Maxwell's First equation (Electrostatics), Vector Operator ▼ and divergence theorem, Numerical Problems (**Text: Chapter 3.2 to 3.7).**
- **Energy, Potential and Conductors**: Energy expended or work done in moving a point charge in an electric field, The line integral, Definition of potential difference and potential, The potential field of point charge, Potential gradient, Numerical Problems **(Text: Chapter 4.1 to 4.4 and 4.6).** Current and Current density, Continuity of current. **(Text: Chapter 5.1, 5.2)**
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Gauss’s law and Divergence

Gauss's Law

- The results of Faraday's experiments with the concentric spheres could be summed up as an experimental law by stating that the electric flux passing through any imaginary spherical surface lying between the two conducting spheres is equal to the charge enclosed within that imaginary surface.

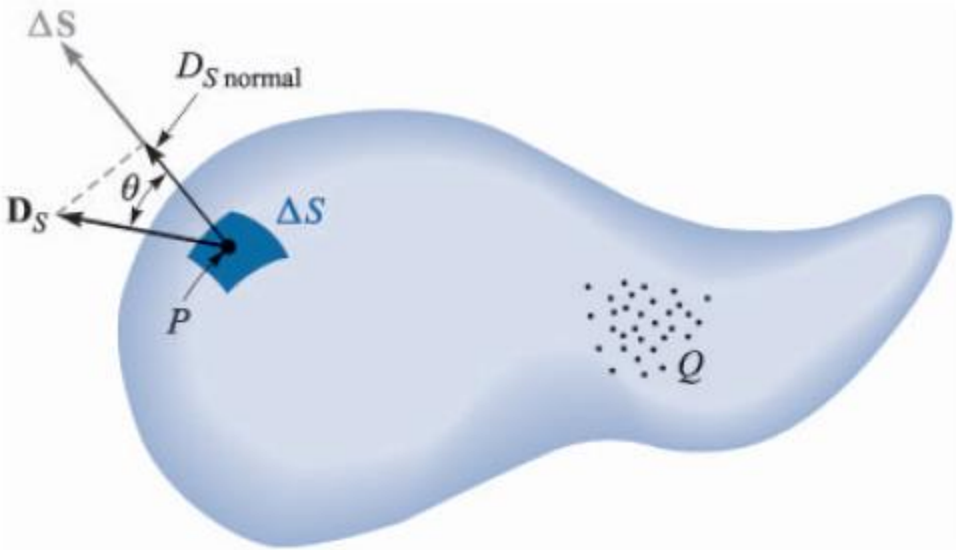
$$\psi = Q$$

- Faraday's experiment can be generalized to the following statement, which is known as Gauss's Law:

"The electric flux passing through <u>any</u> closed surface is equal to the total charge enclosed by that surface."

Gauss's Law

- Imagine a distribution of charge, shown as a cloud of point charges, surrounded by a closed surface of any shape.



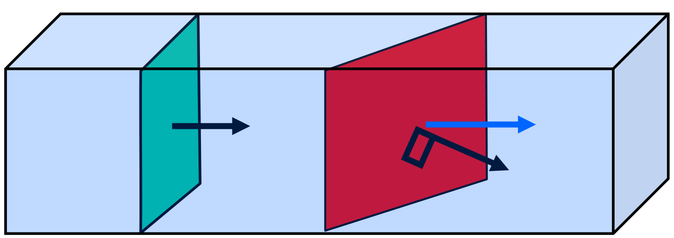
- If the total charge is *Q*, the *Q* coulombs of electric flux will pass through the enclosing surface.
- At every point on the surface the electric-flux-density vector **D** will have some value $\mathbf{D}_S$ (subscript *S* means that **D** must be evaluated at the surface).

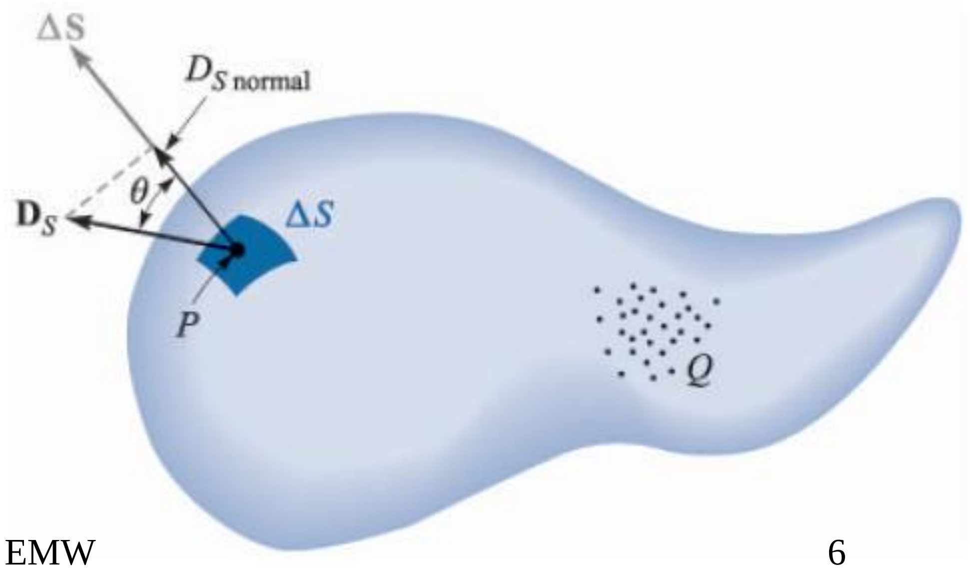
Gauss's Law

- Δ**S** defines an incremental element of area with magnitude of Δ*S* and the direction normal to the plane, or tangent to the surface at the point in question.
- At any point *P*, where $\mathbf{D}_S$ makes an angle *θ* with Δ**S**, then the flux crossing Δ**S** is the product of the normal components of $\mathbf{D}_S$ and Δ**S**.

$$\Delta\psi = \text{flux crossing } \Delta S = D_S \cos\theta \cdot \Delta S = \mathbf{D}_S \cdot \Delta\mathbf{S}$$

$$\psi = \int d\psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$





Gauss's Law

- The resultant integral is a closed surface integral, with *d***S** always involves the differentials of two coordinates
  ► The integral is a double integral.
- We can formulate the Gauss's law mathematically as:

$$\psi = \oint_s \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q$$

- The charge enclosed meant by the formula above might be several point charges, a line charge, a surface charge, or a volume charge distribution.

$$Q = \sum Q_n \qquad Q = \int \rho_L dL \qquad Q = \int_S \rho_S dS \qquad Q = \int_{vol} \rho_v dv$$

