Module 3

Poisson’s and Laplace’s Equations: Derivation of Poisson‘s and Laplace‘s Equations, Uniqueness theorem, Examples of the solution of Laplace‘s equation, Numerical problems on Laplace equation **(Text: Chapter 7.1 to 7.3)**

Steady Magnetic Field: Biot-Savart Law, Ampere‘s circuital law, Curl, Stokes‘ theorem, Magnetic flux and magnetic flux density, Basic concepts Scalar and Vector Magnetic Potentials, Numerical problems. (**Text: Chapter 8.1 to 8.6)**

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Derivation of Poisson‘s and Laplace‘s Equations,

Obtaining Poisson's equation is exceedingly simple, for from the point form of Gauss's law,

$$\nabla \cdot \mathbf{D} = \rho_v \tag{1}$$

the definition of **D**,

$$\mathbf{D} = \epsilon \mathbf{E} \tag{2}$$

and the gradient relationship,

$$\mathbf{E} = -\nabla V \tag{3}$$

by substitution we have

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = -\nabla \cdot (\epsilon \nabla V) = \rho_v$$

or

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon} \tag{4}$$

Equation (4) is *Poisson's equation*, but the "double ∇" operation must be interpreted and expanded, at least in cartesian coordinates, before the equation can be useful. In cartesian coordinates,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

and therefore

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{5}$$

Usually the operation $\nabla \cdot \nabla$ is abbreviated ∇^2 (and pronounced "del squared"), a good reminder of the second-order partial derivatives appearing in (5), and we have

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon} \tag{6}$$

in cartesian coordinates.

If $\rho_v = 0$, indicating zero *volume* charge density, but allowing point charges, line charge, and surface charge density to exist at singular locations as sources of the field, then

$$\nabla^2 V = 0 \tag{7}$$

which is *Laplace's equation*. The ∇^2 operation is called the *Laplacian of V*.

In cartesian coordinates Laplace's equation is

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(cartesian)} \tag{8}$$

and the form of $\nabla^2 V$ in cylindrical and spherical coordinates may be obtained by using the expressions for the divergence and gradient already obtained in those coordinate systems. For reference, the Laplacian in cylindrical coordinates is

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{(cylindrical)} \tag{9}$$

and in spherical coordinates is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \qquad \text{(spherical)} \tag{10}$$

Uniqueness Theorem

Let us assume that we have two solutions of Laplace's equation, V_1 and V_2, both general functions of the coordinates used. Therefore

$$\nabla^2 V_1 = 0$$

and

$$\nabla^2 V_2 = 0$$

from which

$$\nabla^2 (V_1 - V_2) = 0$$

Each solution must also satisfy the boundary conditions, and if we represent the given potential values on the boundaries by V_b, then the value of V_1 on the boundary V_{1b} and the value of V_2 on the boundary V_{2b} must both be identical to V_b,

$$V_{1b} = V_{2b} = V_b$$

or

$$V_{1b} - V_{2b} = 0$$

$$\nabla \cdot (V\mathbf{D}) \equiv V(\nabla \cdot \mathbf{D}) + \mathbf{D} \cdot (\nabla V)$$

which holds for any scalar V and any vector $\mathbf{D}$. For the present application we shall select $V_1 - V_2$ as the scalar and $\nabla(V_1 - V_2)$ as the vector, giving

$$\nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)] \equiv (V_1 - V_2)[\nabla \cdot \nabla(V_1 - V_2)] + \nabla(V_1 - V_2) \cdot \nabla(V_1 - V_2)$$

which we shall integrate throughout the volume *enclosed* by the boundary surfaces specified:

$$\int_{\text{vol}} \nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)]\, dv \equiv \int_{\text{vol}} (V_1 - V_2)[\nabla \cdot \nabla(V_1 - V_2)] dv + \int_{\text{vol}} [\nabla(V_1 - V_2)]^2\, dv \qquad (11)$$

The divergence theorem allows us to replace the volume integral on the left side of the equation by the closed surface integral over the surface surrounding the volume. This surface consists of the boundaries already specified on which $V_{1b} = V_{2b}$, and therefore

$$\int_{\text{vol}} \nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)]\, dv = \oint_S [(V_{1b} - V_{2b})\nabla(V_{1b} - V_{2b})] \cdot d\mathbf{S} = 0$$

One of the factors of the first integral on the right side of (11) is $\nabla \cdot \nabla(V_1 - V_2)$, or $\nabla^2(V_1 - V_2)$, which is zero by hypothesis, and therefore that integral is zero. Hence the remaining volume integral must be zero:

$$\int_{\text{vol}} [\nabla(V_1 - V_2)]^2\, dv = 0$$

There are two reasons why an integral may be zero: either the integrand (the quantity under the integral sign) is everywhere zero, or the integrand is positive in some regions and negative in others, and the contributions cancel algebraically. In this case the first reason must hold because $[\nabla(V_1 - V_2)]^2$ cannot be negative. Therefore

$$[\nabla(V_1 - V_2)]^2 = 0$$

and

$$\nabla(V_1 - V_2) = 0$$

Finally, if the gradient of $V_1 - V_2$ is everywhere zero, then $V_1 - V_2$ cannot change with any coordinates and

$$V_1 - V_2 = \text{constant}$$

$V_1 - V_2 = V_{1b} - V_{2b} = 0$, and we see that the constant is indeed zero, and therefore

$$V_1 = V_2$$

giving two identical solutions.

The uniqueness theorem also applies to Poisson's equation, for if $\nabla^2 V_1 = -\rho_v/\epsilon$ and $\nabla^2 V_2 = -\rho_v/\epsilon$, then $\nabla^2(V_1 - V_2) = 0$ as before. Boundary conditions still require that $V_{1b} - V_{2b} = 0$, and the proof is identical from this point.

Examples of the Solution of Laplace's Equation

Example 1

Assume V is a function only of x – solve

$$V = \frac{V_o \cdot x}{d}$$

Finding the capacitance of a parallel-plate capacitor

Steps

1 – Given V, use E = - DelV to find E
2 – Use D = εE to find D
3 - Evaluate D at either capacitor plate, D = Ds = Dn *an*
4 – Recognize that ρs = Dn
5 – Find Q by a surface integration over the capacitor plate

$$C = \frac{|Q|}{V_o} = \frac{\varepsilon \cdot S}{d}$$

Examples of the Solution of Laplace's Equation

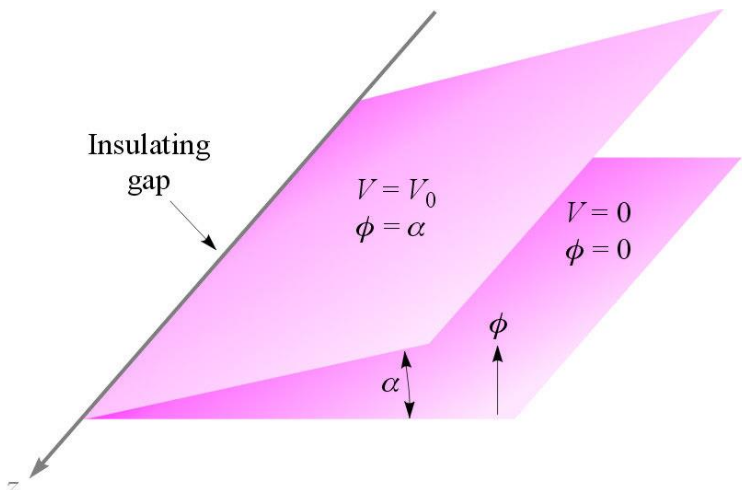
Example 2 - Cylindrical

$$V = V_o \cdot \frac{\ln\left(\frac{b}{\rho}\right)}{\ln\left(\frac{b}{a}\right)}$$

$$C = \frac{2 \cdot \pi \cdot \varepsilon \cdot L}{\ln\left(\frac{b}{a}\right)}$$

Examples of the Solution of Laplace's Equation

Example 3



Examples of the Solution of Laplace's Equation

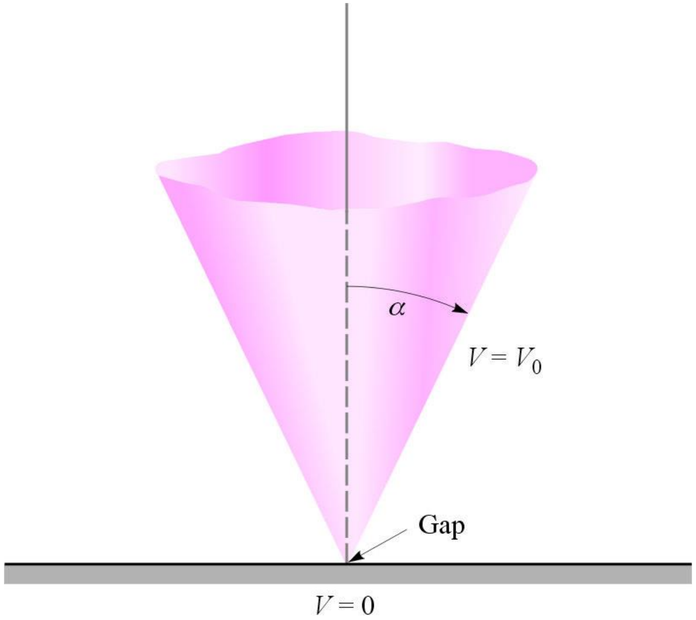
Example 4 (spherical coordinates)

$$V = V_o \cdot \frac{\frac{1}{r} - \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}$$

$$C = \frac{4 \cdot \pi \cdot \varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

Examples of the Solution of Laplace's Equation

Example 5



$$V = V_o \cdot \frac{\ln\left(\tan\left(\frac{\theta}{2}\right)\right)}{\ln\left(\tan\left(\frac{\alpha}{2}\right)\right)}$$

$$C = \frac{2 \cdot \pi \cdot \varepsilon \cdot r_1}{\ln\left(\cot\left(\frac{\alpha}{2}\right)\right)}$$

The Steady Magnetic Field

- At this point, we shall begin our study of the magnetic field with a definition of the magnetic field itself and show how it arises from a current distribution.
- The relation of the steady magnetic field to its source is more complicated than is the relation of the electrostatic field to its source.

- The source of the steady magnetic field may be a permanent magnet, an electric field changing linearly with time, or a direct current.
- Our present concern will be the magnetic field produced by a <u>differential dc element</u> in the free space.

Biot-Savarts Law

- Consider a differential current element as a vanishingly small section of a current-carrying filamentary conductor.
- We assume a current I flowing in a differential vector length of the filament $d\mathbf{L}$.

- The law of Biot-Savart then states that

  "At any point *P* the magnitude of the magnetic field intensity produced by the differential element is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point *P* at which the field is desired; also, the magnitude of the magnetic field intensity is inversely proportional to the square of the distance from the differential element to the point *P*."

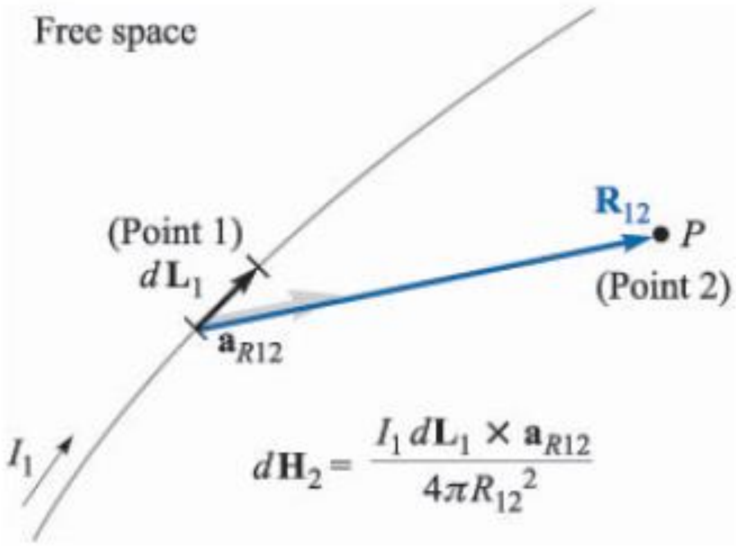
- The Biot-Savart law may be written concisely using vector notation as

$$d\mathbf{H} = \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I d\mathbf{L} \times \mathbf{R}}{4\pi R^3}$$

- The units of the magnetic field intensity **H** are evidently amperes per meter (A/m).

