Module-5

Syllabus

- **Maxwell's equations** Continuity equation, Inconsistency of Ampere's law with continuity equation, displacement current, Conduction current, Derivation of Maxwell‘s equations in point form, and integral form, Maxwell's equations for different media, Numerical problems **(Text: Chapter 10.2 to 10.4)**
- **Uniform Plane Wave**: Plane wave, Uniform plane wave, Derivation of plane wave equations from Maxwell's equations, Solution of wave equation for perfect dielectric, Relation between E and H, Wave propagation in free space, Solution of wave equation for sinusoidal excitation, wave propagation in any conducting media (**γ, α, β, η**) and good conductors, Skin effect or Depth of penetration, Poynting‘s theorem and wave power, Numerical problems. **(Text: Chapter 12.1 to 12.4)**

Displacement Current

- The fundamental postulate for electromagnetic induction assures us that a time-varying magnetic field gives rise to an electric field.

Time-varying case: $\nabla \times \vec{E} = 0 \Rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad 1 \qquad\qquad \nabla \times \vec{H} = \vec{J}, \qquad (2)$$

$$\nabla \cdot \vec{D} = \rho, \qquad 3 \qquad\qquad \nabla \cdot \vec{B} = 0. \qquad 4)$$

- The mathematical expression of charge conservation is the equation of continuity :

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}. \qquad (5)$$

- Divergence of Eq. (7 – 47b) : $\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{J},$ (7 – 49) (null identity)

  since Eq. (7 – 48) asserts that $\nabla \cdot \vec{J}$ does not vanish in a time-varying situation, Eq. (7 – 49) is, in general, not true.

First of all, a term $\partial\rho/\partial t$ must be added to the right side

$$\nabla\cdot(\nabla\times\vec{H})=0=\nabla\cdot\vec{J}+\frac{\partial\rho}{\partial t}. \qquad (6)$$

Using Eq. (3) in Eq. (6), we have

$$\nabla\cdot(\nabla\times\vec{H})=\nabla\cdot(\vec{J}+\frac{\partial\vec{D}}{\partial t}),$$

which implies that

$$\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}. \qquad (7)$$

Eq. (7) indicates that a time-varying electric field will give rise to a magnetic field, even in the absence of a current flow.

The additional term $\partial\vec{D}/\partial t$ is necessary to make Eq. (7 – 52) consistent with the principle of conservation of charge.

The term $\partial\vec{D}/\partial t$ is called *displacement current density*.

Maxwell's Equations in Point Form

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \text{(a)}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \qquad \text{(b)}$$

$$\nabla \cdot \vec{D} = \rho, \qquad \text{(c)}$$

$$\nabla \cdot \vec{B} = 0. \qquad \text{(d)}$$

They are known as *Maxwell's equations*.

INTEGRAL FORM OF MAXWELL'S EQUATIONS.

- The four Maxwell's equations in (a, b, c, d) are differential equations that are valid at every point in space.

In explaining electromagnetic phenomena in a physical environment we must deal with finite objects of specified shapes and boundaries.

It is convenient to convert the differential forms into their integral-form equivalents.

- We take the surface integral of both sides of the curl equations over an open surface S with contour C and apply Stokes's theorem to obtain

$$\oint_c \vec{E} \cdot d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \qquad \text{(a)}$$

$$\oint_c \vec{H} \cdot d\vec{l} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{s}. \qquad \text{(b)}$$

- Taking the volume integral of both sides of the divergence equations over a volume V with a closed surface S and using divergence theorem, we have

$$\oint_s \vec{D} \cdot d\vec{s} = \int_v \rho dv \qquad \text{(c)}$$

$$\oint_s \vec{B} \cdot d\vec{s} = 0. \qquad \text{(d)}$$

The set of four equations in (a, b, c, d) are the integral form of Maxwell's equations.

