Module 4

- **Magnetic Forces**: Force on a moving charge, differential current elements, Force between differential current elements, Numerical problems **(Text: Chapter 9.1 to 9.3).**
- **Magnetic Materials**: Magnetization and permeability, Magnetic boundary conditions, The magnetic circuit, Potential energy and forces on magnetic materials, Inductance and mutual reactance, Numerical problems **(Text: Chapter 9.6 to 9.7).**
- Faraday' law of Electromagnetic Induction –Integral form and Point form, Numerical problems **(Text: Chapter 10.1)**

MAGNETIC FORCES, MATERIALS, AND INDUCTANCE

FORCE ON A MOVING CHARGE

In an electric field the definition of the electric field intensity shows us that the force on a charged particle is

$$\mathbf{F} = Q\mathbf{E}$$

The force is in the same direction as the electric field intensity (for a positive charge) and is directly proportional to both E and Q. If the charge is in motion, the force at any point in its trajectory is then given by (1).

A charged particle in motion in a magnetic field of flux density B is found experimentally to experience a force whose magnitude is proportional to the product of the magnitudes of the charge Q, its velocity v, and the flux density B, and to the sine of the angle between the vectors v and B. The direction of the force is perpendicular to both v and B and is given by a unit vector in the direction of v Â B. The force may therefore be expressed as

$$\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$$

The force on a moving particle due to combined electric and magnetic fields is obtained easily by superposition,

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

This equation is known as the Lorentz force equation, and its solution is required in determining electron orbits in the magnetron, proton paths in the cyclotron, plasma characteristics in a magnetohydrodynamic (MHD) generator, or, in general, charged-particle motion in combined electric and magnetic fields.

FORCE ON A DIFFERENTIAL CURRENT ELEMENT

The force on a charged particle moving through a steady magnetic field may be written as the differential force exerted on a differential element of charge,

$$d\mathbf{F} = dQ\,\mathbf{v} \times \mathbf{B} \qquad \mathbf{J} = \rho_v \mathbf{v} \qquad dQ = \rho_v dv \qquad d\mathbf{F} = \rho_v dv\,\mathbf{v} \times \mathbf{B}$$

$$d\mathbf{F} = \mathbf{J} \times \mathbf{B}\, dv$$

$$\mathbf{J}\, dv = \mathbf{K}\, dS = I\, d\mathbf{L}$$

$$d\mathbf{F} = \mathbf{K} \times \mathbf{B}\, dS$$

the Lorentz force equation may be applied to surface current density,

$$d\mathbf{F} = I\, d\mathbf{L} \times \mathbf{B}$$

or to a differential current filament,

$$\mathbf{F} = \oint I\, d\mathbf{L} \times \mathbf{B} = -I \oint \mathbf{B} \times d\mathbf{L}$$

One simple result is obtained by applying (7) or (10) to a straight conductor in a uniform magnetic field,

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B}$$

FORCE BETWEEN DIFFERENTIAL CURRENT ELEMENTS

The magnetic field at point 2 due to a current element at point 1 was found to be

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$

the differential force on a differential current element is

$$d\mathbf{F} = I\, d\mathbf{L} \times \mathbf{B}$$

the differential amount of our differential force on element 2 as

$$d(d\mathbf{F}_2) = I_2 d\mathbf{L}_2 \times d\mathbf{B}_2$$

we obtain the force between two differential current elements,

$$d(d\mathbf{F}_2) = \mu_0 \frac{I_1 I_2}{4\pi R_{12}^2} d\mathbf{L}_2 \times (d\mathbf{L}_1 \times \mathbf{a}_{R12})$$

The total force between two filamentary circuits is obtained by integrating twice:

$$\mathbf{F}_2 = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[d\mathbf{L}_2 \times \oint \frac{d\mathbf{L}_1 \times \mathbf{a}_{R12}}{R_{12}^2} \right]$$

$$= \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[\oint \frac{\mathbf{a}_{R12} \times d\mathbf{L}_1}{R_{12}^2} \right] \times d\mathbf{L}_2$$