- Using additional subscripts to indicate the point to which each of the quantities refers,

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$



Biot-Savart Law

- It is impossible to check experimentally the law of Biot-Savart as expressed previously, because the differential current element cannot be isolated.
- It follows that only the integral form of the Biot-Savart law can be verified experimentally,

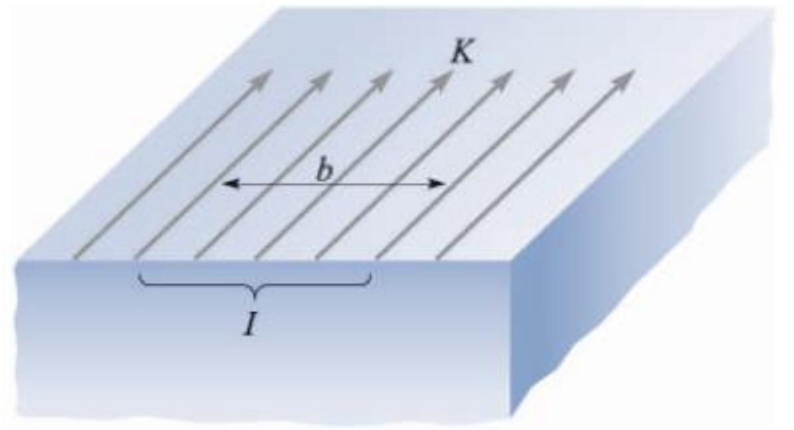
$$\mathbf{H} = \oint \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

- The Biot-Savart law may also be expressed in terms of distributed sources, such as current density **J** (A/m^2) and surface current density **K** (A/m).
- Surface current **K** flows in a sheet of vanishingly small thickness, and the sheet's current density **J** is therefore infinite.

- Surface current density **K**, however, is measured in amperes per meter width. Thus, if the surface current density is uniform, the total current I in any width b is

$$I = Kb$$

where the width b is measured perpendicularly to the direction in which the current is flowing.



Biot-Savart Law

- For a nonuniform surface current density, integration is necessary:

$$I = \int K dN$$

where *dN* is a differential element of the path <u>*across*</u> which the current is flowing.

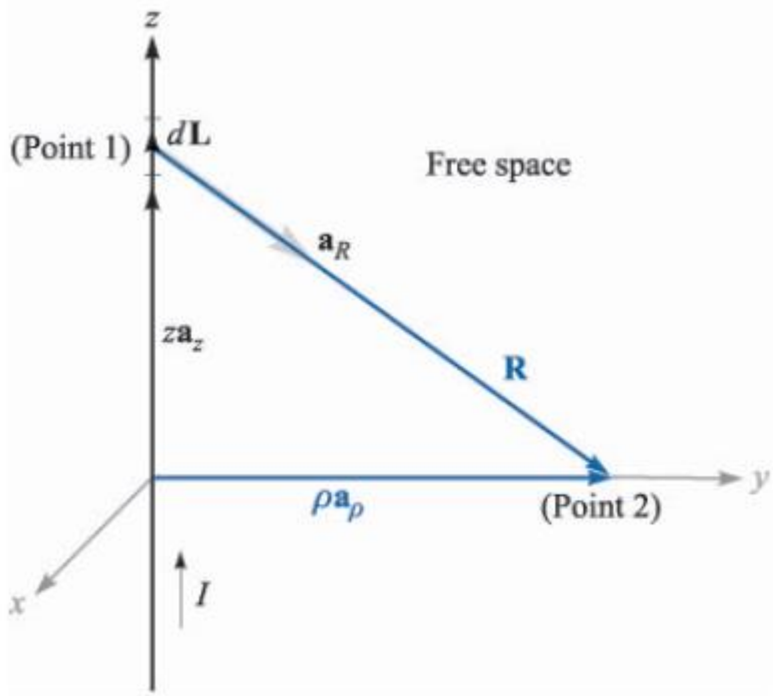
- Thus, the differential current element $I d\mathbf{L}$ may be expressed in terms of surface current density **K** or current density **J**,

$$I d\mathbf{L} = \mathbf{K} dS = \mathbf{J} dv$$

and alternate forms of the Biot-Savart law can be obtained as

$$\mathbf{H} = \int_{s} \frac{\mathbf{K} \times \mathbf{a}_R dS}{4\pi R^2}$$ and $$\mathbf{H} = \int_{\text{vol}} \frac{\mathbf{J} \times \mathbf{a}_R dv}{4\pi R^2}$$

- We may illustrate the application of the Biot-Savart law by considering an infinitely long straight filament.
- Referring to the next figure, we should recognize the symmetry of this field. As we moves along the filament, no variation of *z* or ϕ occur.



- The field point **r** is given by **r** = $\rho\mathbf{a}_\rho$, and the source point **r**' is given by **r**' = $z'\mathbf{a}_z$. Therefore,

$$\mathbf{R}_{12} = \mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

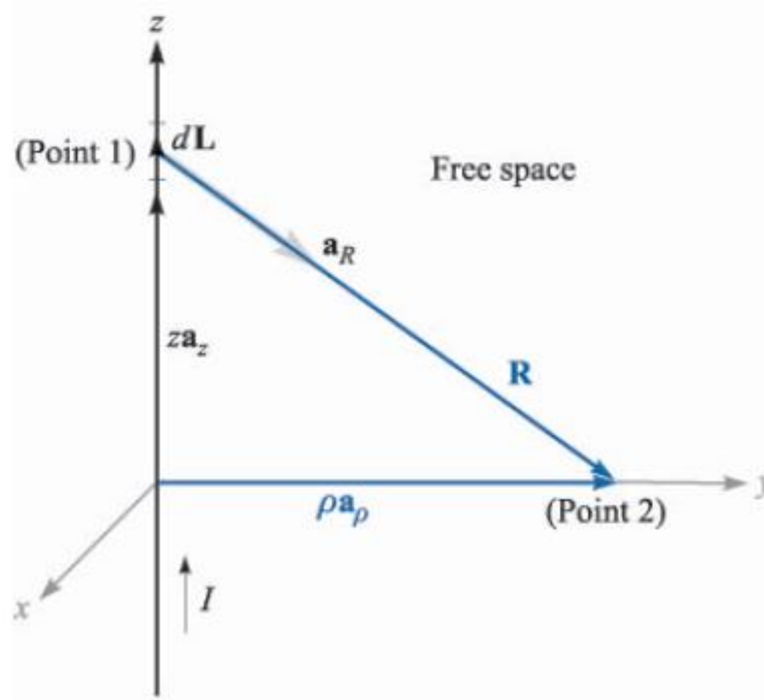
$$\mathbf{a}_{R12} = \frac{\rho\mathbf{a}_\rho - z'\mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$

Biot-Savart Law

- We take $d\mathbf{L} = dz'\mathbf{a}_z$ and the current is directed toward the increasing values of z'. Thus we have

$$\mathbf{H}_2 = \int_{-\infty}^{\infty} \frac{I dz' \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z' \mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho dz' \mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$



- **The resulting magnetic field intensity is directed to $\mathbf{a}_\phi$ direction.**

Biot-Savart Law

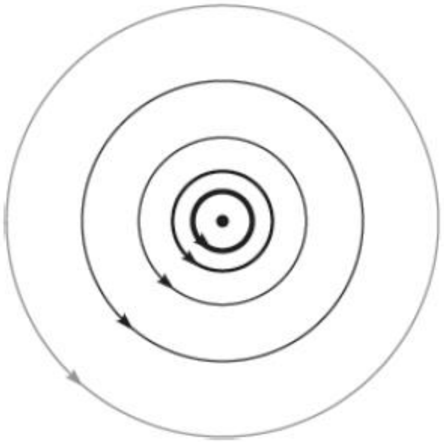
- Continuing the integration with respect to *z*’ only,

$$\mathbf{H}_2 = \frac{I\rho \mathbf{a}_\phi}{4\pi} \int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + z'^2)^{3/2}}$$

$$\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$$

$$= \frac{I\rho \mathbf{a}_\phi}{4\pi} \left. \frac{z'}{\rho^2\sqrt{(\rho^2 + z'^2)}} \right|_{-\infty}^{\infty}$$

$$\mathbf{H}_2 = \frac{I}{2\pi\rho}\mathbf{a}_\phi$$



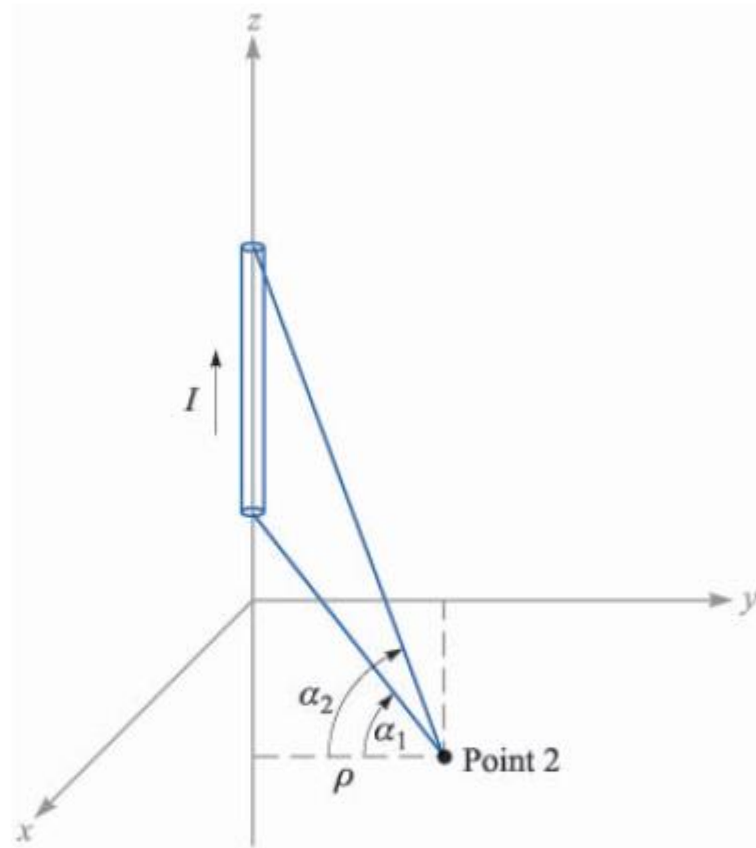
- **The magnitude of the field is not a function of ϕ or *z*.**
- **It varies inversely with the distance from the filament.**
- **The direction of the magnetic-field-intensity vector is circumferential.**

Biot-Savart Law

- The formula to calculate the magnetic field intensity caused by a finite-length current element can be readily used:

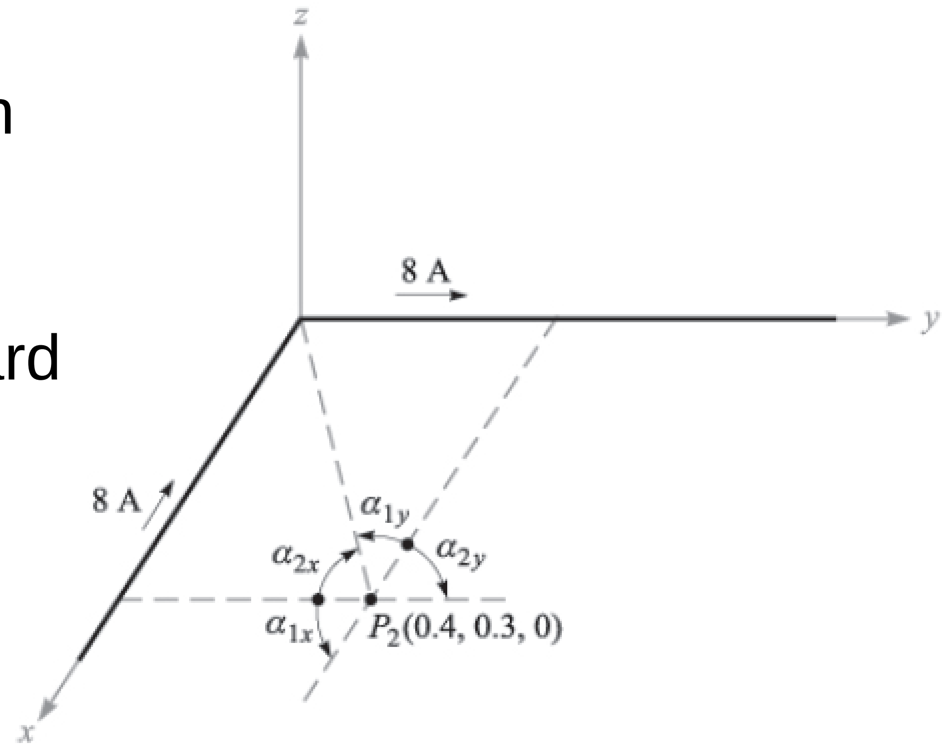
$$\mathbf{H} = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$$

  - **Try to derive this formula**



Biot-Savart Law

- <u>Example</u>
  Determine **H** at $P_2(0.4, 0.3, 0)$ in the field of an 8 A filamentary current directed inward from infinity to the origin on the positive *x* axis, and then outward to infinity along the *y* axis.



$$\alpha_{1x} = -90°,\ \alpha_{2x} = 53.1°$$

$$\alpha_{1y} = -36.9°,\ \alpha_{2y} = 90°$$

$$\mathbf{H}_{2x} = \frac{8}{4\pi(0.3)}(\sin 53.1° - \sin(-90°))\mathbf{a}_\phi = \frac{12}{\pi}\mathbf{a}_\phi \ \Rightarrow \mathbf{H}_{2x} = -\frac{12}{\pi}\mathbf{a}_z \text{ A/m}$$

$$\mathbf{H}_{2y} = \frac{8}{4\pi(0.4)}(\sin 90° - \sin(-36.9°))\mathbf{a}_\phi = \frac{8}{\pi}\mathbf{a}_\phi \ \Rightarrow \mathbf{H}_{2y} = -\frac{8}{\pi}\mathbf{a}_z \text{ A/m}$$

$$\mathbf{H}_2 = \mathbf{H}_{2x} + \mathbf{H}_{2y} = -\frac{20}{\pi}\mathbf{a}_z = \underline{\underline{-6.37\mathbf{a}_z \text{ A/m}}}$$

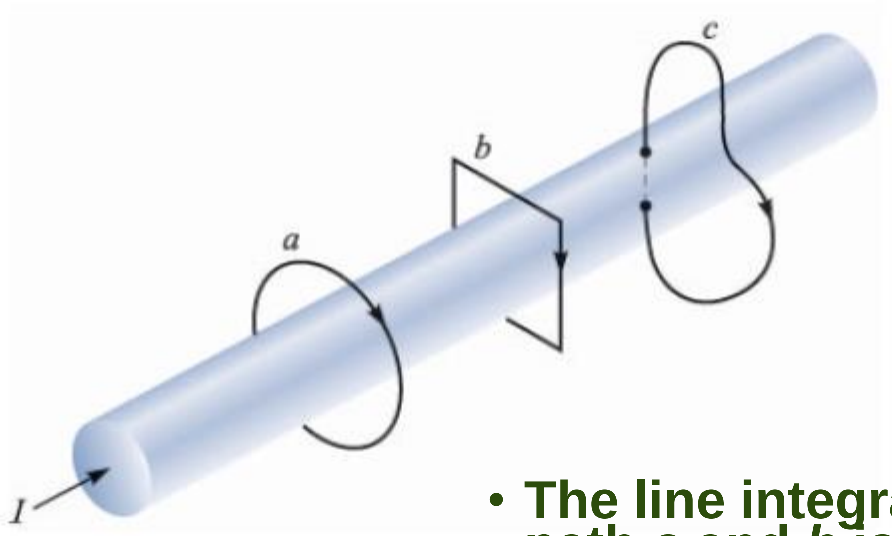
- **What if the line goes onward to infinity along the *z* axis?**

Ampere's Circuital

- In solving electrostatic problems, whenever a high degree of symmetry is present, we found that they could be solved much more easily by using Gauss's law compared to Coulomb's law.
- Again, an analogous procedure exists in magnetic field.
- Here, the law that helps solving problems more easily is known as Ampere's circuital law.

- The derivation of this law will wait until several subsection ahead. For the present we accept Ampere's circuital law as another law capable of experimental proof.
- Ampere's circuital law states that the line integral of magnetic field intensity **H** about any *closed* path is exactly equal to the direct current enclosed by that path,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

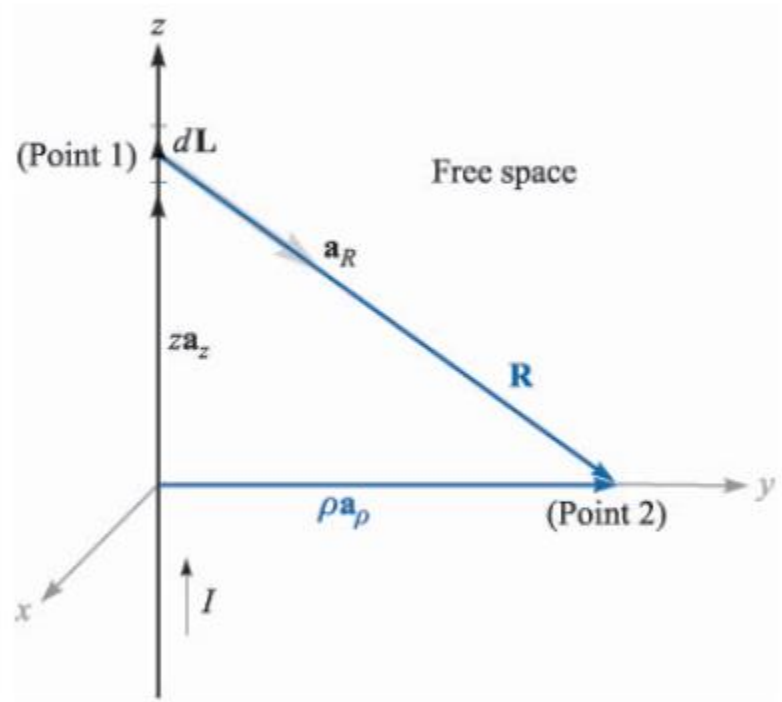


- **The line integral of H about the closed path *a* and *b* is equal to *I***
- **The integral around path *c* is less than *I*.**

- The application of Ampere’s circuital law involves finding the total current enclosed by a closed path.

Ampere's Circuital Law

- Let us again find the magnetic field intensity produced by an infinite long filament carrying a current I. The filament lies on the z axis in free space, flowing to $\mathbf{a}_z$ direction.
- We choose a convenient path to any section of which **H** is either perpendicular or tangential and along which the magnitude H is constant.



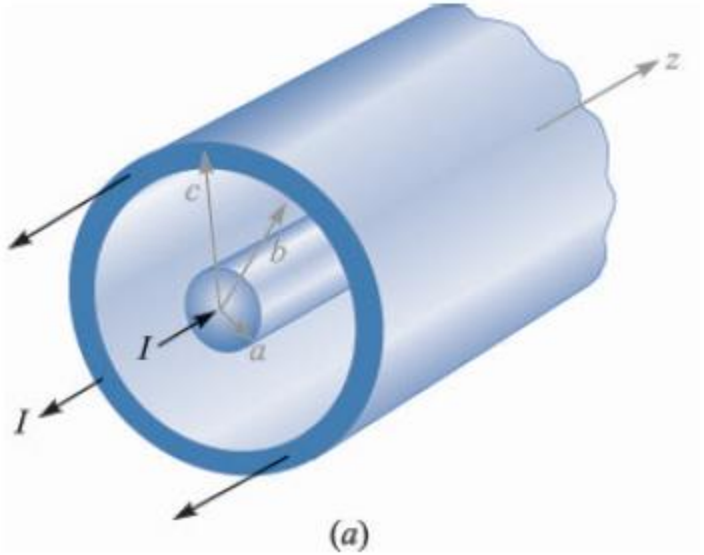
- The path must be a circle of radius ρ, and Ampere's circuital law can be written as

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi\rho = I \quad \Rightarrow \boxed{H_\phi = \frac{I}{2\pi\rho}}$$

Ampere's Circuital Law

- As a second example, consider an infinitely long coaxial transmission line, carrying a uniformly distributed total current ***I*** in the center conductor and –***I*** in the outer conductor.
- A circular path of radius *ρ*, where *ρ* is larger than the radius of the inner conductor *a* but less than the inner radius of the outer conductor *b*, leads immediately to

$$H_\phi = \frac{I}{2\pi\rho} \qquad (a < \rho < b)$$



- If *ρ* < *a*, the current enclosed is

$$I_{\text{encl}} = 2\pi\rho H_\phi = I\frac{\rho^2}{a^2}$$

- Resulting

$$H_\phi = I\frac{\rho}{2\pi a^2} \quad (\rho < a)$$

Ampere’s Circuital Law

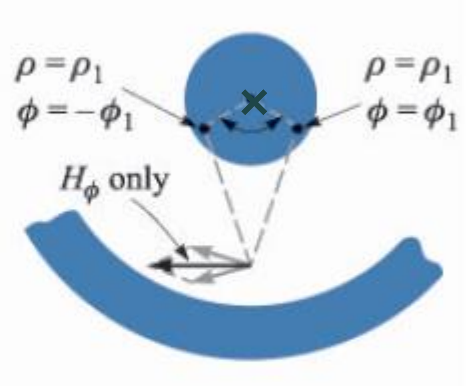
- If the radius ρ is larger than the outer radius of the outer conductor c, no current is enclosed and

$$H_\phi = 0 \qquad (\rho > c)$$

- Finally, if the path lies within the outer conductor, we have

$$2\pi\rho H_\phi = I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)$$

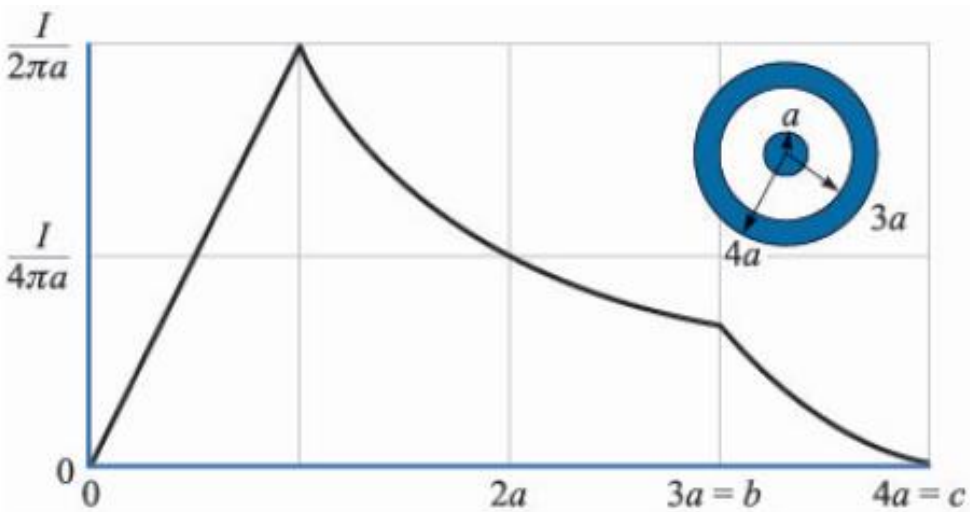
$$H_\phi = \frac{I}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2} \qquad (b < \rho < c)$$