Maxwell's Equations in Diffrential and Integral Form

- Maxwell's Equations

Differential Form	Integral Form	Significance
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_c \vec{E} \cdot d\vec{l} = -\frac{d\vec{\Phi}}{dt}$	Faraday's law
$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_c \vec{H} \cdot d\vec{l} = I + \int_s \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$	Ampere's circuital law
$\nabla \cdot \vec{D} = \rho$	$\oint_s \vec{D} \cdot d\vec{s} = Q$	Gauss's law
$\nabla \cdot \vec{B} = 0$	$\oint_s \vec{B} \cdot d\vec{s} = 0$	No isolated magnetic charge

PROPAGATION & REFLECTION OF PLANE WAVES

Will discuss the effect of propagation of EM wave in four medium : Free space ; Lossy dielectric ; Lossless dielectric (perfect dielectric) and Conducting media.

Also will be discussed the phenomena of reflections at interface between different media.

Ex : EM wave is radio wave, TV signal, radar radiation and optical wave in optical fiber.

Three basics characteristics of EM wave :

- travel at high velocity
- travel following EM wave characteristics
- travel outward from the source

These propagation phenomena for a type traveling wave called plane wave can be explained or derived by Maxwell's equations.

ELECTRIC AND MAGNETIC FIELDS FOR PLANE WAVE

From Maxwell's equations :

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} = -\mu \frac{\partial \overline{H}}{\partial t}$$

$$\nabla \times \overline{H} = \overline{\mathrm{J}} + \frac{\partial \overline{D}}{\partial t} = \overline{\mathrm{J}} + \varepsilon \frac{\partial \overline{E}}{\partial t}$$

$$\nabla \bullet \overline{D} = \rho_v$$

$$\nabla \bullet \overline{B} = 0$$

Assume the medium is free of charge :

$$\rho_v = 0, \bar{J} = 0$$

$$\nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t} \qquad (1)$$

$$\nabla \times \overline{H} = \varepsilon \frac{\partial \overline{E}}{\partial t} \qquad (2)$$

$$\nabla \bullet \overline{D} = 0 \qquad (3)$$

$$\nabla \bullet \overline{B} = 0 \qquad (4)$$

From vector identity and taking the curl of (1)and substituting (1) and (2)

$$\nabla \times (\nabla \times \overline{E}) = \nabla(\nabla \bullet \overline{E}) - \nabla^2 \overline{E}$$

$$where \;\; \nabla(\nabla \bullet \overline{E}) = 0$$

$$\Rightarrow \nabla \times \left(-\mu \frac{\partial \overline{H}}{\partial t} \right) = -\nabla^2 \overline{E}$$

$$\rightarrow \nabla^2 \overline{E} = \mu \frac{\partial}{\partial t} (\nabla \times \overline{H})$$

$$\therefore \;\; \nabla^2 \overline{E} = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2}$$

ie Helmholtz's equation for electric field

$$\nabla^2 \overline{E} = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} \quad Vm^{-3}$$

Similarly in the same way, from vector identity and taking the curl of (2)and substituting (1) and (2)

$$\nabla^2 \overline{H} = \mu\varepsilon \frac{\partial^2 \overline{H}}{\partial t^2} \quad Am^{-3}$$

In Cartesian coordinates :

$$\frac{\partial^2 \overline{E}}{\partial x^2} + \frac{\partial^2 \overline{E}}{\partial y^2} + \frac{\partial^2 \overline{E}}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2}$$

Assume that :

(i) Electric field only has x component

(ii) Propagate in the z direction

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

The solution for this equation :

$$E_x = E_x^+ \cos(\omega t - \beta z) + E_x^- \cos(\omega t + \beta z)$$

Incidence wave propagate in +z direction

Reflected wave propagate in -z direction

To find H field :

$$\nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t}$$

$$\nabla \times \overline{E} = \frac{\partial E_x}{\partial z}\hat{y} - \frac{\partial E_x}{\partial y}\hat{z}$$

$$= \left\{\beta E_x^+ \sin(\omega t - \beta z) - \beta E_x^- \sin(\omega t + \beta z)\right\}\hat{y}$$

On the right side equation :

$$-\mu \frac{\partial \overline{H}}{\partial t} = -\mu \left\{ \frac{\partial H_x}{\partial t}\hat{x} + \frac{\partial H_y}{\partial t}\hat{y} + \frac{\partial H_z}{\partial t}\hat{z} \right\}$$