MAGNETIZATION AND PERMEABILITY

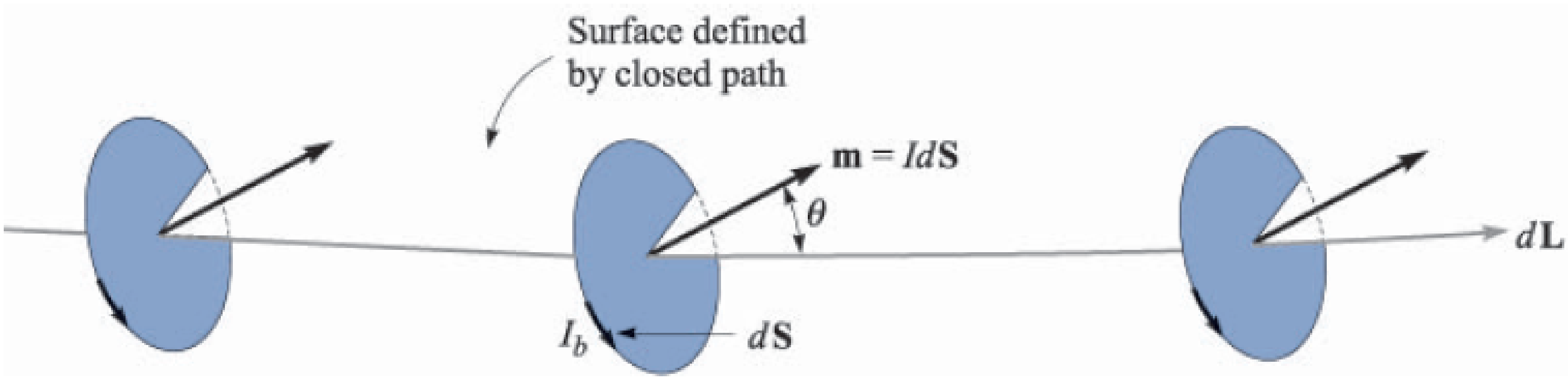


FIGURE 9.9
A section $d\mathbf{L}$ of a closed path along which magnetic dipoles have been partially aligned by some external magnetic field. The alignment has caused the bound current crossing the surface defined by the closed path to increase by $nI_b d\mathbf{S} \cdot d\mathbf{L}$ amperes.

Let us begin by defining the magnetization $\mathbf{M}$ in terms of the magnetic dipole moment $\mathbf{m}$. The bound current I_b circulates about a path enclosing a differential area $d\mathbf{S}$, establishing a dipole moment (A·m^2),

$$\mathbf{m} = I_b d\mathbf{S}$$

If there are n magnetic dipoles per unit volume and we consider a volume Δv, then the total magnetic dipole moment is found by the vector sum

$$\mathbf{m}_{\text{total}} = \sum_{i=1}^{n\Delta v} \mathbf{m}_i \tag{19}$$

Each of the $\mathbf{m}_i$ may be different. Next, we define the *magnetization* $\mathbf{M}$ as the *magnetic dipole moment per unit volume*,

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \mathbf{m}_i \qquad I_b = \oint \mathbf{M} \cdot d\mathbf{L} \qquad \oint \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{L} = I_T$$

$$I_T = I_b + I$$

I is the total free current enclosed by the closed path.

$$I = I_T - I_b = \oint \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) \cdot d\mathbf{L} \qquad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad \boxed{\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})}$$

Utilizing the several current densities, we have

$$I_b = \oint_S \mathbf{J}_b \cdot d\mathbf{S}$$

$$I_T = \oint_S \mathbf{J}_T \cdot d\mathbf{S}$$

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S}$$

$$\nabla \times \mathbf{M} = \mathbf{J}_b$$

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \mathbf{J}_T$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

a magnetic susceptibility m can be defined:

$$\boxed{\mathbf{M} = \chi_m \mathbf{H}}$$

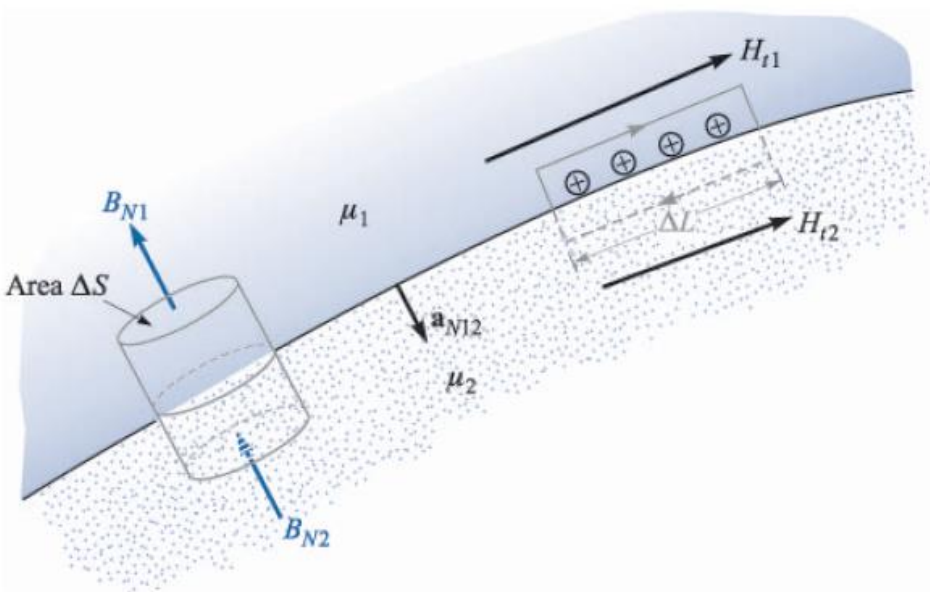
$$\mathbf{B} = \mu_0(\mathbf{H} + \chi_m \mathbf{H}) = \mu_0 \mu_R \mathbf{H}$$

$$\mu_R = 1 + \chi_m$$

is defined as the relative permeability R . We next define the permeability :

$$\mu = \mu_0 \mu_R \qquad \mathbf{B} = \mu \mathbf{H}$$

MAGNETIC BOUNDARY CONDITIONS



where we assume that the boundary may carry a surface current K whose component normal to the plane of the closed path is K. Thus

$$B_{N2} = B_{N1}$$

$$H_{t1} - H_{t2} = K$$

FIGURE 9.10
A gaussian surface and a closed path are constructed at the boundary between media 1 and 2, having permeabilities of μ_1 and μ_2, respectively. From this we determine the boundary conditions $B_{N1} = B_{N2}$ and $H_{t1} - H_{t2} = K$, the component of the surface current density directed into the page.

THE MAGNETIC CIRCUIT

the corresponding relationship between the mmf and the magnetic field intensity,

$$V_{mAB} = \int_A^B \mathbf{H} \cdot d\mathbf{L}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

we see that the magnetic flux density will be the analog of the current density,

the total magnetic flux flowing through the cross section of a magnetic circuit:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

we shall now define reluctance as the ratio of the magnetomotive force to the total flux; thus

$$V_m = \Phi \Re$$

$$\Re = \frac{d}{\mu S}$$

Finally, let us consider the analog of the source voltage in an electric circuit. We know that the closed line integral of **E** is zero,

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

In other words, Kirchhoff's voltage law states that the rise in potential through the source is exactly equal to the fall in potential through the load. The expression for magnetic phenomena takes on a slightly different form,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I_{\text{total}}$$

$$\oint \mathbf{H} \cdot d\mathbf{L} = NI$$

Let us try out some of these ideas on a simple magnetic circuit. In order to avoid the complications of ferromagnetic materials at this time, we shall assume that we have an air-core toroid with 500 turns, a cross-sectional area of 6 cm^2, a mean radius of 15 cm, and a coil current of 4 A. As we already know, the magnetic field is confined to the interior of the toroid, and if we consider the closed path of our magnetic circuit along the mean radius, we link 2000 A·t,

$$V_{m,\,\text{source}} = 2000 \quad \text{A}\cdot\text{t}$$

Although the field in the toroid is not quite uniform, we may assume that it is for all practical purposes and calculate the total reluctance of the circuit as

$$\mathfrak{R} = \frac{d}{\mu S} = \frac{2\pi(0.15)}{4\pi 10^{-7} \times 6 \times 10^{-4}} = 1.25 \times 10^{9} \quad \text{A}\cdot\text{t/Wb}$$

Thus

$$\Phi = \frac{V_{m,S}}{\mathfrak{R}} = \frac{2000}{1.25 \times 10^{9}} = 1.6 \times 10^{-6} \quad \text{Wb}$$

This value of the total flux is in error by less than $\frac{1}{4}$ percent, in comparison with the value obtained when the exact distribution of flux over the cross section is used.