(*b*)

- **ρ components cancel, z component is zero.**
- **Only ϕ component of H does exist.**

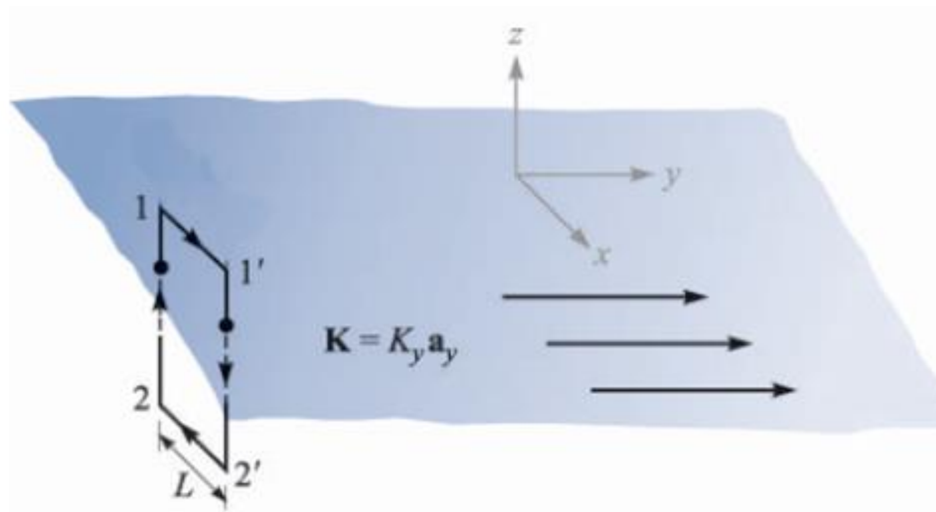
- The magnetic-field-strength variation with radius is shown below for a coaxial cable in which $b = 3a$, $c = 4a$.
- It should be noted that the magnetic field intensity **H** is continuous at all the conductor boundaries → The value of $H\varphi$ does not show sudden jumps.



- **The shielding effect of coaxial cable applies for static and moving charges**

- Outside the coaxial cable, a complete cancellation of magnetic field occurs. Such coaxial cable would not produce any noticeable effect to the surroundings (“shielding”).

- As final example, consider a sheet of current flowing in the positive *y* direction and located in the $z = 0$ plane, with uniform surface current density $\mathbf{K} = K_y\mathbf{a}_y$.
- Due to symmetry, **H** cannot vary with *x* and *y*.
- If the sheet is subdivided into a number of filaments, it is evident that no filament can produce an H_y component.
- Moreover, the Biot-Savart law shows that the contributions to H_z produced by a symmetrically located pair of filaments cancel each other. → H_z is zero also.
- **Thus**, only H_x component is present.



- We therefore choose the path 1-1’-2’-2-1 composed of straight-line segments which are *__either parallel or perpendicular__* to *Hx* and *__enclose__* the current sheet.
- Ampere's circuital law gives

• **K : surface current density [A/m]**

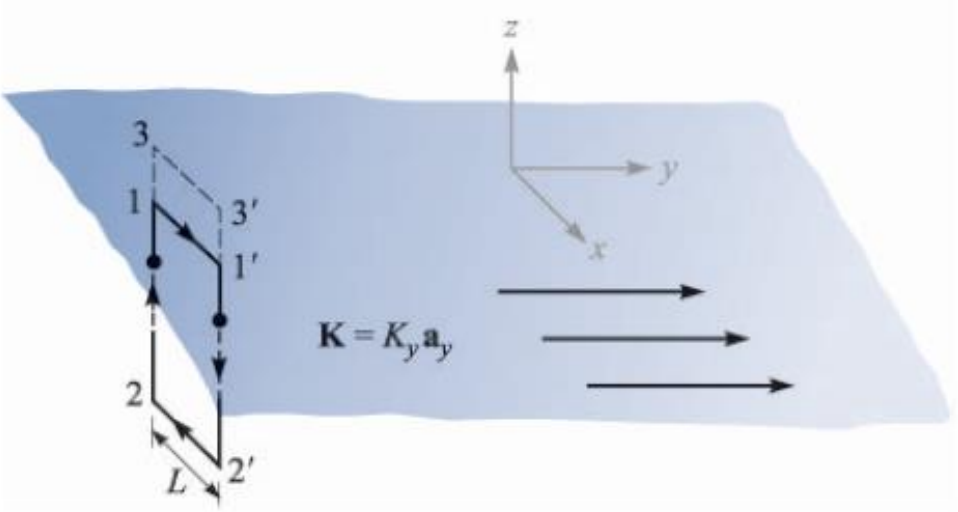
$$H_{x1}L + H_{x2}(-L) = K_y L \qquad \Rightarrow H_{x1} - H_{x2} = K_y$$

- If we choose a new path 3-3’-2’-2’3, the same current is enclosed, giving

$$H_{x3} - H_{x2} = K_y$$

and therefore

$$H_{x3} = H_{x1}$$



Ampere's Circuital Law

- Because of the symmetry, then, the magnetic field intensity on one side of the current sheet is the negative of that on the other side.
- Above the sheet

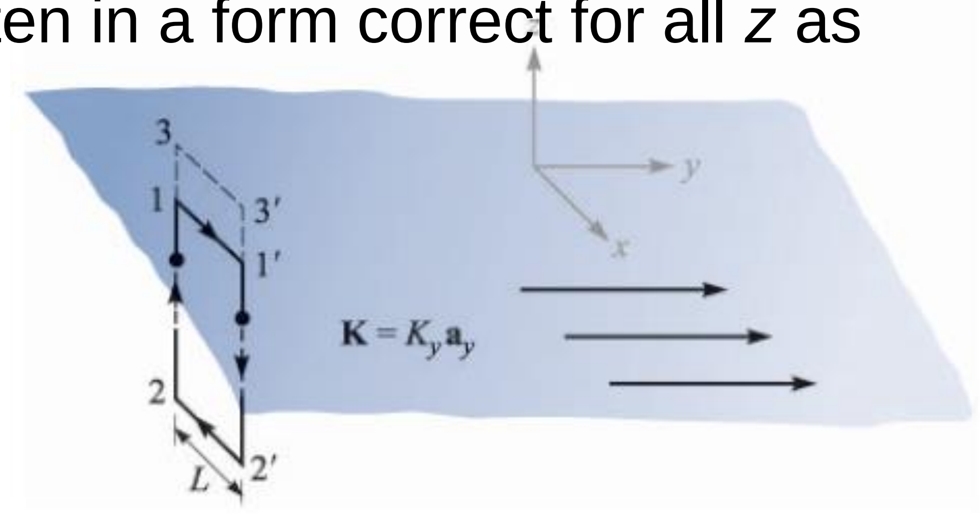
$$H_x = \tfrac{1}{2} K_y \quad (z > 0)$$

while below it

$$H_x = -\tfrac{1}{2} K_y \quad (z < 0)$$

- Letting $\mathbf{a}_N$ be a unit vector normal (outward) to the current sheet, this result may be written in a form correct for all *z* as

$$\mathbf{H} = \tfrac{1}{2} \mathbf{K} \times \mathbf{a}_N$$



- If a second sheet of current flowing in the opposite direction, $\mathbf{K} = -K_y\mathbf{a}_y$, is placed at $z = h$, then the field in the region between the current sheets is

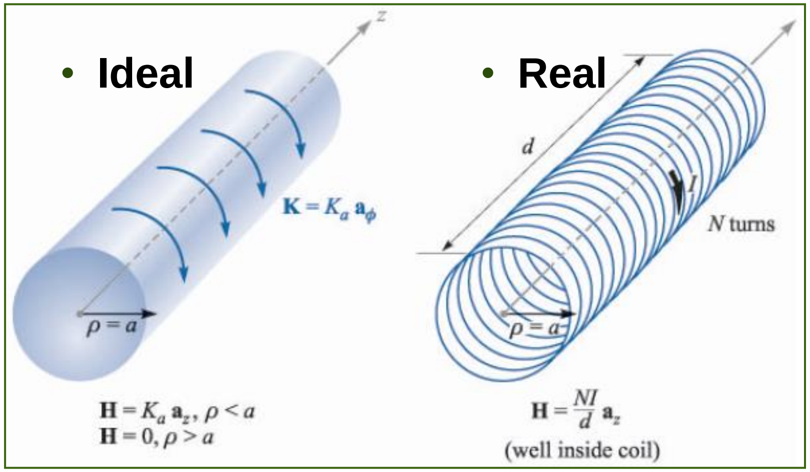
$$\mathbf{H} = \mathbf{K} \times \mathbf{a}_N \quad (0 < z < h)$$

and is zero elsewhere

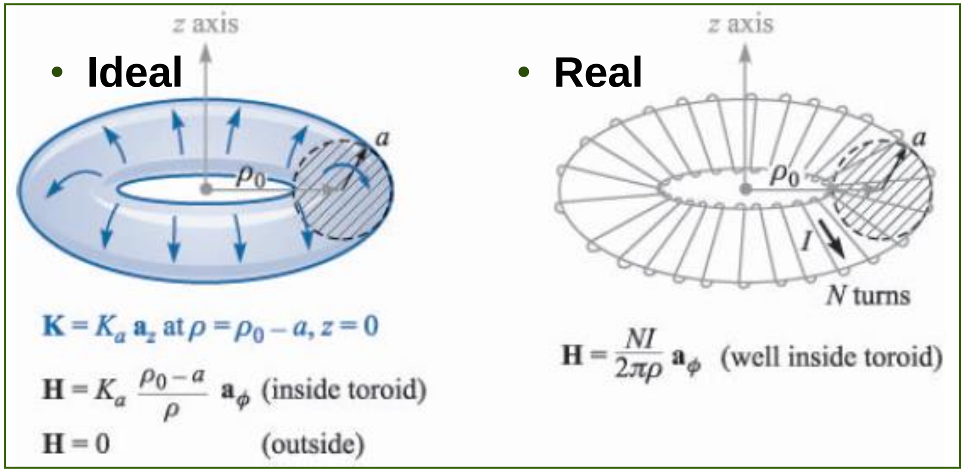
$$\mathbf{H} = 0 \quad (z < 0, z > h)$$

Ampere's Circuital Law

- The **difficult** part of the application of Ampere's circuital law is the determination of the components of the field which are present.
- The surest method is the logical application of the Biot-Savart law and a knowledge of the magnetic fields of simple form (line, sheet of current, "volume of current").