Equating components on both side = y component

$$-\mu \frac{\partial H_y}{\partial t} = \left\{ \beta E_x^+ \sin(\omega t - \beta z) - \beta E_x^- \sin(\omega t + \beta z) \right\}$$

$$-H_y = \int \frac{\beta E_x^+}{\mu} \sin(\omega t - \beta z) dt - \int \frac{\beta E_x^-}{\mu} \sin(\omega t + \beta z) dt$$

$$= -\frac{\beta}{\omega\mu} E_x^+ \cos(\omega t - \beta z) + \frac{\beta}{\omega\mu} E_x^- \cos(\omega t + \beta z)$$

$$H_y = \frac{\beta}{\omega\mu} E_x^+ \cos(\omega t - \beta z) - \frac{\beta}{\omega\mu} E_x^- \cos(\omega t + \beta z)$$

$$= H_y^+ \cos(\omega t - \beta z) - H_y^- \cos(\omega t + \beta z)$$

Hence :

$$E_x = E_x^+ \cos(\omega t - \beta z) + E_x^- \cos(\omega t + \beta z)$$

$$H_y = H_y^+ \cos(\omega t - \beta z) - H_y^- \cos(\omega t + \beta z)$$

These equations of EM wave are called PLANE WAVE.

Main characteristics of EM wave :

(i) Electric field and magnetic field always perpendicular.

(ii) NO electric or magnetic fields component in the direction of propagation.

(iii) $\overline{E} \times \overline{H}$ will provides information on the direction of propagation.

PLANE WAVE IN LOSSY DIELECTRICS – IMPERFECT DIELECTRICS

$$\sigma \neq 0 ; \mu = \mu_0 \mu_r ; \varepsilon = \varepsilon_0 \varepsilon_r$$

Assume a media is charged free , ρ_v =0

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} = (\sigma + j\omega\varepsilon)\overline{E} \qquad (1)$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} = -j\omega\mu\overline{H} \qquad (2)$$

Taking the curl of (2) :

$$\nabla \times \nabla \times \overline{E} = -j\omega\mu(\nabla \times \overline{H})$$

From vector identity :

$$\nabla \times \nabla \times \overline{A} = \nabla(\nabla \bullet \overline{A}) - \nabla^2 \overline{A}$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} = (\sigma + j\omega\varepsilon)\overline{E} \quad (1)$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} = -j\omega\mu\overline{H} \quad (2)$$

$$\nabla \times \nabla \times \overline{E} = -j\omega\mu(\nabla \times \overline{H})$$

$$\nabla(\nabla \bullet \overline{E}) - \nabla^2 \overline{E} = -j\omega\mu(\sigma + j\omega\varepsilon)\overline{E}$$

$$\nabla^2 \overline{E} - j\omega\mu(\sigma + j\omega\varepsilon)\overline{E} = 0$$

$$\nabla^2 \overline{E} - \gamma^2 \overline{E} = 0$$

Where :

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) = -\omega^2\mu\varepsilon + j\omega\mu\sigma \quad (4)$$

$$\gamma = \text{propagation constant}$$

Define :

$$\gamma = \alpha + j\beta$$

$$\gamma^2 = (\alpha^2 - \beta^2) + 2j\alpha\beta \quad (5)$$

Equating (4) and (5) for Re and Im parts :

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \quad (\text{Re}) \ (6)$$

$$2\alpha\beta = \omega\mu\sigma \quad (\text{Im}) \ (7)$$

Magnitude for (5) ;

$$|\gamma^2| = \alpha^2 + \beta^2 \quad (8)$$

Magnitude for (4) ;

$$|\gamma^2| = \sqrt{(-\omega^2\mu\varepsilon)^2 + (\omega\sigma\mu)^2}$$
$$= \omega\mu\sqrt{\omega^2\varepsilon + \sigma^2} \quad (9)$$

Equate (8) and (9) :

$$\alpha^2 + \beta^2 = \omega\mu\sqrt{\omega^2\varepsilon + \sigma^2} \quad (10)$$

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \quad (\mathrm{Re}) \quad (6)$$

Add (10) and (6) :

Hence :

$$2\alpha^2 = \omega\mu\sqrt{\omega^2\varepsilon + \sigma^2} - \omega^2\mu\varepsilon$$
$$= \omega^2\mu\varepsilon\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - \omega^2\mu\varepsilon$$

$$\alpha^2 = \frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]} \; Np/m \quad (11)$$

α is known as attenuation constant as a measure of the wave is attenuated while traveling in a medium.