Gauss's Law

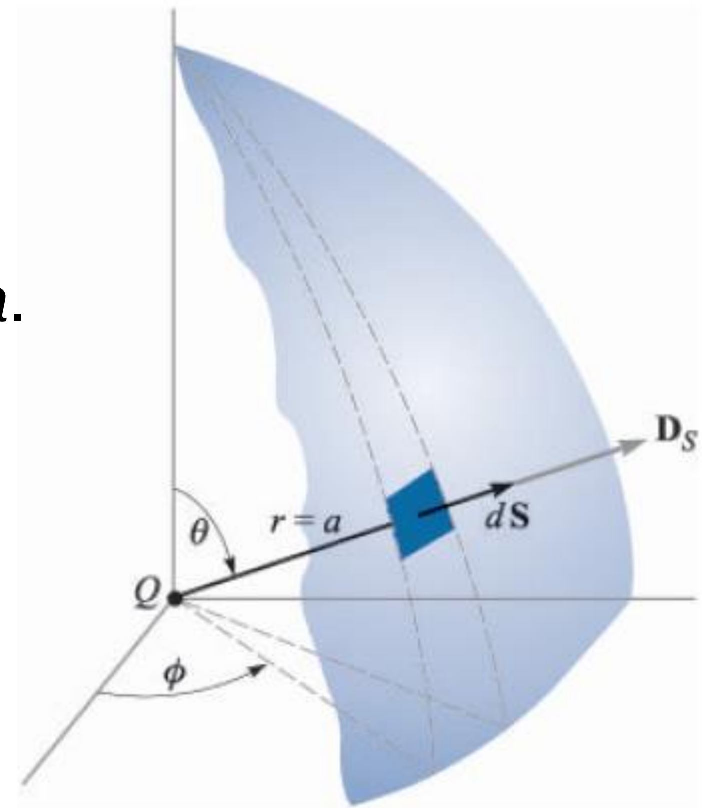
- We now take the last form, written in terms of the charge distribution, to represent the other forms:

$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v dv$$

- **Illustration.** Let a point charge *Q* be placed at the origin of a spherical coordinate system, and choose a closed surface as a sphere of radius *a*.
- The electric field intensity due to the point charge has been found to be:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \;\Rightarrow \mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$



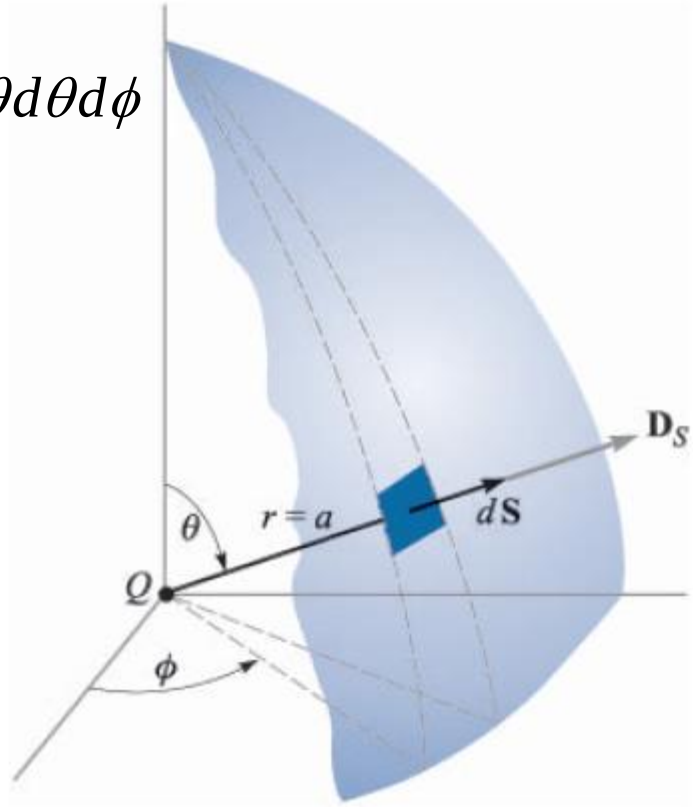
Gauss's Law

- At the surface, $r = a$,

$$\mathbf{D}_S = \frac{Q}{4\pi a^2}\mathbf{a}_r$$

$$d\mathbf{S} = a^2 \sin\theta \, d\theta d\phi \, \mathbf{a}_r$$

$$\mathbf{D}_S \cdot d\mathbf{S} = \frac{Q}{4\pi a^2} a^2 \sin\theta d\theta d\phi \, \mathbf{a}_r \cdot \mathbf{a}_r = \frac{Q}{4\pi}\sin\theta d\theta d\phi$$



$$\psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{Q}{4\pi}\sin\theta d\theta d\phi \bigg|_{r=a}$$

$$= -\frac{Q}{4\pi}\cos\theta\bigg|_{\theta=0}^{\pi} \; \theta\big|_{\phi=0}^{2\pi}$$

$$= \underline{\underline{Q}}$$

Application of Gauss's Law: Some Symmetrical Charge Distributions

- Let us now consider how to use the Gauss's law to calculate the electric field intensity $\mathbf{D}_S$:

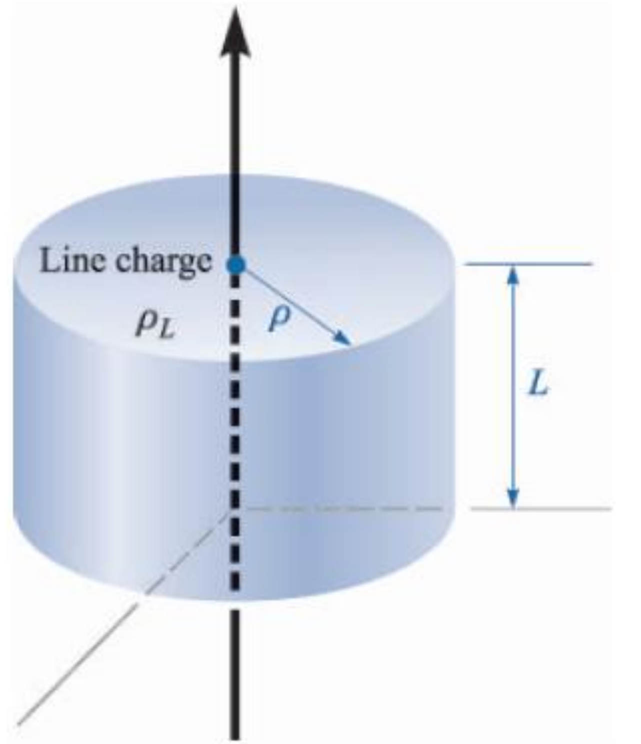
$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

- The solution will be easy if we are able to choose a closed surface which satisfies two conditions:
  1. $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S dS$ or zero, respectively.
  2. On that portion of the closed surface for which $\mathbf{D}_S \cdot d\mathbf{S}$ is not zero, D_S is constant.

- For point charge ► The surface of a sphere.
- For line charge ► The surface of a cylinder.

Application of Gauss's Law: Some Symmetrical Charge Distributions

- From the previous discussion of the uniform line charge, only the radial component of **D** is present:
  $\mathbf{D} = D_\rho \mathbf{a}_\rho$
- The choice of a surface that fulfill the requirement is simple: a cylindrical surface.
- D_ρ is every normal to the surface of a cylinder. It may then be closed by two plane surfaces normal to the *z* axis.