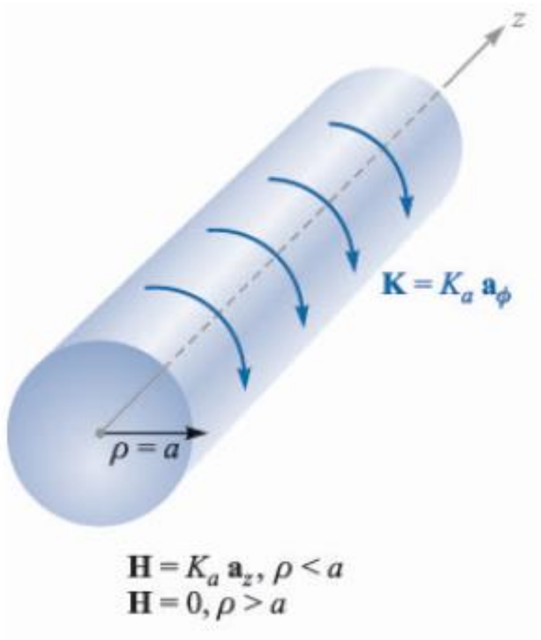


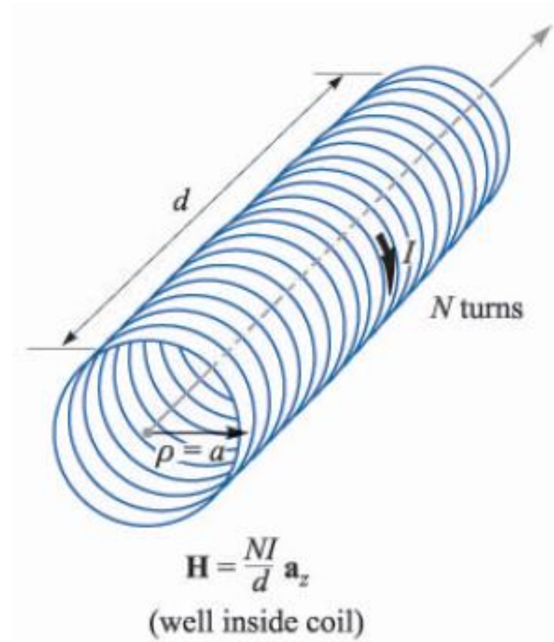
- Solenoid



- Toroid

Ampere's Circuital Law





- For an ideal solenoid, infinitely long with radius *a* and uniform current density $K_a\mathbf{a}_\varphi$, the result is

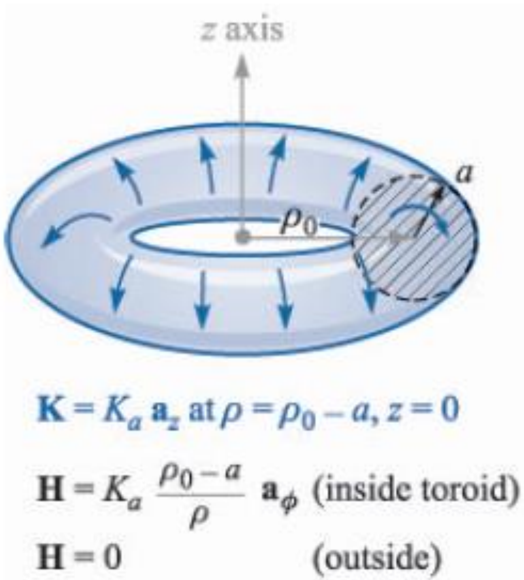
$$\mathbf{H} = K_a\mathbf{a}_z \quad (\rho < a)$$

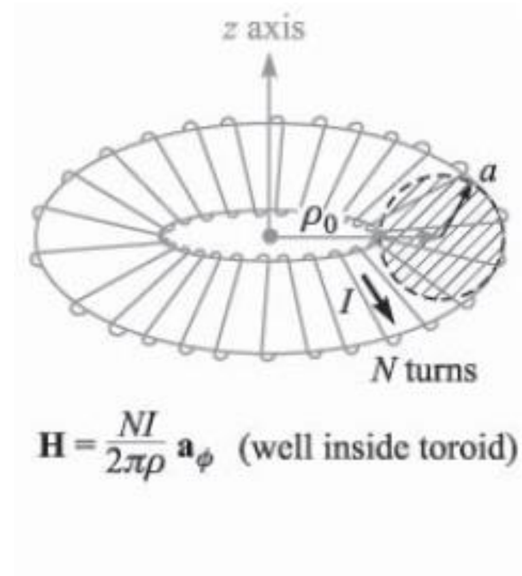
$$\mathbf{H} = 0 \quad (\rho > a)$$

- If the solenoid has a finite length d and consists of *N* closely wound turns of a filament that carries a current ***I***, then

$$\mathbf{H} = \frac{NI}{d}\mathbf{a}_z \quad \text{(well within the solenoid)}$$

Ampere's Circuital Law





- For a toroid with ideal case

$$\mathbf{H} = K_a \frac{\rho_0 - a}{\rho}\mathbf{a}_\phi \quad \text{(inside toroid)}$$

$$\mathbf{H} = 0 \quad \text{(outside toroid)}$$

- For the *N*-turn toroid, we have the good approximations

$$\mathbf{H} = \frac{NI}{2\pi\rho}\mathbf{a}_\phi \quad \text{(inside toroid)}$$

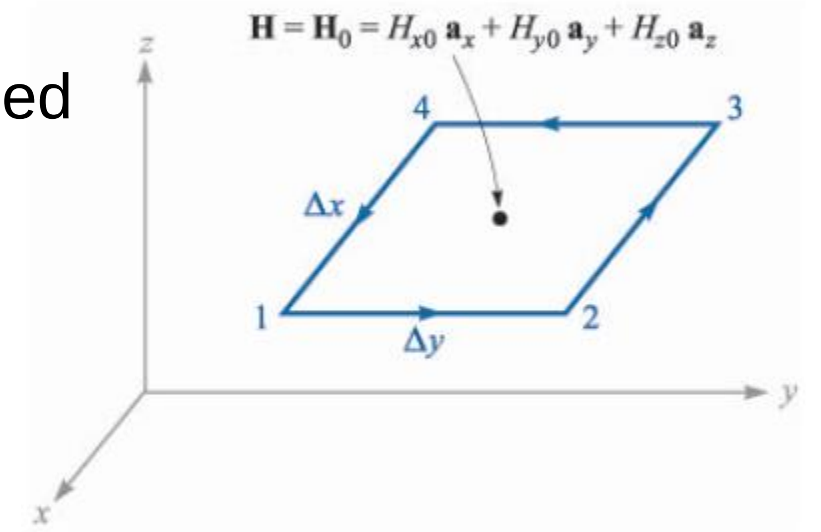
$$\mathbf{H} = 0 \quad \text{(outside toroid)}$$

Curl

- In our study of Gauss's law, we applied it to a differential volume element which led to the "Concept of Divergence."
- We now apply Ampere's circuital law to the <u>perimeter</u> of a differential surface element and discuss the *third* and last of the special derivatives of vector analysis, the ***curl***.
- Our immediate objective is to obtain the point form of Ampere's circuital law.

Curl

- Again, we choose rectangular coordinate, and an incremental closed path of sides Δx and Δy is selected.
- We assume that some current produces a reference value for **H** at the center of this small rectangle, given by



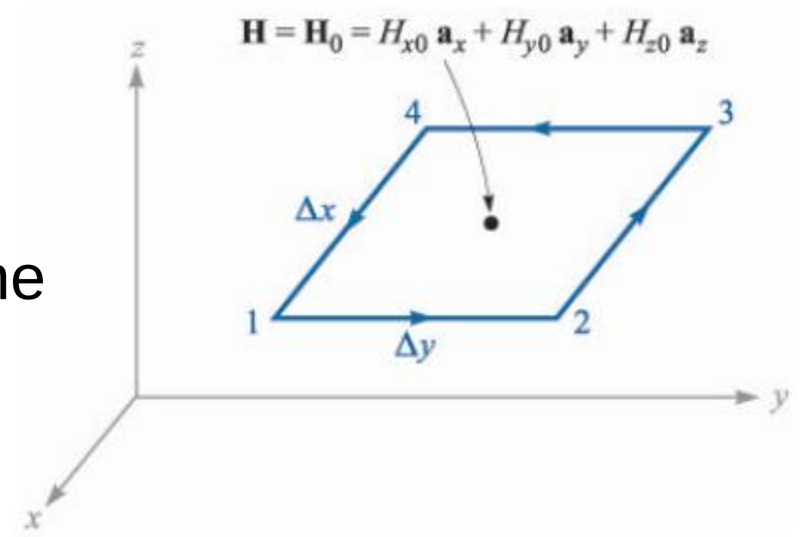
$$\mathbf{H}_0 = H_{x0}\mathbf{a}_x + H_{y0}\mathbf{a}_y + H_{z0}\mathbf{a}_z$$

- The closed line integral of **H** about this path is then approximately the sum of the four values of $\mathbf{H}\cdot\Delta\mathbf{L}$ on each side.
- We choose the direction of traverse as 1-2-3-4-1, which corresponds to a current in the $\mathbf{a}_z$ direction.
- The first contribution, from section 1-2, is therefore

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{1-2} = H_{y,1-2}\Delta y$$

Curl

- The value of H_y on section 1-2 may be given in terms of the reference value H_{y0} at the center of the rectangle, the rate of change of H_y with x, and the distance $\Delta x/2$ from the center to the midpoint of side 1-2.



$$H_{y,1-2} \doteq H_{y0} + \frac{\partial H_y}{\partial x}\left(\frac{1}{2}\Delta x\right)$$

- Thus,

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{1-2} \doteq \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)\Delta y$$

- The next contribution, from section 2-3, is given as

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{2-3} \doteq H_{x,2-3}(-\Delta x) \doteq -\left(H_{x0} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)\Delta x$$

Curl

- Further, section 3-4 will give

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{3-4} \doteq \left(H_{y0} - \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)(-\Delta y)$$

- Finally, section 4-1 will give

$$(\mathbf{H}\cdot\Delta\mathbf{L})_{4-1} \doteq \left(H_{x0} - \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)\Delta x$$

- Adding the results from all 4 sections, we obtain

$$\oint \mathbf{H}\cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x\Delta y$$

Curl

- By Ampere's circuital law, this closed path integration of magnetic field intensity **H** must be equal to the current <u>enclosed</u> by the path, or the current crossing any surface <u>bounded</u> by the path.
- If we assume a general current density **J**, the enclosed current is then

$$\Delta I \doteq J_z \Delta x \Delta y$$

and

$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \Delta x \Delta y \doteq J_z \Delta x \Delta y$$

- Finally,

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} \doteq \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \doteq J_z$$

Curl

- As we let the closed path shrink, the approximation becomes more nearly exact, and we have the current density in *z* direction.

$$\lim_{\Delta x,\Delta y\to 0}\frac{\oint \mathbf{H}\cdot d\mathbf{L}}{\Delta x\Delta y}=\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=J_z$$

- If we choose closed paths which are oriented perpendicularly to each of the remaining two coordinate axes, analogous processes lead to expression for the *x* and *y* components of the current density

$$\lim_{\Delta y,\Delta z\to 0}\frac{\oint \mathbf{H}\cdot d\mathbf{L}}{\Delta y\Delta z}=\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=J_x$$

$$\lim_{\Delta z,\Delta x\to 0}\frac{\oint \mathbf{H}\cdot d\mathbf{L}}{\Delta z\Delta x}=\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=J_y$$

Curl

- Comparing all equation in the previous slide, we can conclude that "a component of a current density in a certain direction is given by the limit of the quotient of the closed line integral of **H** about a small path in a plane normal to that component as the area enclosed by the path shrinks to zero."
- This limit has its counterpart in other fields of science and received the name of ***curl***.

- The mathematical form of the curl is

$$(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N}$$

- ΔS_N is the planar area enclosed by the closed line integral, while N subscript indicated that the component of the curl is normal to the surface enclosed by the closed path.