Substract (10) and (6) :

$$2\beta^2 = \omega\mu\sqrt{\omega^2\varepsilon + \sigma^2} + \omega^2\mu\varepsilon$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}}+1\right]} \quad rad/m \quad (12)$$

β is phase constant

If the electric field propagate in +z direction and has component x, the equation of the wave is given by :

$$\overline{E}(z,t) = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\hat{x} \qquad (13)$$

And the magnetic field :

$$\overline{H}(z,t) = H_0 e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)\hat{y} \qquad (14)$$

where ; $$H_0 = \frac{E_0}{|\eta|} \qquad (15)$$

$$\bar{E}(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\hat{x} \quad (14)$$

$$\bar{H}(z,t) = H_0 e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\hat{y} \quad (15)$$

Intrinsic impedance :

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| \angle \theta_\eta = |\eta| e^{j\theta_\eta}, (\Omega) \qquad (16)$$

where ;

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}, \quad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}, \quad 0 \leq \theta_\eta \leq 45^0 \qquad (17)$$

Conclusions that can be made for the wave propagating in lossy dielectrics material :

(i) E and H fields amplitude will be attenuated by $e^{-\alpha z}$

(ii) E leading H by θ_η

Wave velocity ;

$$u = \omega / \beta \;;\; \lambda = 2\pi / \beta$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}, \quad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}, \quad 0 \le \theta_\eta \le 45^0 \quad (17)$$

Loss tangent ;

$$\frac{|\bar{J}|}{|\bar{J}_d|} = \frac{\sigma\bar{E}}{|j\omega\varepsilon\bar{E}|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta \qquad (18)$$

From (17) and (18)

$$\theta = 2\theta_\eta$$

Loss tangent values will determine types of media :

tan θ small ($\sigma / \omega\varepsilon < 0.1$) – good dielectric – low loss

tan θ large ($\sigma / \omega\varepsilon > 10$) - good conductor – high loss

Another factor that determined the characteristic of the media is operating frequency. A medium can be regarded as a good conductor at low frequency might be a good dielectric at higher frequency.

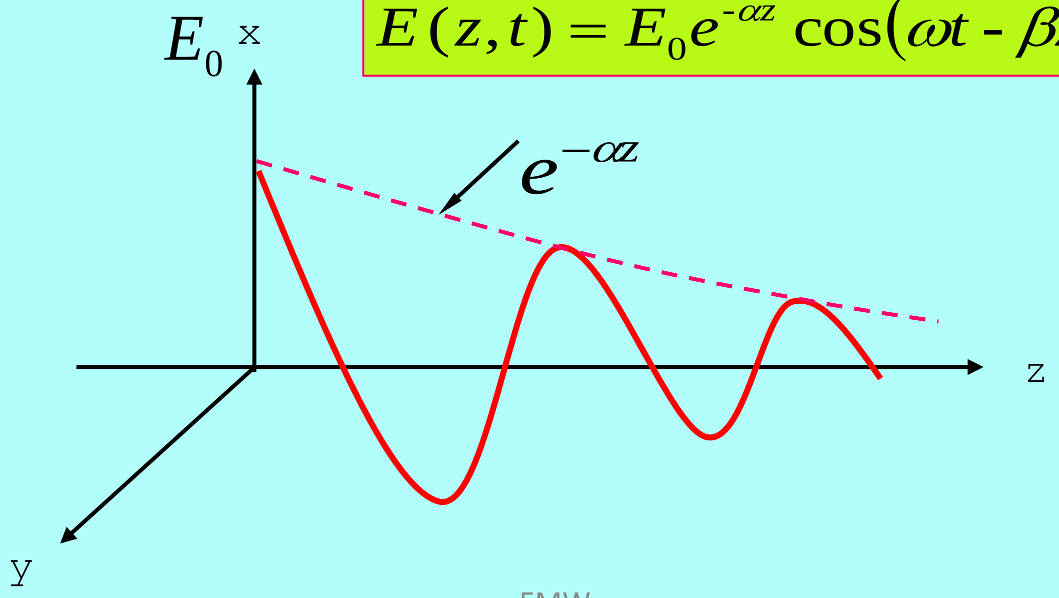
$$\overline{E}(z,t) = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\hat{x} = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}\hat{x} \quad (14)$$