Hence

$$B = \frac{\Phi}{S} = \frac{1.6 \times 10^{-6}}{6 \times 10^{-4}} = 2.67 \times 10^{-3} \quad \text{T}$$

and finally,

$$H = \frac{B}{\mu} = \frac{2.67 \times 10^{-3}}{4\pi 10^{-7}} = 2120 \quad \text{A}\cdot\text{t/m}$$

As a check, we may apply Ampère's circuital law directly in this symmetrical problem,

$$H_\phi 2\pi r = NI$$

and obtain

$$H_\phi = \frac{NI}{2\pi r} = \frac{500 \times 4}{6.28 \times 0.15} = 2120 \quad \text{A/m}$$

at the mean radius.

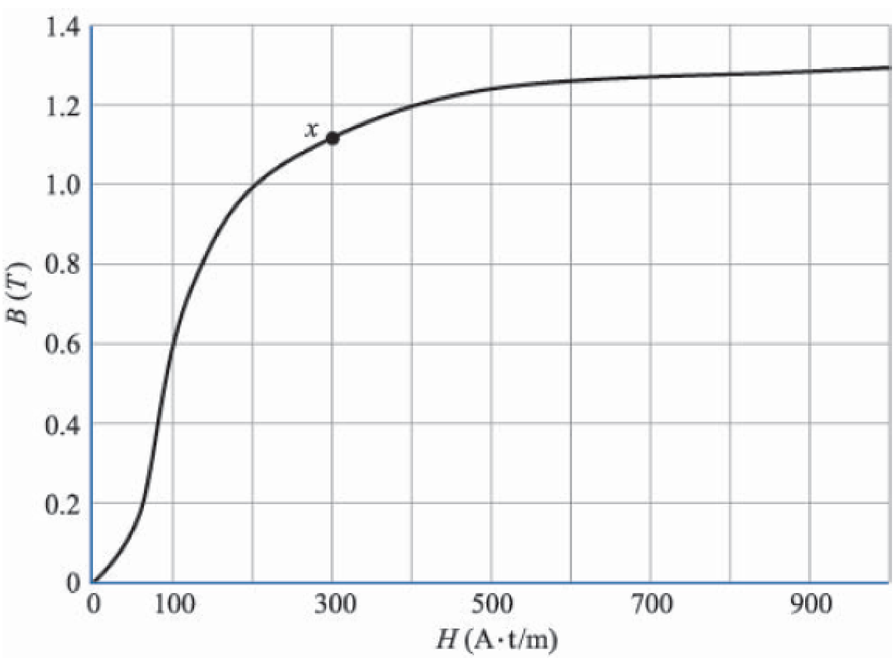


FIGURE 9.11
Magnetization curve of a sample of silicon sheet steel.

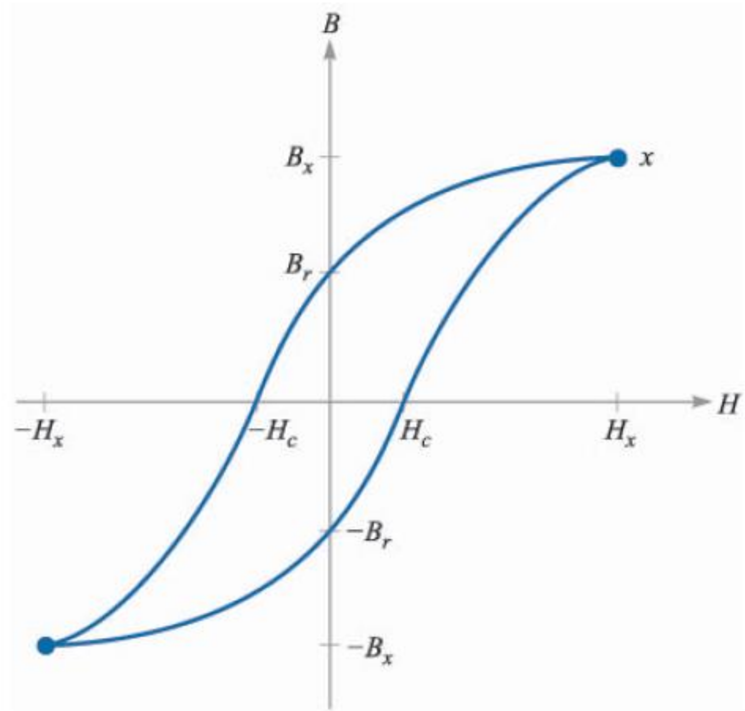


FIGURE 9.12
A hysteresis loop for silicon steel. The coercive force H_c and remnant flux density B_r are indicated.