Curl

- In rectangular coordinates, the curl **H** is given by

$$\text{curl } \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{a}_z$$

Rectangular

- This result may be written in the form of a determinant or in terms of the vector operator, as follows

$$\text{curl } \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H}$$

Curl

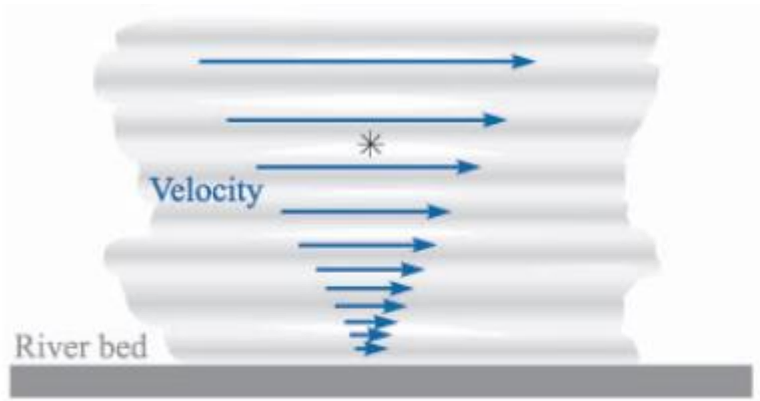
$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right) \mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} \right) \mathbf{a}_\phi + \frac{1}{\rho} \left(\frac{\partial (\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi} \right) \mathbf{a}_z$$

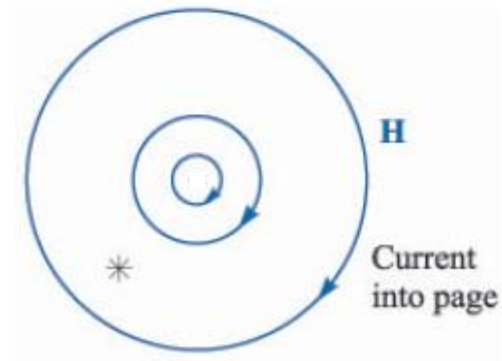
Cylindrical

$$\nabla \times \mathbf{H} = \frac{1}{r \sin\theta} \left(\frac{\partial (H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi} \right) \mathbf{a}_r + \frac{1}{r} \left(\frac{1}{\sin\theta} \frac{\partial H_r}{\partial \phi} - \frac{\partial (r H_\phi)}{\partial r} \right) \mathbf{a}_\theta + \frac{1}{r} \left(\frac{\partial (r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta} \right) \mathbf{a}_\phi$$

Spherical

Small Paddle Wheel as a Curl Meter



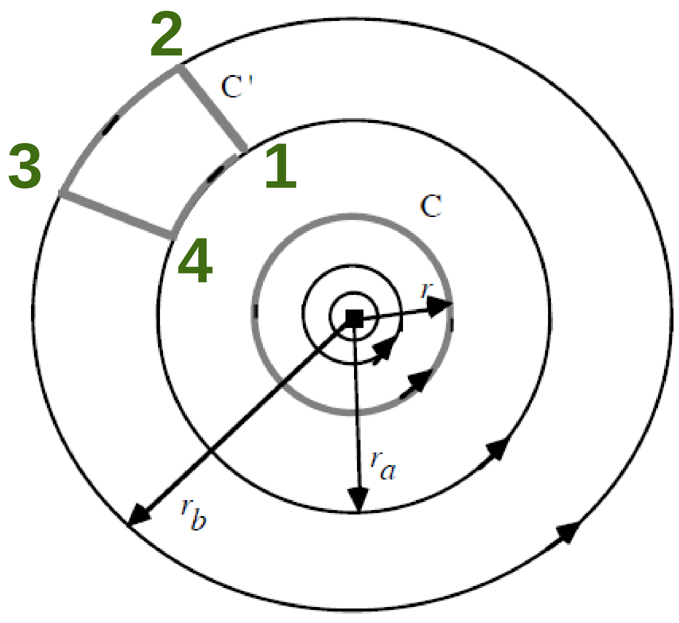


- **Clockwise rotation**
- **No rotation**

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi$$

$$\text{curl } \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho}\mathbf{a}_z = 0$$

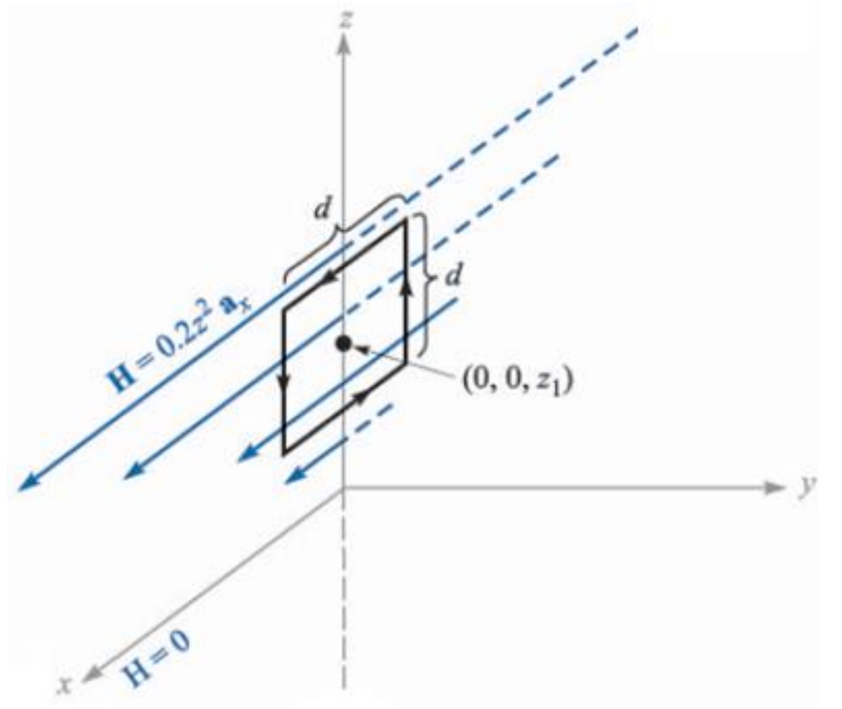
Small Paddle Wheel as a Curl Meter



$$\oint \mathbf{H}\cdot d\mathbf{L} = \underbrace{\int_{1-2} \mathbf{H}\cdot d\mathbf{L}}_{0} + \int_{2-3} \mathbf{H}\cdot d\mathbf{L} + \underbrace{\int_{3-4} \mathbf{H}\cdot d\mathbf{L}}_{0} + \int_{4-1} \mathbf{H}\cdot d\mathbf{L}$$

$$= \int_{\phi_A}^{\phi_B} \frac{I}{2\pi r_B}\cdot r_B d\phi + \int_{\phi_B}^{\phi_A} \frac{I}{2\pi r_A}\cdot r_A d\phi \quad = 0$$

- <u>Example</u>
  Let $\mathbf{H}=0.2z^2\mathbf{a}_x$ for $z>0$, and $\mathbf{H}=0$ elsewhere, as shown in the next figure. Calculate $\oint \mathbf{H}\cdot d\mathbf{L}$ about a square path with side d, centered at $(0,0,z_1)$ in the $y=0$ plane where $z_1>d/2$.



First $\oint \mathbf{H}\cdot d\mathbf{L} = 0.2(z_1+\tfrac{1}{2}d)^2 d + 0$

$$- 0.2(z_1-\tfrac{1}{2}d)^2 d + 0$$

$$= 0.4 z_1 d^2,$$

then $(\nabla\times\mathbf{H})_y = \lim_{d\to 0}\dfrac{\oint \mathbf{H}\cdot d\mathbf{L}}{d^2}$

$$= \lim_{d\to 0}\frac{0.4z_1 d^2}{d^2} = \underline{\underline{0.4z_1}}$$

Or $\nabla\times\mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0.2z^2 & 0 & 0 \end{vmatrix}$

$$= \frac{\partial}{\partial z}(0.2z^2)\mathbf{a}_y - \frac{\partial}{\partial y}(0.2z^2)\mathbf{a}_z$$

$$= \underline{\underline{0.4z\mathbf{a}}}_y$$

- To complete our original examination of the application of Ampere’s circuital law to a differential-sized path, we may write

$$\text{curl}\ \mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{a}_z = \mathbf{J}$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

- **Point Form of Ampere’s Circuital law**
- **Second Maxwell’s Equation, non-time varying condition.**

$$\nabla \times \mathbf{E} = 0$$

- **Point Form of Potential Difference law**
- **Third Maxwell’s Equation, non-time varying condition.**

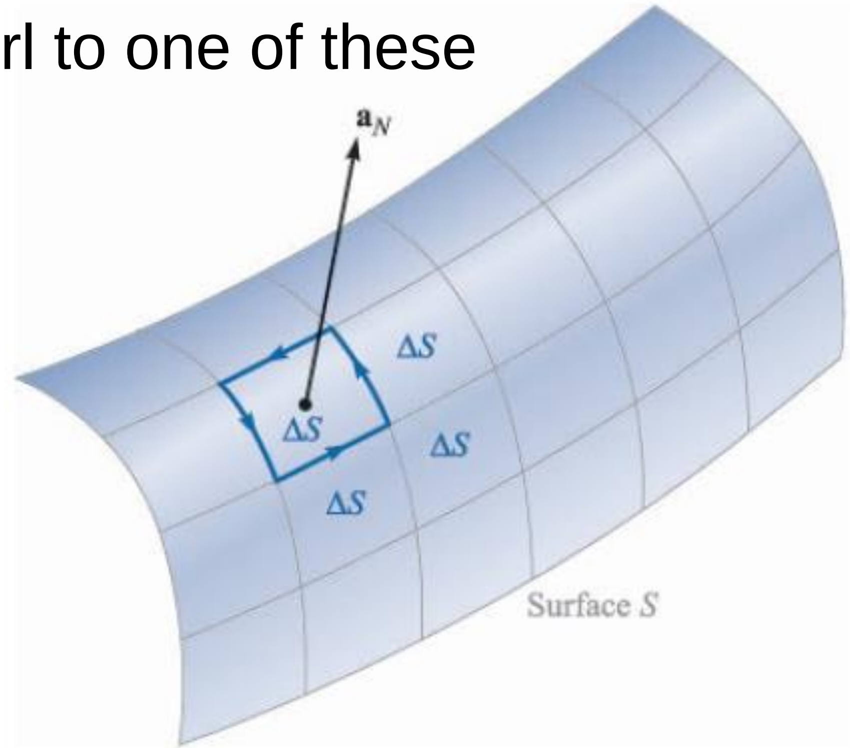
Stokes' Theorem

- Previously, from Ampere's circuital law, we derive one of Maxwell's equations, $\nabla \times \mathbf{H} = \mathbf{J}$.
- This equation is to be considered as the point form of Ampere's circuital law and applies on a "per-unit-area" basis.

- Now, we shall devote the material to a mathematical theorem known as Stokes' theorem.
- In the process, we shall show that we may obtain Ampere's circuital law from $\nabla \times \mathbf{H} = \mathbf{J}$.

Stokes' Theorem

- Consider the surface *S* of the next figure, which is broken up into incremental surfaces of area Δ*S*.
- If we apply the definition of the curl to one of these incremental surfaces, then:



$$\frac{\oint \mathbf{H}\cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla\times\mathbf{H})_N$$

or

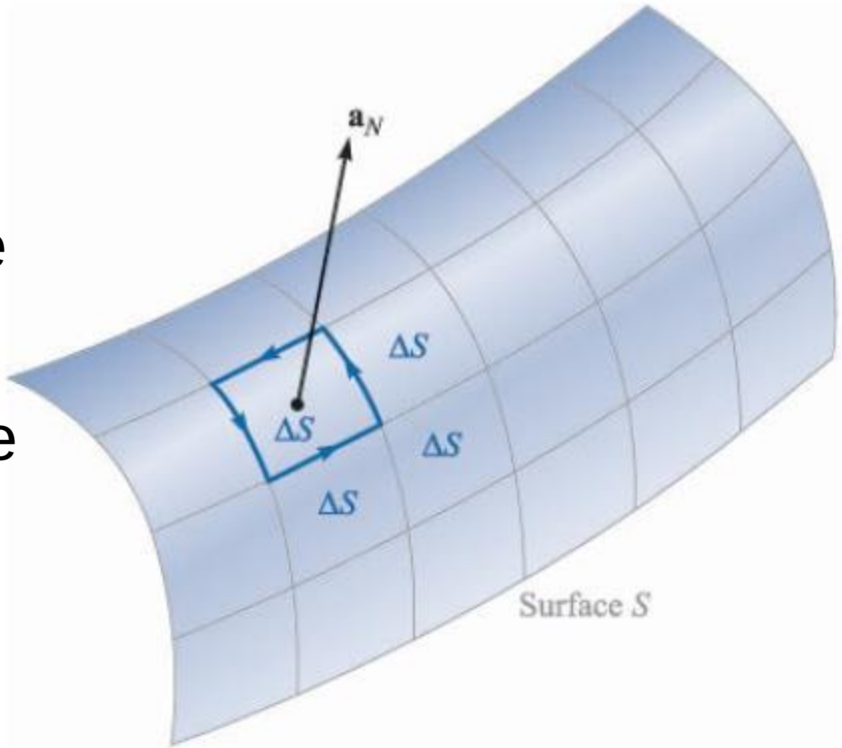
$$\frac{\oint \mathbf{H}\cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla\times\mathbf{H})\cdot\mathbf{a}_N$$

or

$$\oint \mathbf{H}\cdot d\mathbf{L}_{\Delta S} \doteq (\nabla\times\mathbf{H})\cdot\mathbf{a}_N\Delta S = (\nabla\times\mathbf{H})\cdot\Delta\mathbf{S}$$

Stokes' Theorem

- Let us now perform the circulation for every Δ*S* comprising *S* and sum the results.
- As we evaluate the closed line integral for each Δ*S*, some cancellation will occur because every interior wall is covered once in each direction.
- The only boundaries on which cancellation cannot occur form the outside boundary, the path enclosing *S*.