$$\overline{H}(z,t) = H_0 e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)\hat{y} = \frac{E_0}{|\eta|} e^{-\alpha z} e^{j(\omega t - \beta z - \theta_\eta)}\hat{y} \quad (15)$$

$$H_0 = \frac{E_0}{|\eta|}$$

Graphical representation of E field in lossy dielectric

$$\overline{E}(z,t) = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\hat{x}$$



PLANE WAVE IN LOSSLESS (PERFECT) DIELECTRICS

Characteristics: $\sigma = 0, \varepsilon = \varepsilon_0 \varepsilon_r, \mu = \mu_0 \mu_r$ (19)

$$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]} \quad Np/m \quad (11)$$

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right]} \quad rad/m \quad (12)$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| \angle \theta_\eta = |\eta| e^{j\theta_\eta}, (\Omega)$$

Substitute in (11) and (12) :

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu\varepsilon} \quad (20)$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \quad \lambda = \frac{2\pi}{\beta} \quad (21)$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \angle 0^o \quad (22)$$

The zero angle means that E and H fields are in phase at each fixed location.

PLANE WAVE IN FREE SPACE

Free space is nothing more than the perfect dielectric media :

Characteristics: $$\sigma = 0, \varepsilon = \varepsilon_0, \mu = \mu_0 \quad (23)$$

Substitute in (20) and (21) :

$$\alpha = 0 \ , \quad \beta = \omega\sqrt{\mu_0\varepsilon_0} = \omega / c \quad (24)$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c \ , \quad \lambda = \frac{2\pi}{\beta} \quad (25)$$

where $$u = c \approx 3\times10^8 \ m/s$$

$$\eta = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \ \Omega \quad (26)$$

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu\varepsilon} \quad (20)$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \quad \lambda = \frac{2\pi}{\beta} \quad (21)$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\angle 0^o \quad (22)$$

$$\mu = \mu_0 = 4\pi\times10^{-7} \ H/m$$

$$\varepsilon = \varepsilon_0 = 8.854\times10^{-12} \approx \frac{1}{36\pi}\times10^{-9} \ F/m$$

The field equations for E and H obtained :

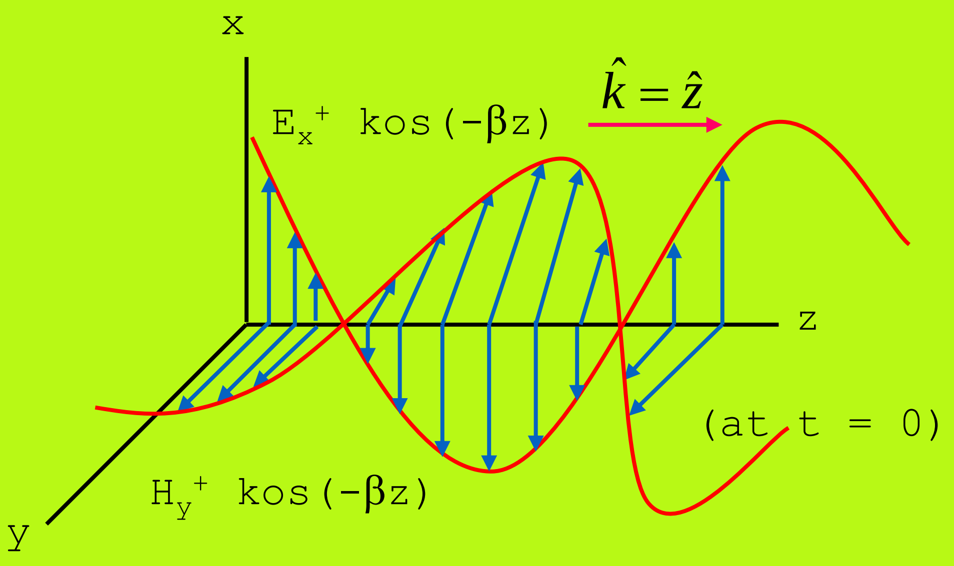
$$\overline{E} = E_0 \cos(\omega t - \beta z)\hat{x} \quad (27)$$