Application of Gauss's Law: Some Symmetrical Charge Distributions

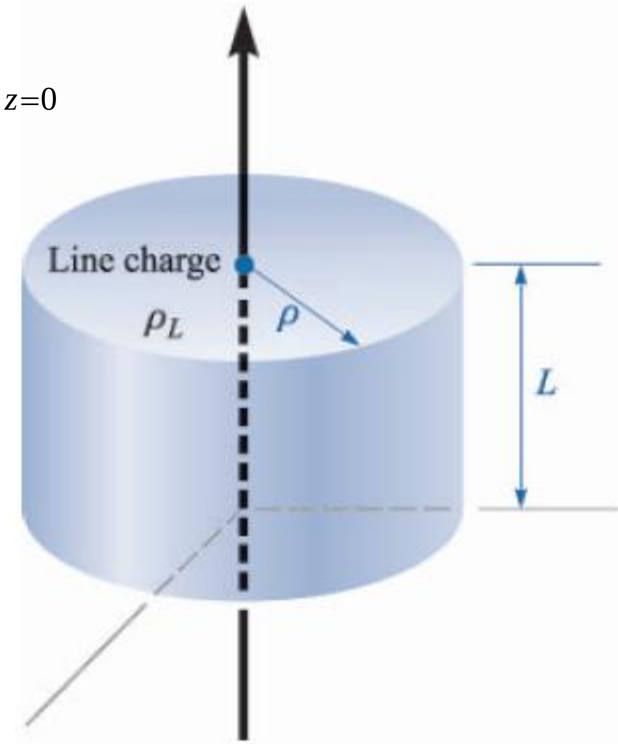
$$Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}$$

$$= D_\rho \int_{\text{sides}} dS_\rho \Big|_{\rho=\rho'} + D_z \int_{\text{top}} dS_z \Big|_{z=L} + D_z \int_{\text{bottom}} dS_z \Big|_{z=0}$$

$$= D_\rho \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz$$

$$= D_\rho 2\pi\rho L$$

$$\Rightarrow D_\rho = \frac{Q}{2\pi\rho L}$$

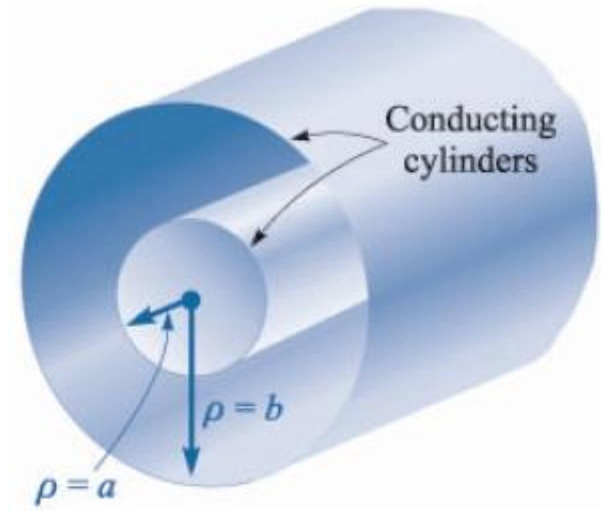


- We know that the charge enclosed is $\rho_L L$,

$$D_\rho = \frac{\rho_L}{2\pi\rho}$$

$$E_\rho = \frac{\rho_L}{2\pi\varepsilon_0\rho}$$

Application of Gauss's Law: Some Symmetrical Charge Distributions



- The problem of a coaxial cable is almost identical with that of the line charge.
- Suppose that we have two coaxial cylindrical conductors, the inner of radius *a* and the outer of radius *b*, both with infinite length.
- We shall assume a charge distribution of ρ_S on the outer surface of the inner conductor.
- Choosing a circular cylinder of length *L* and radius *ρ*, $a < \rho < b$, as the gaussian surface, we find:

$$Q = D_S 2\pi\rho L$$

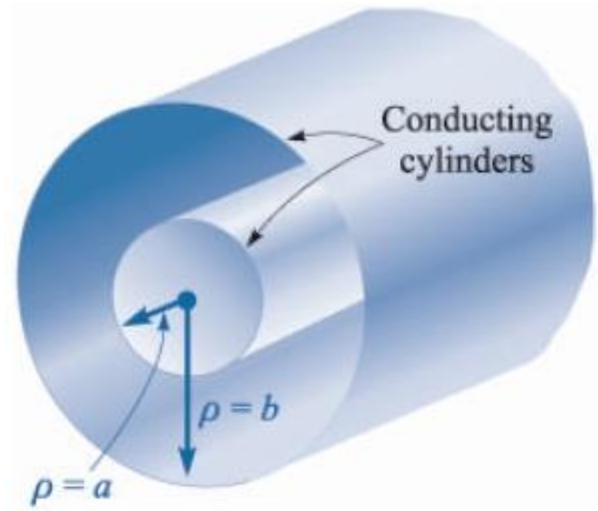
- The total charge on a length *L* of the inner conductor is:

$$Q = \int_{z=0}^{L}\int_{\phi=0}^{2\pi} \rho_S a\, d\phi\, dz = 2\pi a L \rho_S \Rightarrow D_S = \frac{a\rho_S}{\rho}$$

Application of Gauss's Law: Some Symmetrical Charge Distributions

- For one meter length, the inner conductor has $2\pi a\rho_S$ coulombs, hence $\rho_L = 2\pi a\rho_S$,

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho$$



- Everly line of electrix flux starting from the inner cylinder must terminate on the inner surface of the outer cylinder:

$$Q_{\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

$$2\pi b L \rho_{S,\text{outer cyl}} = -2\pi a L \rho_{S,\text{inner cyl}}$$

$$\rho_{S,\text{outer cyl}} = -\frac{a}{b}\rho_{S,\text{inner cyl}}$$

- If we use a cylinder of radius $\rho > b$, then the total charge enclosed will be zero.
  ► There is no external field,

$$D_S = 0$$

- **Due to simplicity, noise immunity and broad bandwidth, coaxial cable is still the most common means of data transmission over short distances.**

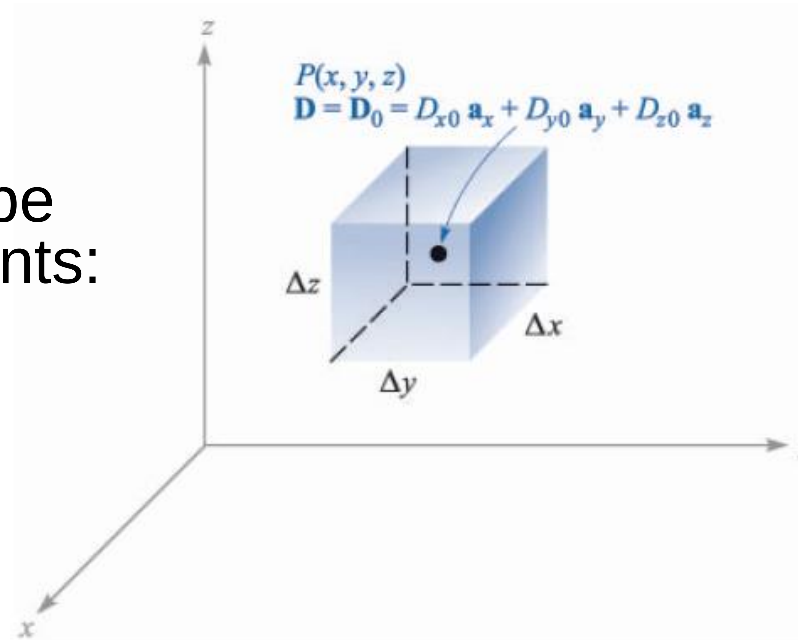
Application of Gauss's Law: Differential Volume Element

- We are now going to apply the methods of Gauss's law to a slightly different type of problem: a surface without symmetry.
- We have to choose such a very small closed surface that **D** is almost constant over the surface, and the small change in **D** may be adequately represented by using the first two terms of the Taylor's-series expansion for **D**.
- The result will become more nearly correct as the volume enclosed by the gaussian surface decreases.