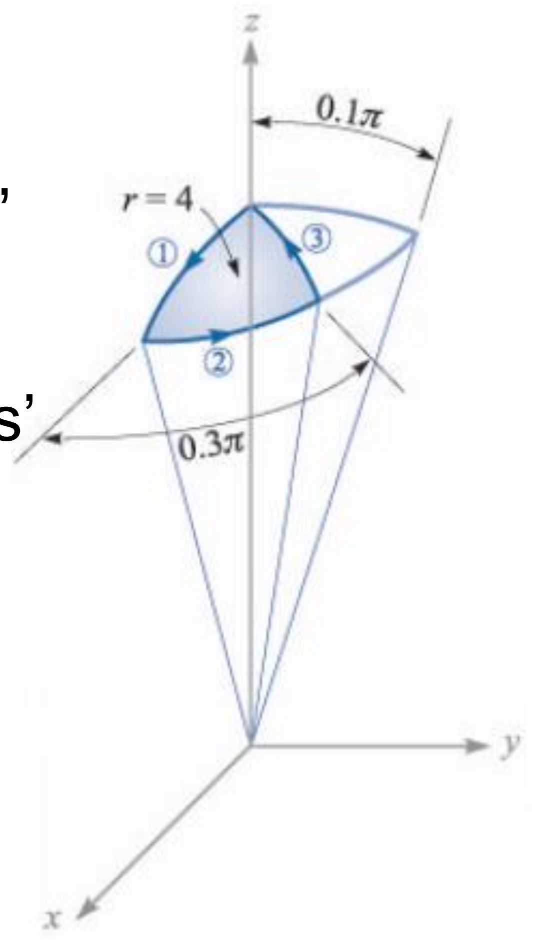
- Therefore,

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$

where $d\mathbf{L}$ is taken only on the perimeter of *S*.

Stokes' Theorem

- <u>Example</u>
  Consider the portion of a sphere as shown. The surface is specified by $r = 4$, $0 \le \theta \le 0.1\pi$, $0 \le \phi \le 0.3\pi$. The closed path forming its perimeter is composed of three circular arcs. Given the magnetic field $\mathbf{H} = 6r\sin\phi\,\mathbf{a}_r + 18r\sin\theta\cos\phi\,\mathbf{a}_\phi$ A/m, evaluate each side of Stokes' theorem.



$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$$

$$\oint \mathbf{H}\cdot d\mathbf{L} = \int_1 H_\theta r\,d\theta + \int_2 H_\phi r\sin\theta\,d\phi + \int_3 H_\theta r\,d\theta$$

$$\oint \mathbf{H}\cdot d\mathbf{L} = \int_0^{0.3\pi} \left[18(4)\sin 0.1\pi\cos\phi\right] 4\sin 0.1\pi\,d\phi$$

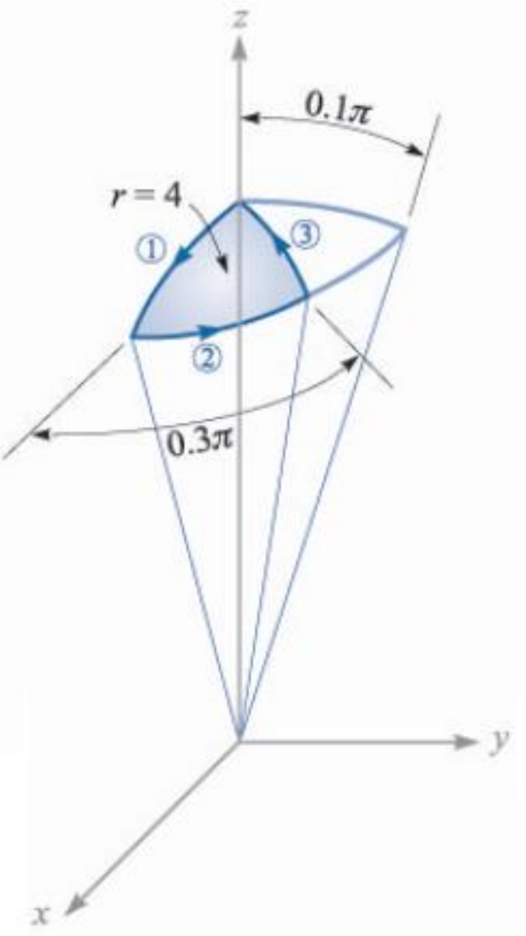
$$= 288\sin^2 0.1\pi\sin 0.3\pi \quad = \underline{\underline{22.2\text{ A}}}$$

$$\oint \mathbf{H}\cdot d\mathbf{L} = \int_S (\nabla\times\mathbf{H})\cdot d\mathbf{S}$$

Stokes' Theorem

$$d\mathbf{S} = r^2 \sin\theta d\theta d\phi\, \mathbf{a}_r$$

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\left(36r\sin\theta\cos\theta\cos\phi\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}6r\cos\phi - 36r\sin\theta\cos\phi\right)\mathbf{a}_\theta$$



$$\int_S (\nabla\times\mathbf{H})\cdot d\mathbf{S} = \int_0^{0.3\pi}\int_0^{0.1\pi}(36\cos\theta\cos\phi)(16\sin\theta d\theta d\phi)$$

$$= \int_0^{0.3\pi} 576(\tfrac{1}{2}\sin^2\theta)\Big|_0^{0.1\pi}\cos\phi d\phi$$

$$= 288\sin^2 0.1\pi \sin 0.3\pi$$

$$= \underline{\underline{22.2\text{ A}}}$$

$$\oint \mathbf{H}\cdot d\mathbf{L} = \int_S (\nabla\times\mathbf{H})\cdot d\mathbf{S}$$

Magnetic Flux and Magnetic Flux Density

- In free space, let us define magnetic flux density **B** as

$$\mathbf{B} = \mu_0 \mathbf{H}$$

where **B** is measured in webers per square meter (Wb/m^2) or tesla (T).

- The constant μ_0 is not dimensionless and has a defined value for free space, in henrys per meter (H/m), of

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$

- The magnetic-flux-density vector **B**, as the name weber per square meter implies, is a member of the flux-density family of vector fields.
- Comparing the laws of Biot-Savart and Coulomb, one can find analogy between **H** and **E** that leads to an analogy between **B** and **D**; $\mathbf{D} = \varepsilon_0\mathbf{E}$ and $\mathbf{B} = \mu_0\mathbf{H}$.

Magnetic Flux and Magnetic Flux Density

- If **B** is measured in teslas or webers per square meter, then magnetic flux Φ should be measured in webers.
- Let us represent magnetic flux by Φ and define Φ as the flux passing through any designated area,

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \quad \text{Wb}$$

$$\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

- We remember that Gauss's law states that the total electric flux passing through any closed surface is equal to the charge enclosed. This charge is the source of the electric flux **D**.
- For magnetic flux, no current source can be enclosed, since the current is considered to be in closed circuit.

Magnetic Flux and Magnetic Flux Density

- For this reason, the Gauss's law for the magnetic field can be written as

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0$$

- Through the application of the divergence theorem, we can also find that

$$\nabla \cdot \mathbf{B} = 0$$

- **Fourth Maxwell's Equation, static electric fields and steady magnetic fields.**

Magnetic Flux and Magnetic Flux

- Collecting all equations we have until now of static electric fields and steady magnetic fields,

$$\nabla \cdot \mathbf{D} = \rho_v \qquad \nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \times \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

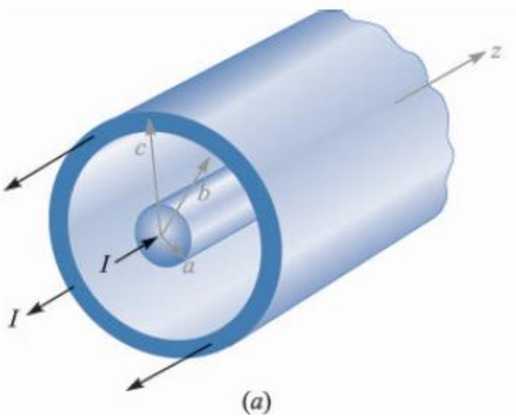
- The corresponding set of four integral equations that apply to static electric fields and steady magnetic fields is

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv \qquad \oint \mathbf{H} \cdot d\mathbf{L} = I = \oint_S \mathbf{J} \cdot d\mathbf{S}$$

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0 \qquad \oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Example

Find the flux between the conductors of the coaxial line we have discussed previously, for $a \le \rho \le b$, $0 \le z \le d$.



$$H_\phi = \frac{I}{2\pi\rho} \qquad (a < \rho < b)$$

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi \cdot d\rho dz\, \mathbf{a}_\phi$$

$$\Phi = \frac{\mu_0 I d}{2\pi} \ln \frac{b}{a}$$

Scalar Magnetic Potential

We are already familiar with the relation between the scalar electric potential and electric field:

$$\mathbf{E} = -\nabla V$$

So it is tempting to define a scalar magnetic potential such that:

$$\mathbf{H} = -\nabla V_m$$

This rule must be consistent with Maxwell's equations, so therefore:

$$\nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla V_m)$$

But the curl of the gradient of any function is identically zero! Therefore, the scalar magnetic potential is valid only in regions where the current density is zero (such as in free space).