$$\overline{H} = \frac{E_0}{\eta_0} \cos(\omega t - \beta z)\hat{y} \quad (28)$$

$$\overline{E}(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\hat{x} \quad (14)$$

$$\overline{H}(z,t) = H_0 e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\hat{y} \quad (15)$$

E and H fields and the direction of propagation :



Generally :

$$\hat{E} \times \hat{H} = \hat{k}$$

PLANE WAVE IN CONDUCTORS

In conductors : $\sigma >> \omega\varepsilon$ $\qquad \dfrac{\sigma}{\omega\varepsilon} \to \infty$

With the characteristics : $\sigma \sim> \infty, \varepsilon = \varepsilon_0, \mu = \mu_0\mu_r$ (29)

Substitute in (11 and (12) :

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma} \qquad (30)$$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^o$$

E leads H by 45^0 (31)

The field equations for E and H obtained :

$$\overline{E} = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\hat{x} \qquad (32)$$

$$\overline{H} = \frac{E_0}{\eta_0} e^{-\alpha z} \cos(\omega t - \beta z - 45^o)\hat{y} \qquad (33)$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]} \; Np/m \quad (11)$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right]} \; rad/m \quad (12)$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta|\angle\theta_\eta = |\eta| e^{j\theta_\eta}, (\Omega)$$

It is seen that in conductors and $\bar{E}$ and $\bar{H}$ waves are attenuated by $e^{-\alpha z}$

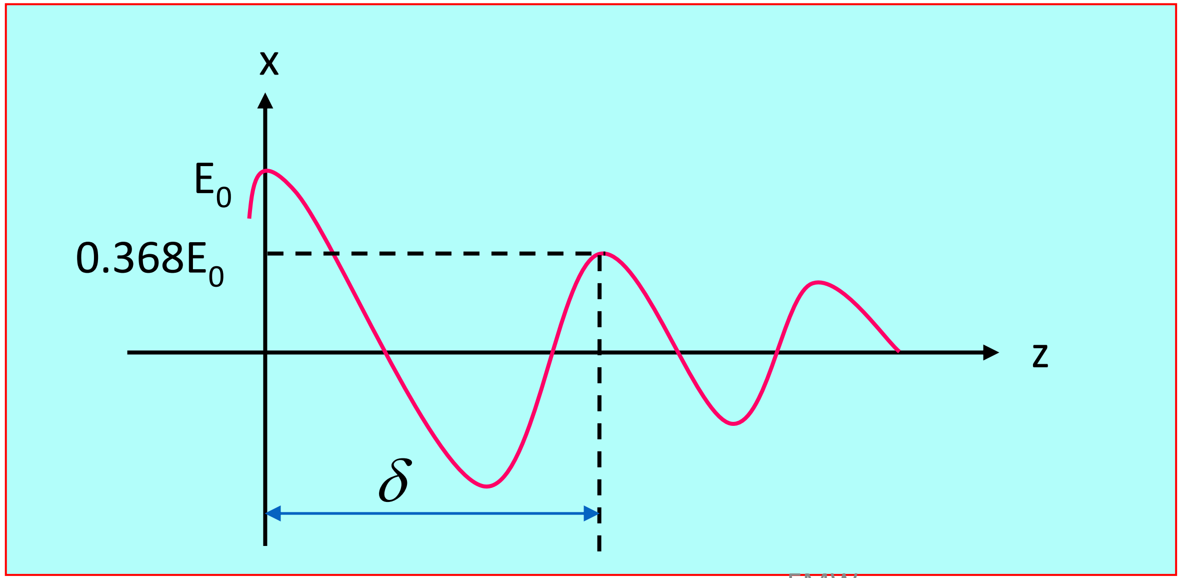
From the diagram δ is referred to as the skin depth. It refers to the amplitude of the wave propagate to a conducting media is reduced to e^{-1} or 37% from its initial value.