POTENTIAL ENERGY AND FORCES ON MAGNETIC MATERIALS

$$W_E = \frac{1}{2}\int_{\text{vol}} \mathbf{D}\cdot\mathbf{E}\, dv$$

an expression for the energy in an electrostatic field by establishing the work necessary to bring the prerequisite point charges from infinity to their final resting places. The general expression for energy is

The total energy stored in a steady magnetic

$$W_H = \frac{1}{2}\int_{\text{vol}} \mathbf{B}\cdot\mathbf{H}\,dv$$

In spite of the fact that these results are valid only for linear media, we may use them to calculate the forces on nonlinear magnetic materials if we focus our attention on the linear media (usually air) which may surround them. For example, suppose that we have a long solenoid with a silicon-steel core. A coil containing n turns/m with a current I surrounds it. The magnetic field intensity in the core is therefore nI A t/m, and the magnetic flux density can be obtained from the magnetization curve for silicon steel. Let us call this value Bst .

$$dW_H = F\,dL = \frac{1}{2}\frac{B_{\text{st}}^2}{\mu_0} S\,dL \qquad F = \frac{B_{\text{st}}^2 S}{2\mu_0}$$

If, for example, the magnetic field intensity is sufficient to produce saturation in the steel, approximately 1.4 T, the force is

$$F = 7.80 \times 10^5 S \quad \text{N}$$

or about $113\,\text{lb}_f/\text{in}^2$.

INDUCTANE AND MUTUAL INDUCTANCE

We now define inductance (or self-inductance) as the ratio of the total flux linkages to the current which they link,

$$L = \frac{N\Phi}{I}$$

Let us apply (49) in a straightforward way to calculate the inductance per meter length of a coaxial cable of inner radius a and outer radius b. We may take the expression for total flux developed as Eq. (42) in Chap. 8,

$$\Phi = \frac{\mu_0 I d}{2\pi} \ln \frac{b}{a}$$

As an example of the use of flux and flux density in magnetic fields, let us find the flux between the conductors of the coaxial line of Fig. 8.8*a*. The magnetic field intensity was found to be

$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

and therefore

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi\rho} \mathbf{a}_\phi$$

The magnetic flux contained between the conductors in a length d is the flux crossing any radial plane extending from $\rho = a$ to $\rho = b$ and from, say, $z = 0$ to $z = d$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho} \mathbf{a}_\phi \cdot d\rho\, dz\, \mathbf{a}_\phi$$

or

$$\Phi = \frac{\mu_0 I d}{2\pi} \ln \frac{b}{a} \qquad (42)$$

This expression will be used later to obtain the inductance of the coaxial transmission line.

the inductance rapidly for a length d,

$$L = \frac{\mu_0 d}{2\pi} \ln \frac{b}{a} \quad \text{H} \qquad L = \frac{\mu_0}{2\pi} \ln \frac{b}{a} \quad \text{H/m}$$