Application of Gauss's Law: Differential Volume Element

- Consider any point *P*, located by a rectangular coordinate system.
- The value of **D** at the point *P* may be expressed in rectangular components:

$$\mathbf{D}_0 = D_{x0}\mathbf{a}_x + D_{y0}\mathbf{a}_y + D_{z0}\mathbf{a}_z$$



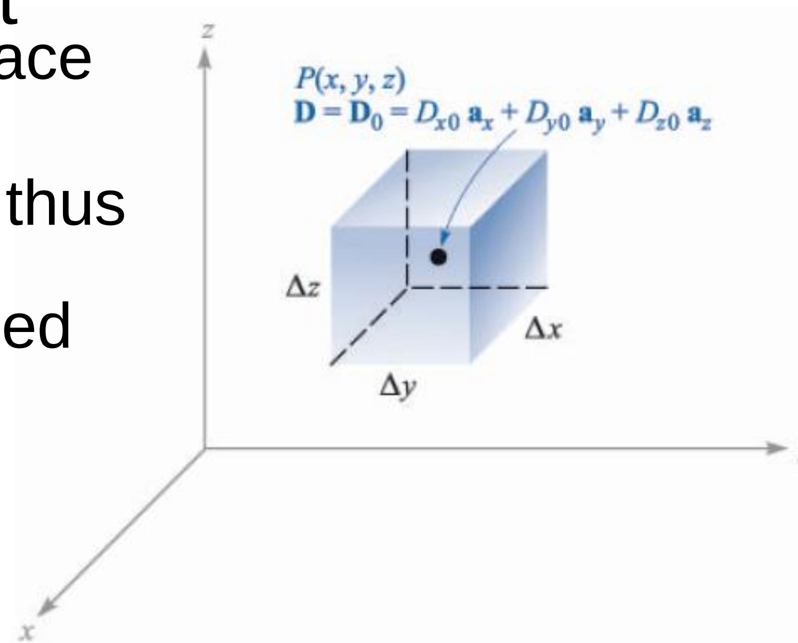
- We now choose as our closed surface, the small rectangular box, centered at *P*, having sides of lengths Δ*x*, Δ*y*, and Δ*z*, and apply Gauss's law:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

Application of Gauss's Law: Differential Volume Element

- We will now consider the front surface in detail.
- The surface element is very small, thus **D** is essentially constant over this surface (a portion of the entire closed surface):



$$\int_{\text{front}} \doteq \mathbf{D}_{\text{front}} \cdot \Delta\mathbf{S}_{\text{front}}$$

$$\doteq \mathbf{D}_{\text{front}} \cdot \Delta y \Delta z \ \mathbf{a}_x$$

$$\doteq D_{x,\text{front}} \Delta y \Delta z$$

- The front face is at a distance of Δ*x*/2 from *P*, and therefore:

$$D_{x,\text{front}} \doteq D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x$$

$$\doteq D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

Application of Gauss's Law: Differential Volume Element

- We have now, for front surface:

$$\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \Delta z$$

- In the same way, the integral over the back surface can be found as:

$$\int_{\text{back}} \doteq \mathbf{D}_{\text{back}} \cdot \Delta \mathbf{S}_{\text{back}}$$

$$\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y \Delta z \ \mathbf{a}_x)$$

$$\doteq -D_{x,\text{back}} \Delta y \Delta z$$

$$D_{x,\text{back}} \doteq D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

$$\int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \Delta z$$

Application of Gauss's Law: Differential Volume Element

- If we combine the two integrals over the front and back surface, we have:

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x \Delta y \Delta z$$

- Repeating the same process to the remaining surfaces, we find:

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta y \Delta x \Delta z$$

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta z \Delta x \Delta y$$

- These results may be collected to yield:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \Delta y \Delta z$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

Application of Gauss's Law: Differential Volume Element

- The previous equation is an approximation, which becomes better as Δv becomes smaller.
- For the moment, we have applied Gauss's law to the closed surface surrounding the volume element Δv, with the result:

$$\text{Charge enclosed in volume } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \Delta v$$

Divergence

- We shall now obtain an exact relationship, by allowing the volume element Δv to shrink to zero.

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)\doteq\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\frac{Q}{\Delta v}$$

$$\Downarrow$$

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\lim_{\Delta v\to 0}\frac{Q}{\Delta v}$$

- The last term is the volume charge density ρ_v, so that:

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\rho_v$$

Divergence

- Let us no consider one information that can be obtained from the last equation:

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_s \mathbf{D}\cdot d\mathbf{S}}{\Delta v}$$

- This equation is valid not only for electric flux density **D**, but also to any vector field **A** to find the surface integral for a small closed surface.

$$\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_s \mathbf{A}\cdot d\mathbf{S}}{\Delta v}$$

Divergence

- This operation received a descriptive name, *divergence*. The divergence of **A** is defined as:

$$\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

"The divergence of the vector flux density A is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero."

- A positive divergence of a vector quantity indicates **a source** of that vector quantity at that point.
- Similarly, a negative divergence indicates **a sink**.

Divergence

$$\text{div}\,\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Rectangular

$$\text{div}\,\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

Cylindrical

$$\text{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

Spherical

Maxwell's First Equation- (Electrostatics)

- We may now rewrite the expressions developed until now:

$$\text{div } \mathbf{D} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v}$$

$$\text{div } \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\text{div } \mathbf{D} = \rho_v$$

Maxwell's First Equation
Point Form of Gauss's Law

- This first of Maxwell's four equations applies to electrostatics and steady magnetic field.
- Physically it states that the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there.

The Vector Operator ∇ and The Divergence Theorem

- Divergence is **an operation on a vector yielding a scalar**, just like the dot product.
- We define the **del operator** ∇ as a vector operator:

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

- Then, treating the del operator as an ordinary vector, we can write:

$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z \right) \cdot (D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z)$$

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\text{div}\ \mathbf{D} = \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

The Vector Operator ∇ and The Divergence Theorem

- The ∇ operator does not have a specific form in other coordinate systems than rectangular coordinate system.
- Nevertheless,

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

Cylindrical

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

Spherical

The Vector Operator ∇ and The Divergence Theorem

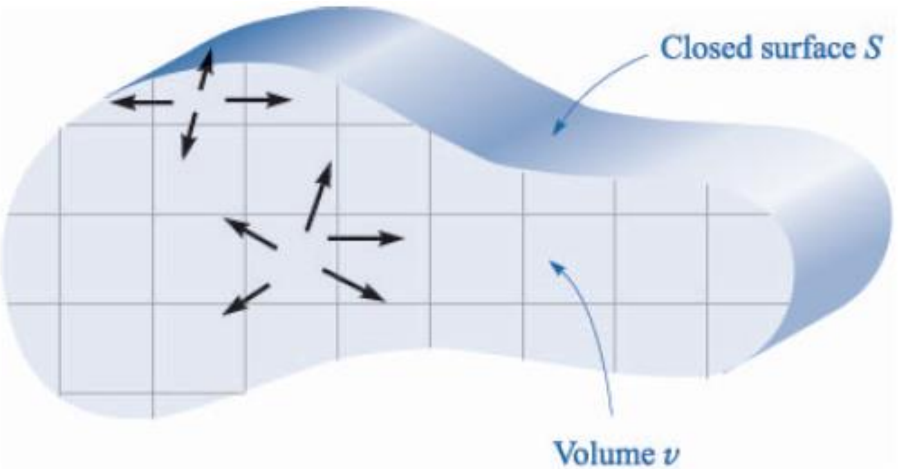
- We shall now give name to a theorem that we actually have obtained, the **Divergence Theorem**:

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv = \int_{\text{vol}} \nabla\cdot\mathbf{D} dv$$

- The first and last terms constitute the divergence theorem:

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_{\text{vol}} \nabla\cdot\mathbf{D}\, dv$$

"The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface."