In a distance :

$$E_0 e^{-\alpha\delta} = E_0 e^{-1}$$

$$\therefore \ \delta = 1/\alpha = \frac{1}{\sqrt{\pi f \mu \sigma}} \qquad (34)$$

It can be seen that at higher frequencies δ is decreasing.



POWER AND THE POYNTING VECTOR

$$\nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t} \qquad (35)$$

$$\nabla \times \overline{H} = \sigma \overline{E} + \varepsilon \frac{\partial \overline{E}}{\partial t} \qquad (36)$$

Dot product (36) with $:\overline{E}$

$$\overline{E} \bullet \left(\nabla \times \overline{H}\right) = \sigma E^2 + \overline{E} \bullet \varepsilon \frac{\partial \overline{E}}{\partial t} \qquad (37)$$

From vector identity:

$$\nabla \bullet \left(\overline{A} \times \overline{B}\right) = \overline{B} \bullet \left(\nabla \times \overline{A}\right) - \overline{A} \bullet \left(\nabla \times \overline{B}\right) \qquad (38)$$

Change $\overline{A} = \overline{H}, \overline{B} = \overline{E}$ in (37) and use (38) , equation (37) becomes :

$$\overline{H} \bullet \left(\nabla \times \overline{E}\right) + \nabla \bullet \left(\overline{H} \times \overline{E}\right) = \sigma E^2 + \overline{E} \bullet \varepsilon \frac{\partial \overline{E}}{\partial t} \qquad (39)$$

$$\overline{H}\bullet\left(\nabla\times\overline{E}\right)+\nabla\bullet\left(\overline{H}\times\overline{E}\right)=\sigma E^2+\overline{E}\bullet\varepsilon\frac{\partial\overline{E}}{\partial t}\quad(39)$$

$$\nabla\times\overline{E}=-\mu\frac{\partial\overline{H}}{\partial t}\quad(35)$$

And from (35):

$$\overline{H}\bullet\left(\nabla\times\overline{E}\right)=\overline{H}\bullet\left(-\mu\frac{\partial\overline{H}}{\partial t}\right)=-\frac{\mu}{2}\frac{\partial}{\partial t}\overline{H}\bullet\overline{H}\quad(40)$$

Therefore (39) becomes:

$$-\frac{\mu}{2}\frac{\partial H^2}{\partial t}-\nabla\bullet\left(\overline{E}\times\overline{H}\right)=\sigma\overline{E}^2+\overline{E}\bullet\varepsilon\frac{\partial\overline{E}}{\partial t}\quad(41)$$

where:

$$\nabla\bullet\left(\overline{H}\times\overline{E}\right)=-\nabla\bullet\left(\overline{E}\times\overline{H}\right)$$

Integration (41) throughout volume V :

$$\int_v\nabla\bullet\left(\overline{E}\times\overline{H}\right)dv=-\frac{\partial}{\partial t}\int_v\left[\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right]dv-\int_v\sigma E^2dv\quad(42)$$

$$\int_v \nabla \bullet \left(\bar{E} \times \bar{H}\right) dv = -\frac{\partial}{\partial t}\int_v \left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right] dv - \int_v \sigma E^2 dv \quad (42)$$

Using divergence theorem to (42):

$$\oint_s \left(\bar{E} \times \bar{H}\right) \bullet d\bar{S} = -\frac{\partial}{\partial t}\int_v \left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right] dv - \int_v \sigma E^2 dv \quad (43)$$