So we define scalar magnetic potential with a condition:

$$\mathbf{H} = -\nabla V_m \quad (\mathbf{J} = 0)$$

Further Requirements on the Scalar Magnetic Potential

The other Maxwell equation involving magnetic field must also be satisfied. This is:

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \mathbf{H} = 0$$ **in free space**

Therefore: $$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

..and so the scalar magnetic potential satisfies Laplace's equation (again with the restriction that current density must be zero:

$$\nabla^2 V_m = 0 \quad (\mathbf{J} = 0)$$

With the center conductor current flowing out of the screen, we have

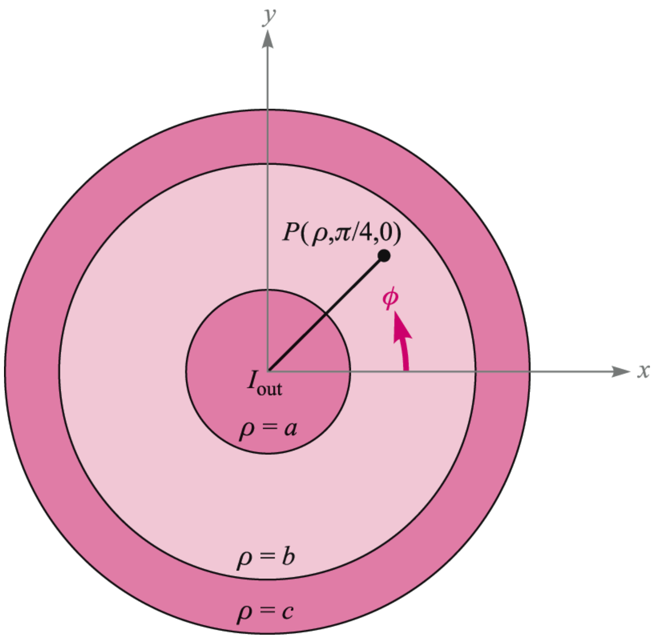
$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi$$

Thus: $$\frac{I}{2\pi\rho} = -\nabla V_m\Big|_\phi = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$

So we solve: $$\frac{\partial V_m}{\partial \phi} = -\frac{I}{2\pi}$$

.. and obtain: $$V_m = -\frac{I}{2\pi}\phi$$

where the integration constant has been set to zero



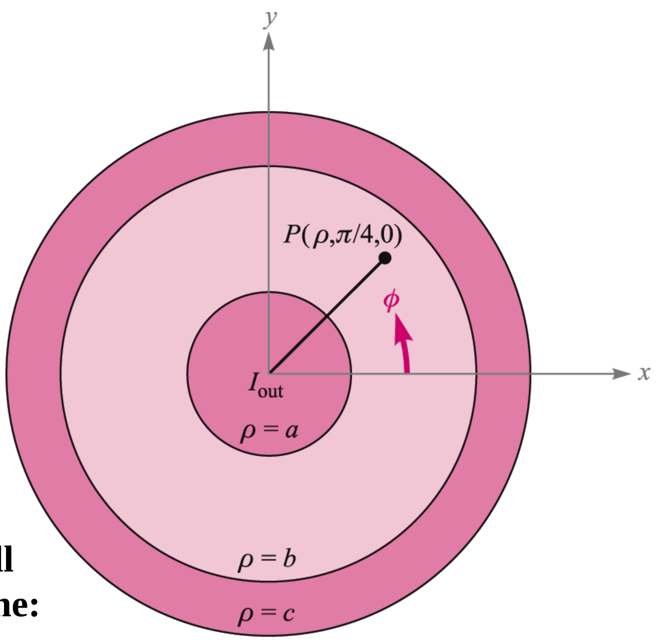
The scalar potential is now:

$$V_m = -\frac{I}{2\pi}\phi$$

where the potential is zero at $\phi = 0$

At point P ($\phi = \pi/4$**) the potential is**

$$V_{mP}(\phi = \pi/4) = -I/8$$



But wait! As ϕ **increases t** $\phi = 2\pi$
we have returned to the same physical location, and the potential has a new value of -I.

In general, the potential at P will be multivalued, and will acquire a new value after each full rotation in the xy plane:

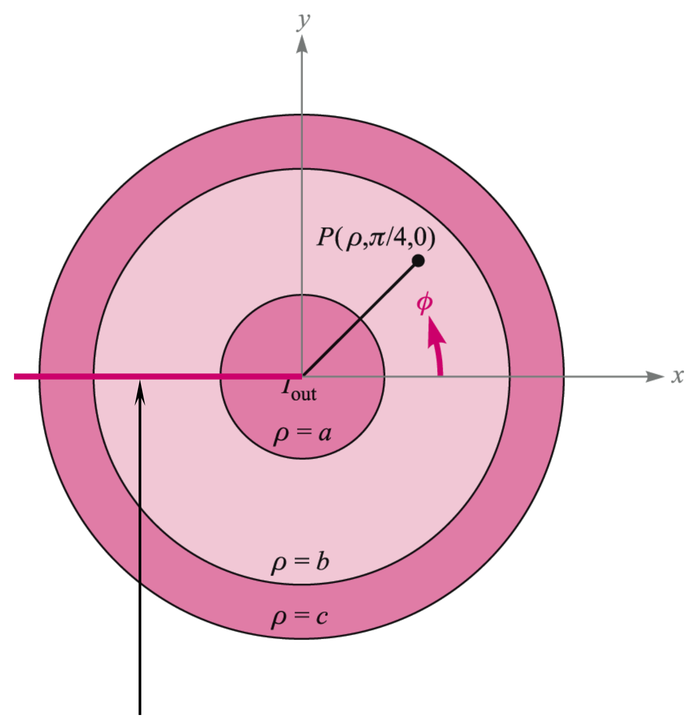
$$V_{mP} = \frac{I}{2\pi}\left(2n - \tfrac{1}{4}\right)\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$$

To remove the ambiguity, we construct a mathematical barrier at any value of phi. The angle domain cannot cross this barrier in either direction, and so the potential function is restricted to angles on either side. In the present case we choose the barrier to lie $\phi = \pi$ so that

$$V_m = -\frac{I}{2\pi}\phi \quad (-\pi < \phi < \pi)$$

The potential at point P is now single-valued:

$$V_{mP} = -\frac{I}{8} \quad \left(\phi = \frac{\pi}{4}\right)$$



Barrier at $\phi = \pi$

Vector Magnetic Potential

We make use of the Maxwell equation: $\nabla \cdot \mathbf{B} = 0$

.. and the fact that the divergence of the curl of any vector field is identically zero (show this!)

$$\nabla \cdot \nabla \times \mathbf{A} = 0$$

This leads to the definition of the magnetic vector potential, A:

$$\underline{\mathbf{B} = \nabla \times \mathbf{A}}$$

Thus: $\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A}$

and Ampere's Law becomes $\nabla \times \mathbf{H} = \mathbf{J} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A}$

Equation for the Vector Potential

We start with: $$\nabla \times \mathbf{H} = \mathbf{J} = \frac{1}{\mu_0} \nabla \times \nabla \times \mathbf{A}$$

Then, introduce a vector identity that defines the vector Laplacian:

$$\nabla^2 \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

Using a (lengthy) procedure (see Sec. 7.7) it can be proven that $\nabla \cdot \mathbf{A} = 0$

We are therefore left with

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

The Direction of **A**

We now have $$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

In rectangular coordinates:

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$$

(not so simple in the other coordinate systems)

The equation separates to give: $$\nabla^2 A_x = -\mu_0 J_x$$

$$\nabla^2 A_y = -\mu_0 J_y$$

$$\nabla^2 A_z = -\mu_0 J_z$$

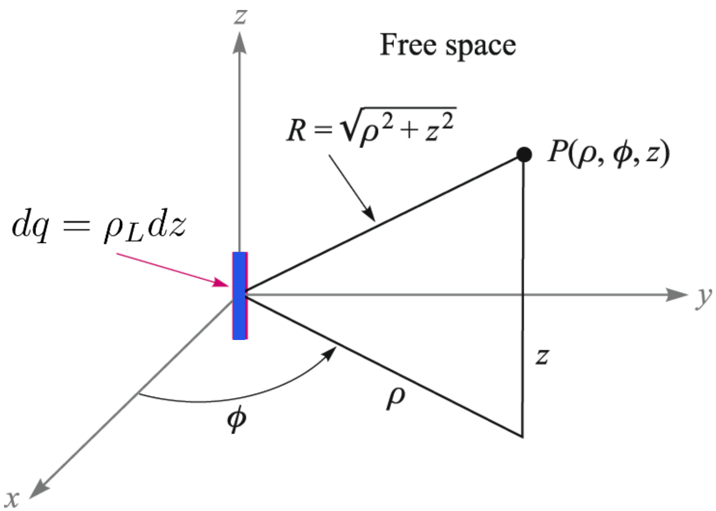
This indicates that the direction of A will be the same as that of the current to which it is associated.

The vector field, A, existing in all space, is sometimes described as being a "fuzzy image" of its generating current.

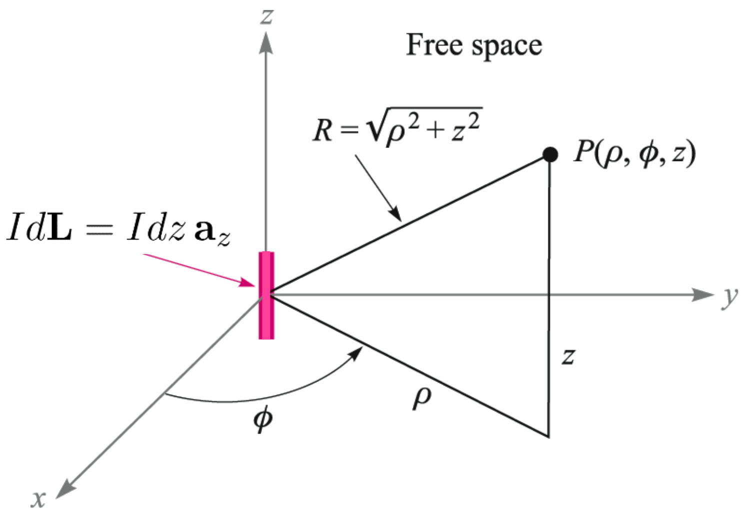
Expressions for Potential

Consider a differential elements, shown here. On the left is a point charge represented by a differential length of line charge. On the right is a differential current element. The setups for obtaining potential are identical between the two cases.

Line Charge



Line Current



Scalar Electrostatic Potential

$$dV = \frac{dq}{4\pi\epsilon_0 R} = \frac{\rho_L dL}{4\pi\epsilon_0 R}$$

Vector Magnetic Potential

$$d\mathbf{A} = \frac{\mu_0 I d\mathbf{L}}{4\pi R} = \frac{\mu_0 I dz\,\mathbf{a}_z}{4\pi R}$$

General Expressions for Vector Potential

For large scale charge or current distributions, we would sum the differential contributions by integrating over the charge or current, thus:

$$V = \int \frac{\rho_L dL}{4\pi\epsilon_0 R}$$

and

$$\mathbf{A} = \oint \frac{\mu_0 I \, d\mathbf{L}}{4\pi R}$$

The closed path integral is taken because the current must close on itself to form a complete circuit.

For surface or volume current distributions, we would have, respectively:

$$\mathbf{A} = \int_S \frac{\mu_0 \mathbf{K} \, dS}{4\pi R}$$

or

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu_0 \mathbf{J} \, dv}{4\pi R}$$

in the same manner that we used for scalar electric potential.

Example

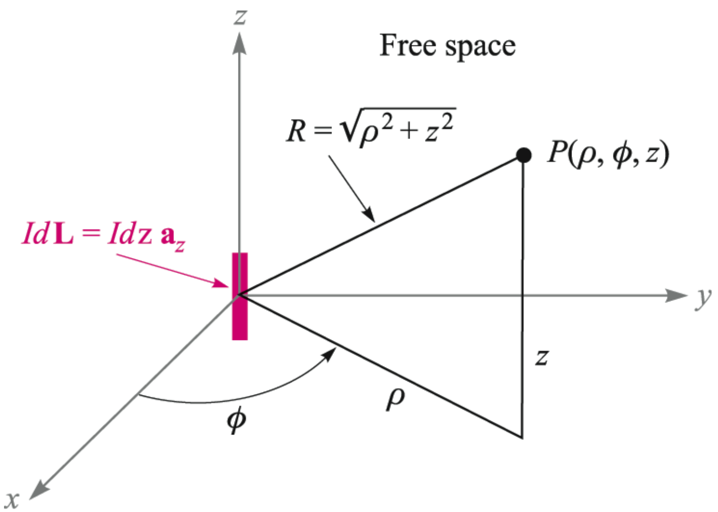
We continue with the differential current element as shown here:

In this case

$$dA = \frac{\mu_0 I \, d\mathbf{L}}{4\pi R}$$

becomes at point P:

$$d\mathbf{A} = \frac{\mu_0 I \, dz \, \mathbf{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$$



Now, the curl is taken in cylindrical coordinates:

$$d\mathbf{H} = \frac{1}{\mu_0}\nabla \times d\mathbf{A} = \frac{1}{\mu_0}\left(-\frac{\partial dA_z}{\partial \rho}\right)\mathbf{a}_\phi = \frac{I\,dz}{4\pi}\frac{\rho}{(\rho^2 + z^2)^{3/2}}\mathbf{a}_\phi$$

This is the same result as found using the Biot-Savart Law (as it should be)


Good
Luck