Energy, Potential and Conductors

Energy Expended in Moving a Point Charge in an Electric Field

- The electric field intensity was defined as the force on a unit test charge at that point where we wish to find the value of the electric field intensity.
- To move the test charge against the electric field, we have to exert a force equal and opposite in magnitude to that exerted by the field. ► We must expend energy or do work.
- To move the charge in the direction of the electric field, our energy expenditure turns out to be negative. ► We do not do the work, the field does.

Energy Expended in Moving a Point Charge in an Electric Field

- To move a charge *Q* a distance *d***L** in an electric field **E**, the force on *Q* arising from the electric field is:

$$\mathbf{F}_E = Q\mathbf{E}$$

- The component of this force in the direction *d***L** is:

$$F_{EL} = \mathbf{F}_E \cdot \mathbf{a}_L = Q\mathbf{E} \cdot \mathbf{a}_L$$

- The force that we apply must be equal and opposite to the force exerted by the field:

$$F_{\mathrm{appl}} = -Q\mathbf{E} \cdot \mathbf{a}_L$$

- Differential work done by external source to *Q* is equal to:

$$dW = -Q\mathbf{E} \cdot \mathbf{a}_L dL = -Q\mathbf{E} \cdot d\mathbf{L}$$

- **If E and L are perpendicular, the differential work will be zero**

Energy Expended in Moving a Point Charge in an Electric Field

- The work required to move the charge a finite distance is determined by integration:

$$W = \int_{\text{init}}^{\text{final}} dW$$

$$W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

- **The path must be specified beforehand**
- **The charge is assumed to be at rest at both initial and final positions**

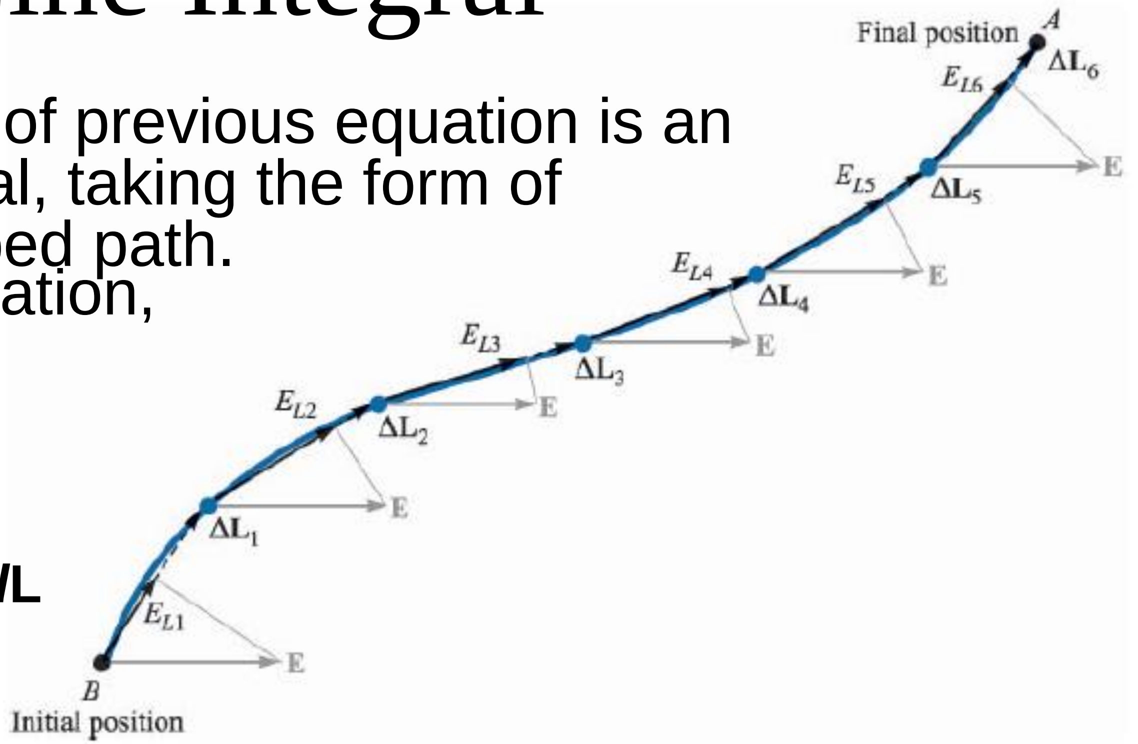
- ***W* > 0 means we expend energy or do work**
- ***W* < 0 means the field expends energy or do work**

The Line Integral

- The integral expression of previous equation is an example of a line integral, taking the form of integral along a prescribed path.
- Without using vector notation, we should have to write:

$$W = -Q\int_{\text{init}}^{\text{final}} E_L \, dL$$

- ***E_L*: component of E along *d*L**



- The work involved in moving a charge *Q* from *B* to *A* is approximately:

$$W = -Q(E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \cdots + E_{L6}\Delta L_6)$$

$$W = -Q(\mathbf{E}_1 \cdot \Delta\mathbf{L}_1 + \mathbf{E}_2 \cdot \Delta\mathbf{L}_2 + \cdots + \mathbf{E}_6 \cdot \Delta\mathbf{L}_6)$$

The Line Integral

- If we assume that the electric field is uniform,

$$\mathbf{E}_1 = \mathbf{E}_2 = \cdots = \mathbf{E}_6$$

$$W = -Q\mathbf{E}\cdot\underbrace{(\Delta\mathbf{L}_1 + \Delta\mathbf{L}_2 + \cdots + \Delta\mathbf{L}_6)}_{\mathbf{L}_{BA}}$$

- Therefore,

$$W = -Q\mathbf{E}\cdot\mathbf{L}_{BA} \qquad \text{(uniform } \mathbf{E}\text{)}$$

- Since the summation can be interpreted as a line integral, the exact result for the uniform field can be obtained as:

$$W = -Q\int_B^A \mathbf{E}\cdot d\mathbf{L}$$

$$W = -Q\mathbf{E}\cdot\int_B^A d\mathbf{L} \qquad \text{(uniform } \mathbf{E}\text{)}$$

$$W = -Q\mathbf{E}\cdot\mathbf{L}_{BA} \qquad \text{(uniform } \mathbf{E}\text{)}$$

- **For the case of uniform E, *W* does not depend on the particular path selected along which the charge is carried**

Differential Length

$$d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$$

Rectangular

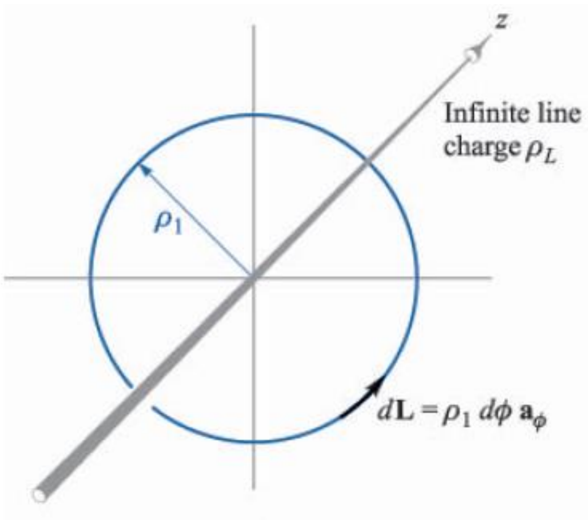
$$d\mathbf{L} = d\rho\mathbf{a}_\rho + \rho d\phi\mathbf{a}_\phi + dz\mathbf{a}_z$$