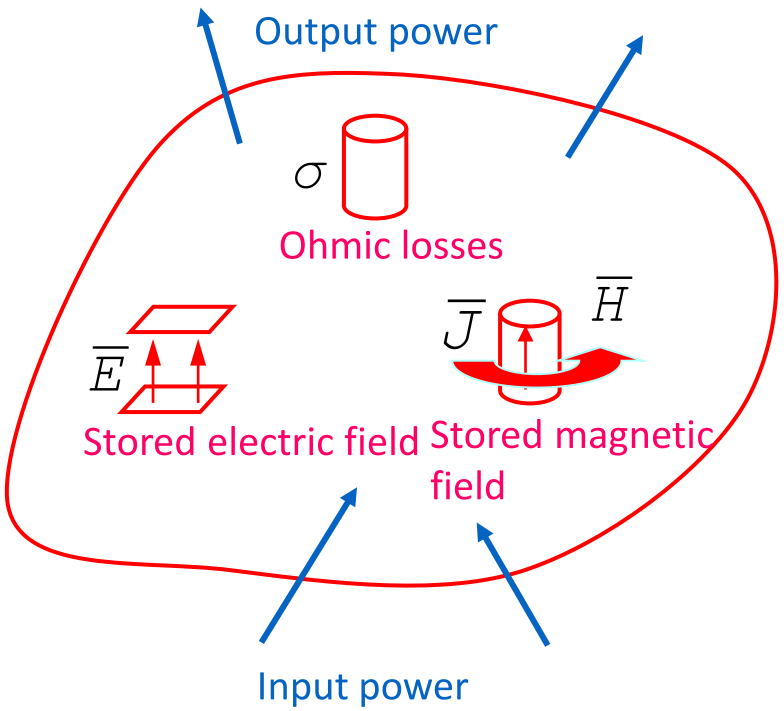
- $\oint_s \left(\bar{E} \times \bar{H}\right) \bullet d\bar{S}$: Total energy flow leaving the volume
- $-\frac{\partial}{\partial t}\int_v \left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right] dv$: The decrease of the energy densities of energy stored in the electric and magnetic fields
- $\int_v \sigma E^2 dv$: Dissipated ohmic power

Equation (43) shows Poynting Theorem and can be written as :

$$\wp = \bar{E} \times \bar{H} \quad W/m^2$$

Poynting theorem states that the total power flow leaving the volume is equal to the decrease of the energy densities of energy stored in the electric and magnetic fields and the dissipated ohmic power.

The theorem can be explained as shown in the diagram below :



Given for lossless dielectric, the electric and magnetic fields are :

$$\overline{E} = E_0 \cos(\omega t - \beta z)\hat{x}$$

$$\overline{H} = \frac{E_0}{\eta} \cos(\omega t - \beta z)\hat{y}$$

The Poynting vector becomes:

$$\wp = \overline{E} \times \overline{H} \quad W/m^2$$

$$\wp = \frac{E^2{}_0}{\eta} \cos^2(\omega t - \beta z)\hat{y}$$

To find average power density :

Integrate Poynting vector and divide with interval T = 1/f :

$$P_{ave} = \frac{1}{T}\int_0^T \frac{E_0^2}{\eta}\cos^2(\omega t - \beta z)dt$$

$$= \frac{1}{2T}\frac{E_0^2}{\eta}\int_0^T \left[1 + \cos(2\omega t - 2\beta z)\right]dt$$

$$= \frac{1}{2T}\frac{E_0^2}{\eta}\left[t + \frac{1}{2\omega}\sin(2\omega t - 2\beta z)\Big|_0^T\right]$$

$$\therefore P_{ave} = \frac{1}{2}\frac{E_0^2}{\eta} \quad W/m^2$$

Average power through area S :

$$P_{ave} = \frac{1}{2}\frac{E_0^2}{\eta} S \ (W)$$

Given for lossy dielectric, the electric and magnetic fields are :

$$\overline{E} = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\hat{x}$$

$$\overline{H} = \frac{E_0}{\eta_0} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\hat{y}$$

The Poynting vector becomes:

$$\wp = \frac{E^2{}_0}{\eta} e^{-2\alpha z} \cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta_\eta)$$

Average power :

$$P_{ave} = \frac{1}{2}\frac{E_0^2}{\eta} e^{-2\alpha z} \cos\theta_\eta$$