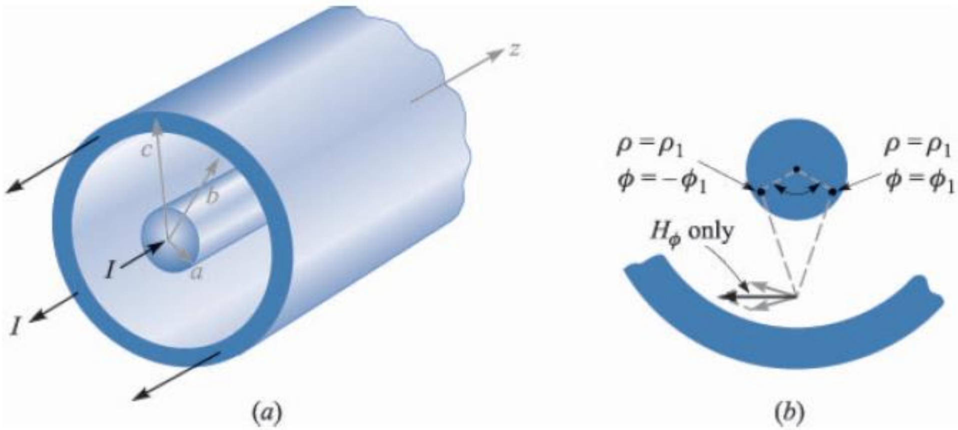


FIGURE 8.8
(*a*) Cross section of a coaxial cable carrying a uniformly distributed current I in the inner conductor and $-I$ in the outer conductor. The magnetic field at any point is most easily determined by applying Ampère's circuital law about a circular path. (*b*) Current filaments at $\rho = \rho_1$, $\phi = \pm\phi_1$, produces $\mathbf{H}_\rho$ components which cancel. For the total field, $\mathbf{H} = \mathbf{H}_\phi \mathbf{a}_\phi$.

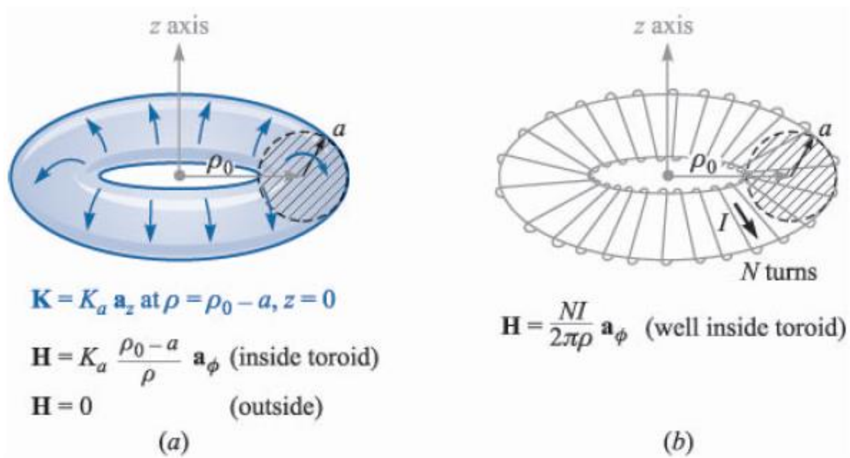


FIGURE 8.12
(*a*) An ideal toroid carrying a surface current **K** in the direction shown. (*b*) An N-turn toroid carrying a filamentary current I.

In the problem of a toroidal coil of N turns and a current I, as shown in Fig. 8.12b, we have

$$B_\phi = \frac{\mu_0 NI}{2\pi\rho}$$

If the dimensions of the cross section are small compared with the mean radius of the toroid ρ_0 then the total flux is

$$\Phi = \frac{\mu_0 NIS}{2\pi\rho_0}$$

where S is the cross-sectional area. Multiplying the total flux by N, we have the flux linkages, and dividing by I, we have the inductance

$$L = \frac{\mu_0 N^2 S}{2\pi\rho_0} \tag{51}$$

An equivalent definition for inductance may be made using an energy point of view,

$$\boxed{L = \frac{2W_H}{I^2}}$$

where I is the total current flowing in the closed path and W_H is the energy in the magnetic field produced by the current. After using (52) to obtain several other general expressions for inductance, we shall show that it is equivalent to (49). We first express the potential energy W_H in terms of the magnetic fields,