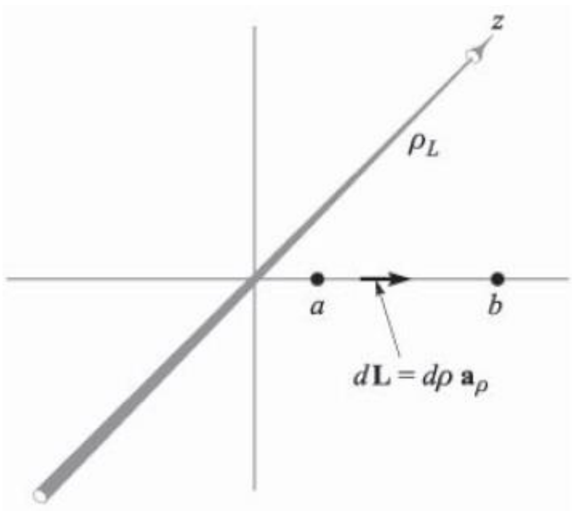
Cylindrical

$$d\mathbf{L} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi$$

Spherical

Work and Path Near an Infinite Line Charge





$$\mathbf{E} = E_\rho \mathbf{a}_\rho = \frac{\rho_L}{2\pi\varepsilon_0 \rho}\mathbf{a}_\rho$$

$$d\mathbf{L} = d\rho \mathbf{a}_\rho + \rho d\phi \mathbf{a}_\phi + dz\mathbf{a}_z$$

$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\varepsilon_0\rho_1}\mathbf{a}_\rho \cdot \rho_1 d\phi \mathbf{a}_\phi$$

$$= -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\varepsilon_0} d\phi \mathbf{a}_\rho \cdot \mathbf{a}_\phi$$

$$= 0$$

$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\varepsilon_0\rho}\mathbf{a}_\rho \cdot d\rho \mathbf{a}_\rho$$

$$= -Q\int_a^b \frac{\rho_L}{2\pi\varepsilon_0}\frac{d\rho}{\rho}$$

$$= -\frac{Q\rho_L}{2\pi\varepsilon_0}\ln\frac{b}{a}$$

Definition of Potential Difference and Potential

- We already find the expression for the work *W* done by an external source in moving a charge *Q* from one point to another in an electric field **E**:

$$W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}\cdot d\mathbf{L}$$

- *Potential difference V* is defined as the work done by an external source in moving a unit positive charge from one point to another in an electric field:

$$\text{Potential difference} = V = -\int_{\text{init}}^{\text{final}} \mathbf{E}\cdot d\mathbf{L}$$

- We shall now set an agreement on the direction of movement. V_{AB} signifies the potential difference between points *A* and *B* and is the work done in moving the unit charge <u>from</u> *B* (last named) <u>to</u> *A* (first named).

Definition of Potential Difference and Potential

- Potential difference is measured in joules per coulomb (J/C). However, volt (V) is defined as a more common unit.
- The potential difference between points *A* and *B* is:

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} \text{ V}$$

  - **V_{AB} is positive if work is done in carrying the positive charge from *B* to *A***

- From the line-charge example, we found that the work done in taking a charge *Q* from $\rho = a$ to $\rho = b$ was:

$$W = -\frac{Q\rho_L}{2\pi\varepsilon_0} \ln \frac{b}{a}$$

- Or, from $\rho = b$ to $\rho = a$,

$$W = -\frac{Q\rho_L}{2\pi\varepsilon_0} \ln \frac{a}{b} = \frac{Q\rho_L}{2\pi\varepsilon_0} \ln \frac{b}{a}$$

- Thus, the potential difference between points at $\rho = a$ to $\rho = b$ is:

$$V_{ab} = \frac{W}{Q} = \frac{\rho_L}{2\pi\varepsilon_0} \ln \frac{b}{a}$$

Definition of Potential Difference and Potential

- For a point charge, we can find the potential difference between points *A* and *B* at radial distance r_A and r_B, choosing an origin at *Q*:

$$\mathbf{E} = E_r \mathbf{a}_r = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r$$

$$d\mathbf{L} = dr\,\mathbf{a}_r$$

$$\begin{aligned} V_{AB} &= -\int_B^A \mathbf{E}\cdot d\mathbf{L} \\ &= -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2}\,dr \\ &= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) \end{aligned}$$

- **$r_B > r_A$ → $V_{AB} > 0$, $W_{AB} > 0$, Work expended by the external source (us)**
- **$r_B < r_A$ → $V_{AB} < 0$, $W_{AB} < 0$, Work done by the electric field**

Definition of Potential Difference and Potential

- It is often convenient to speak of *potential*, or *absolute potential*, of a point rather than the potential difference between two points.
- For this purpose, we must first specify the reference point which we consider to have zero potential.
- The most universal zero reference point is “ground”, which means the potential of the surface region of the earth.
- Another widely used reference point is “infinity.”
- For cylindrical coordinate, in discussing a coaxial cable, the outer conductor is selected as the zero reference for potential.

- If the potential at point *A* is V_A and that at *B* is V_B, then:

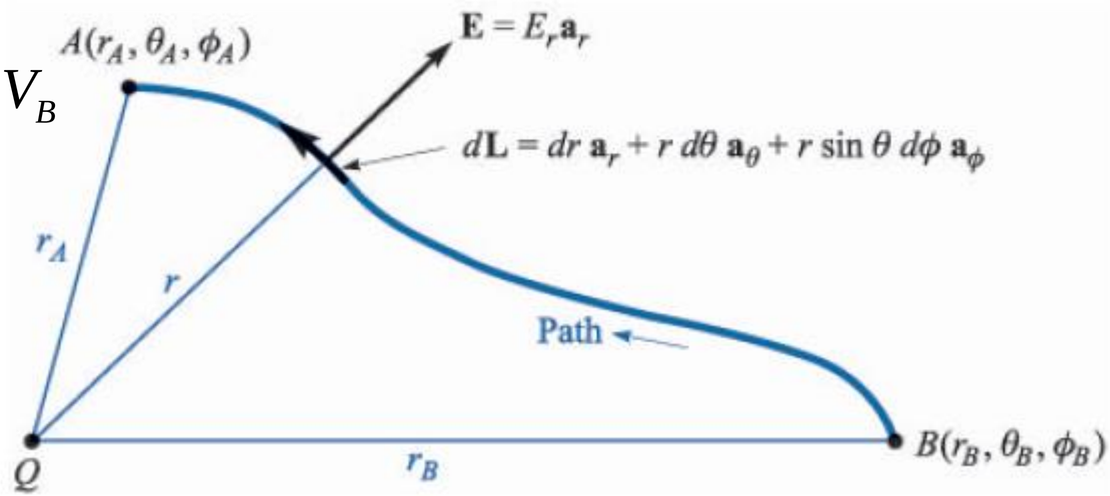
$$V_{AB} = V_A - V_B$$

The Potential Field of a Point Charge

- In previous section we found an expression for the potential difference between two points located at $r = r_A$ and $r = r_B$ in the field of a point charge Q placed at the origin:

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = V_A - V_B$$

$$V_{AB} = -\int_{r_B}^{r_A} E_r \, dr$$



- Any initial and final values of θ or Φ will not affect the answer. As long as the radial distance between r_A and r_B is constant, any complicated path between two points will not change the results.
- This is because although d**L** has r, θ, and Φ components, the electric field **E** only has the radial r component.

The Potential Field of a Point Charge

- The potential difference between two points in the field of a point charge depends only on the distance of each point from the charge.
- Thus, the simplest way to define a zero reference for potential in this case is to let $V = 0$ at infinity.
- As the point $r = r_B$ recedes to infinity, the potential at r_A becomes:

$$V_{AB} = V_A - V_B$$

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_A} - \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_B}$$

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_A} - \frac{Q}{4\pi\varepsilon_0}\frac{1}{\infty}$$

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_A} = V_A$$

The Potential Field of a Point Charge

- Generally,

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

- Physically, $Q/4\pi\varepsilon_0 r$ joules of work must be done in carrying 1 coulomb charge from infinity to any point in a distance of *r* meters from the charge *Q*.

- We can also choose any point as a zero reference:

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C_1$$

with C_1 may be selected so that $V = 0$ at any desired value of *r*.

Equipotential Surface

- Equipotential surface is a surface composed of all those points having the same value of potential.
- No work is involved in moving a charge around on an equipotential surface.

- The equipotential surfaces in the potential field of a point charge are spheres centered at the point charge.
- The equipotential surfaces in the potential field of a line charge are cylindrical surfaces axed at the line charge.
- The equipotential surfaces in the potential field of a sheet of charge are surfaces parallel with the sheet of charge.

Potential Gradient

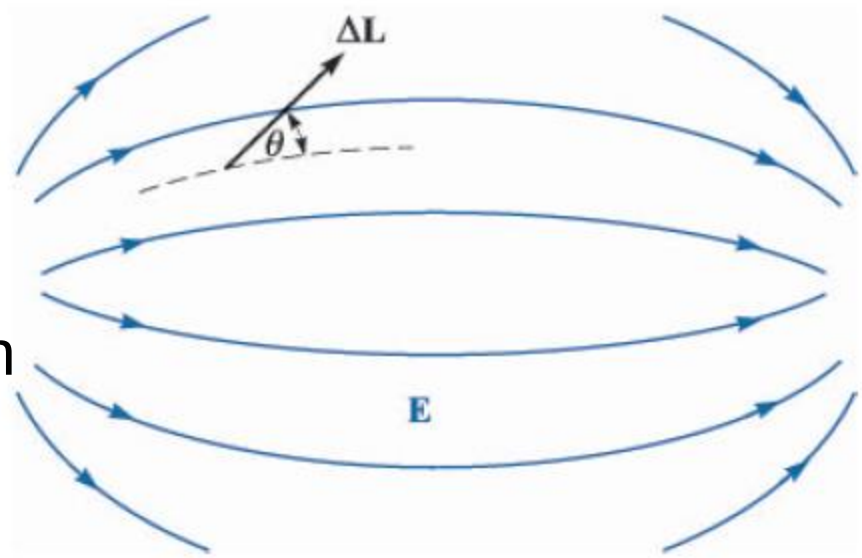
- We have discussed two methods of determining potential: directly from the electric field intensity by means of a line integral, or from the basic charge distribution itself by a volume integral.
- In practical problems, however, we rarely know **E** or ρ_v.
- Preliminary information is much more likely to consist a description of two equipotential surface, and the goal is to find the electric field intensity.

Potential Gradient

- The general line-integral relationship between *V* and **E** is:

$$V = -\int \mathbf{E} \cdot d\mathbf{L}$$

$$dV = -\mathbf{E} \cdot d\mathbf{L}$$



- For a very short element of length **ΔL**, **E** is essentially constant:

$$\Delta V \doteq -\mathbf{E} \cdot \Delta\mathbf{L}$$

$$\Delta V \doteq -E \Delta L \cos\theta$$

$$\frac{\Delta V}{\Delta L} \doteq -E \cos\theta$$

- Assuming a conservative field, for a given reference and starting point, the result of the integration is a function of the end point (*x*,*y*,*z*). We may pass to the limit and obtain:

$$\frac{dV}{dL} = -E \cos\theta$$

Potential Gradient

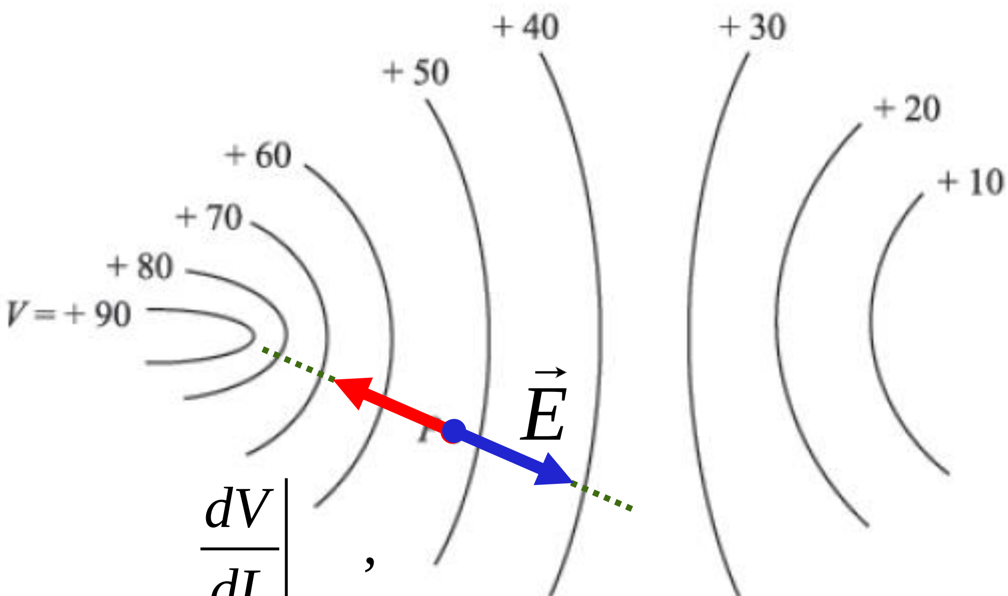
- From the last equation, the maximum positive increment of potential, $\Delta v_{max,}$ will occur when $\cos\theta = -1$, or $\Delta \mathbf{L}$ points in the direction opposite to **E**.

$$\left.\frac{dV}{dL}\right|_{\max} = E$$

- We can now conclude two characteristics of the relationship between **E** and V at any point:
  1. The magnitude of **E** is given by the maximum value of the rate of change of V with distance L.
  2. This maximum value of V is obtained when the direction of the distance increment is opposite to **E**.