$$L = \frac{\int_{\text{vol}} \mathbf{B} \cdot \mathbf{H}\, dv}{I^2} \tag{53}$$

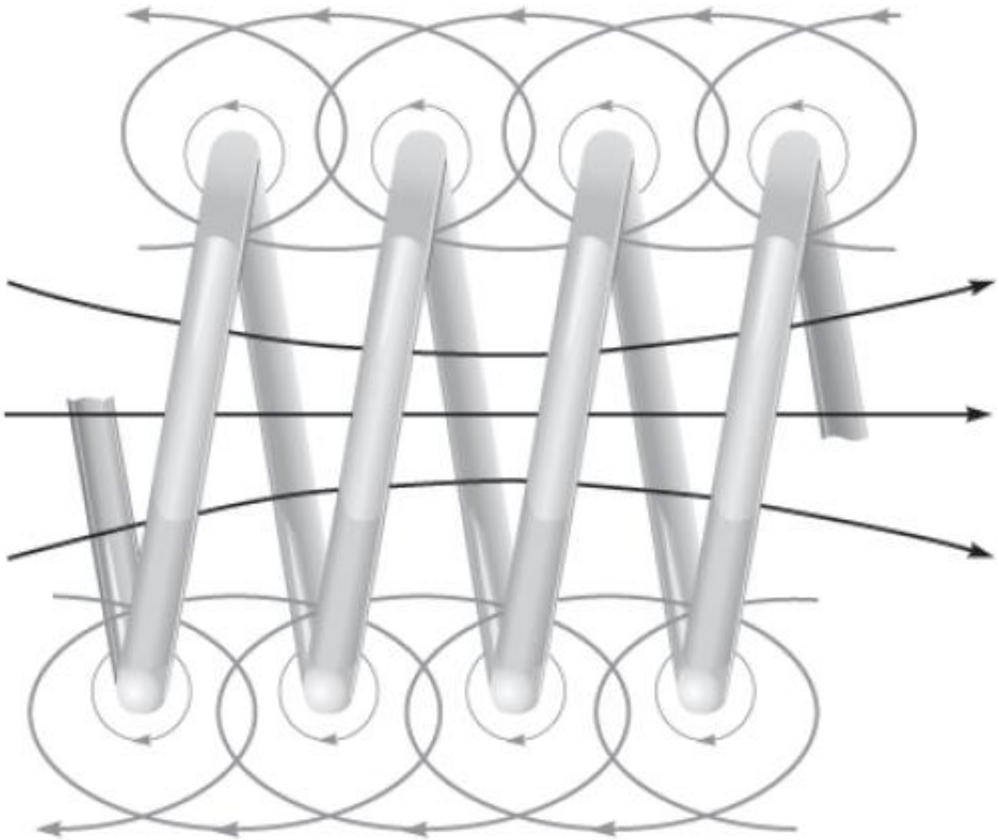


FIGURE 9.14
A portion of a coil showing partial flux linkages. The total flux linkages are obtained by adding the fluxes linking each turn.

and then replace **B** by $\nabla \times \mathbf{A}$,

$$L = \frac{1}{I^2} \int_{\text{vol}} \mathbf{H} \cdot (\nabla \times \mathbf{A}) dv$$

The vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{H}) \equiv \mathbf{H} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{H}) \qquad (54)$$

may be proved by expansion in cartesian coordinates. The inductance is then

$$L = \frac{1}{I^2} \left[\int_{\text{vol}} \nabla \cdot (\mathbf{A} \times \mathbf{H}) dv + \int_{\text{vol}} \mathbf{A} \cdot (\nabla \times \mathbf{H})\, dv \right] \qquad (55)$$

After applying the divergence theorem to the first integral and letting $\nabla \times \mathbf{H} = \mathbf{J}$ in the second integral, we have

$$L = \frac{1}{I^2} \left[\oint_S (\mathbf{A} \times \mathbf{H}) \cdot d\mathbf{S} + \int_{\text{vol}} \mathbf{A} \cdot \mathbf{J}\, dv \right]$$

The surface integral is zero, since the surface encloses the volume containing all the magnetic energy, and this requires that **A** and **H** be zero on the bounding surface. The inductance may therefore be written as

$$L = \frac{1}{I^2} \int_{\text{vol}} \mathbf{A} \cdot \mathbf{J}\, dv \qquad (56)$$

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu \mathbf{J}}{4\pi R} dv \qquad L = \frac{1}{I^2} \int_{\text{vol}} \left(\int_{\text{vol}} \frac{\mu \mathbf{J}}{4\pi R} dv \right) \cdot \mathbf{J}\, dv \qquad L = \frac{1}{I^2} \oint \left(\oint \frac{\mu I\, d\mathbf{L}}{4\pi R} \right) \cdot I\, d\mathbf{L}$$

$$= \frac{\mu}{4\pi} \oint \left(\oint \frac{d\mathbf{L}}{R} \right) \cdot d\mathbf{L}$$