Potential Gradient

- For the equipotential surfaces below, find the direction of **E** at *P*.



$$\left.\frac{dV}{dL}\right|_{\max},$$

$$\theta = 180°$$

Potential Gradient

- Since the potential field information is more likely to be determined first, let us describe the direction of $\Delta\mathbf{L}$ (which leads to a maximum increase in potential) in term of potential field.
- Let $\mathbf{a}_N$ be a unit vector normal to the equipotential surface and directed toward the higher potential.
- The electric field intensity is then expressed in terms of the potential as:

$$\mathbf{E} = -\left.\frac{dV}{dL}\right|_{\max} \mathbf{a}_N$$

- The maximum magnitude occurs when $\Delta\mathbf{L}$ is in the $\mathbf{a}_N$ direction. Thus we define dN as incremental length in $\mathbf{a}_N$ direction,

$$\left.\frac{dV}{dL}\right|_{\max} = \frac{dV}{dN}$$

$$\mathbf{E} = -\frac{dV}{dN}\mathbf{a}_N$$

Potential Gradient

- The mathematical operation to find the rate of change in a certain direction is called gradient.
- Now, the gradient of a scalar field T is defined as:

$$\text{Gradient of } T = \text{grad } T = \frac{dT}{dN}\mathbf{a}_N$$

- Using the new term,

$$\mathbf{E} = -\frac{dV}{dN}\mathbf{a}_N = -\text{grad } V$$

Potential Gradient

- Since *V* is a function of *x*, *y*, and *z*, the total differential is:

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

- But also,

$$dV = -\mathbf{E}\cdot d\mathbf{L} = -E_x dx - E_y dy - E_z dz$$

- Both expression are true for any *dx*, *dy*, and *dz*. Thus:

$$E_x = -\frac{\partial V}{\partial x}$$

$$E_y = -\frac{\partial V}{\partial y}$$

$$E_z = -\frac{\partial V}{\partial z}$$

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right)$$

$$\text{grad } V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

- **Note**: Gradient of a **scalar** is a **vector**.

Potential Gradient

- Introducing the vector operator for gradient:

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

We now can relate **E** and *V* as:

$$\mathbf{E} = -\nabla V$$

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

Rectangular

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z$$

Cylindrical

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

Spherical

Current and Current Density

- Electric charges in motion constitute a *current*.
- The unit of current is the ampere (A), defined as a rate of movement of charge passing a given reference point (or crossing a given reference plane).

$$I = \frac{dQ}{dt}$$

- Current is *defined* as the motion of positive charges, although conduction in metals takes place through the motion of electrons.
- Current density **J** is defined, measured in amperes per square meter (A/m^2).

Current and Current Density

- The increment of current ΔI crossing an incremental surface ΔS normal to the current density is:

$$\Delta I = J_N \Delta S$$

- If the current density is not perpendicular to the surface,

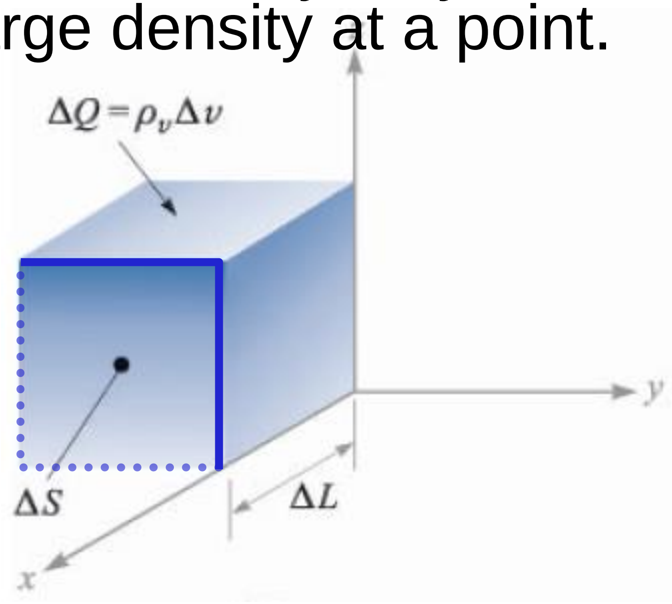
$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{S}$$

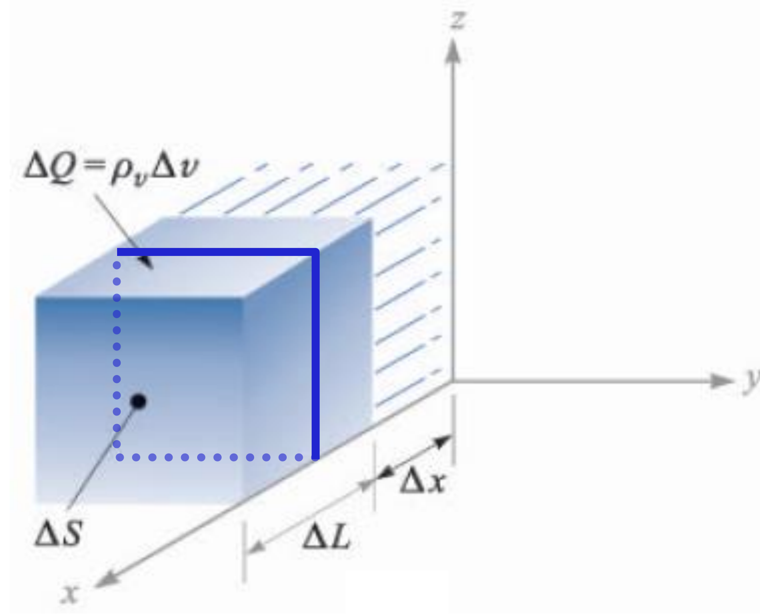
- Through integration, the total current is obtained:

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

Current and Current Density

- Current density may be related to the velocity of volume charge density at a point.





- **An element of charge $\Delta Q = \rho_v \Delta S \Delta L$ moves along the *x* axis**
- **In the time interval Δt, the element of charge has moved a distance Δx**
- **The charge moving through a reference plane perpendicular to the direction of motion is $\Delta Q = \rho_v \Delta S \Delta x$**

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta x}{\Delta t}$$

Current and Current Density

- The limit of the moving charge with respect to time is:

$$\Delta I = \rho_v \Delta S v_x$$

- In terms of current density, we find:

$$J_x = \rho_v v_x$$

$$\mathbf{J} = \rho_v \mathbf{v}$$

- This last result shows clearly that charge in motion constitutes a current. We name it here *convection current*.
- $\mathbf{J} = \rho_v \mathbf{v}$ is then called *convection current density*.

Continuity of Current

- The principle of conservation of charge:
  "Charges can be neither created nor destroyed."
- But, equal amounts of positive and negative charge (pair of charges) may be simultaneously created, obtained by separation, destroyed, or lost by recombination.

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S}$$

- **The Continuity Equation in Closed Surface**

- Any *outward* flow of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface.
- If the charge inside the closed surface is denoted by Q_i, then the rate of decrease is $-dQ_i/dt$ and the principle of conservation of charge requires:

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt}$$

- **The Integral Form of the Continuity Equation**

Continuity of Current

- The differential form (or point form) of the continuity equation is obtained by using the divergence theorem:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_{\text{vol}} (\nabla \cdot \mathbf{J}) dv$$

- We next represent Q_i by the volume integral of ρ_v:

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J}) dv = -\frac{d}{dt} \int_{\text{vol}} \rho_v dv$$

- If we keep the surface constant, the derivative becomes a partial derivative. Writing it within the integral,

$$\int_{\text{vol}} (\nabla \cdot \mathbf{J}) dv = \int_{\text{vol}} -\frac{\partial \rho_v}{\partial t} dv$$

$$(\nabla \cdot \mathbf{J}) \Delta v = -\frac{\partial \rho_v}{\partial t} \Delta v$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

- **The Differential Form (Point Form) of the Continuity Equation**