To obtain our original definition of inductance (49) let us hypothesize a uniform current distribution in a filamentary conductor of small cross section so that $\mathbf{J}\, dv$ in (56) becomes $I\, d\mathbf{L}$,

$$L = \frac{1}{I} \oint \mathbf{A} \cdot d\mathbf{L} \tag{59}$$

$$L = \frac{1}{I} \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \qquad L = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{S} \qquad L = \frac{\Phi}{I}$$

If we now let the filament make N identical turns about the total flux, an idealization which may be closely realized in some types of inductors, the closed line integral must consist of N laps about this common path and (60) becomes

$$L = \frac{N\Phi}{I} \tag{61}$$

The flux Φ is now the flux crossing any surface whose perimeter is the path occupied by any *one* of the N turns. The inductance of an N-turn coil may still be obtained from (60), however, if we realize that the flux is that which crosses the complicated surface[4] whose perimeter consists of all N turns.

The interior of any conductor also contains magnetic flux, and this flux links a variable fraction of the total current, depending on its location. These flux linkages lead to an *internal inductance*, which must be combined with the external inductance to obtain the total inductance. The internal inductance of a long straight wire of circular cross section, radius a, and uniform current distribution is

$$L_{a,\text{int}} = \frac{\mu}{8\pi} \quad \text{H/m}$$

the mutual inductance between circuits 1 and 2, M12 , in terms of mutual flux linkages,

$$M_{12} = \frac{N_2 \Phi_{12}}{I_1}$$

where È12 signifies the flux produced by I1 which links the path of the filamentary current I2 , and N2 is the number of turns in circuit 2. The mutual inductance, therefore, depends on the magnetic interaction between two currents. With either current alone, the total energy stored in the magnetic field can be found in terms of a single inductance, or self-inductance; with both currents having nonzero values, the total energy is a function of the two self-inductances

and the mutual inductance.

$$M_{12} = \frac{1}{I_1 I_2} \int_{\text{vol}} (\mathbf{B}_1 \cdot \mathbf{H}_2) dv$$

$$M_{12} = \frac{1}{I_1 I_2} \int_{\text{vol}} (\mu \mathbf{H}_1 \cdot \mathbf{H}_2) dv$$

$$\boxed{M_{12} = M_{21}}$$

Faradays Law

- Faradays law is stated as

$$\text{emf} = -\frac{d\Phi}{dt} \quad \text{V} \qquad (1)$$

Equation (1) implies a closed path, although not necessarily a closed conducting path; the closed path, for example, might include a capacitor, or it might be a purely imaginary line in space. The magnetic flux is that flux which passes through any and every surface whose perimeter is the closed path, and $d\Phi/dt$ is the time rate of change of this flux.

A nonzero value of $d\Phi/dt$ may result from any of the following situations:

- (a) time changing flux linkage a stationary closed path.
- (b) relative motion between a steady flux a closed path.
- (c) a combination of the above two cases.

If the closed path is that taken by an N-turn filamentary conductor, it is often sufficiently accurate to consider the turns as coincident and let

$$\text{emf} = -N\frac{d\Phi}{dt} \tag{2}$$

where Φ is now interpreted as the flux passing through any one of N coincident paths.

We need to define emf as used in (1) or (2). The emf is obviously a scalar, and (perhaps not so obviously) a dimensional check shows that it is measured in volts. We define the emf as

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} \tag{3}$$

Replacing Φ in (1) by the surface integral of **B**, we have

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \qquad (4)$$

We first consider a stationary path. The magnetic flux is the only time-varying quantity on the right side of (4), and a partial derivative may be taken under the integral sign,

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \qquad (5)$$

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

where the surface integrals may be taken over identical surfaces. The surfaces are perfectly general and may be chosen as differentials,

$$(\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

and

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \qquad (6)$$

This is one of Maxwell's four equations as written in differential, or point, form, the form in which they are most generally used. Equation (5) is the integral form of this equation and is equivalent to Faraday's law as applied to a fixed path. If **B** is not a function of time, (5) and (6) evidently reduce to the electrostatic equations,

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0 \quad \text{(electrostatics)}$$

and

$$\nabla \times \mathbf{E} = 0 \quad \text{(electrostatics)}$$