Module 2: Series expansion and multivariable Calculus

Introduction to series expansion and partial differentiation in the field of ME

The infinite series expansion of a differentiable function $y = f(x)$ about the point $x = 0$ will be series in powers of $(x-a)$ known as Taylor's series expansion. If particular if $a = 0$, The series will be in ascending power of x, known as Maclaurin's series expansion.

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

This is called Taylor's series expansion of $f(x)$ about the point a.

In particular if $a = 0$ we have.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

This is Maclaurin's series expansion of $f(x)$.

we are taking an alternative notation.

$y(x)$ for $f(x)$ and $y_1(x)$, $y_2(x)$, $y_3(x)$ ....

$f'(x)$, $f''(x)$, $f'''(x)$, ... so that we have.

$$y(x) = y(a) + (x-a) y_1(a) + \frac{(x-a)^2}{2!} y_2(a) + \cdots \quad \text{[Taylors]}$$

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \cdots \quad \text{[Maclaurins]}$$

Problem

Dec 2014

1) Expand $e^{\sin x}$ as Maclaurins series upto the terms containing x^4.

Soln $y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0)$

Let ~~y = sin~~

$y = e^{\sin x} \rightarrow ①$ $\qquad \frac{d}{dx}(e^x) = e^x$

$y_1 = e^{\sin x} \cdot \cos x \Rightarrow y \cdot \cos x \rightarrow ②$

$y_2 = y \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(y)$

$y_2 = y(-\sin x) + \cos x \cdot y_1$

$y_2 = -y \sin x + \cos x \, y_1 \rightarrow ③$

$y_3 = -\left[y \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(y)\right] + \cos x \frac{d}{dx}(y_1) + y' \frac{d}{dx}(\cos x)$

$y_3 = -[y \cos x + \sin x \, y_1] + \cos x \, y_2 + y_1(-\sin x)$

$y_3 = -y \cos x - \sin x \, y_1 + \cos x \, y_2 - \sin x \, y_1$

$y_3 = -y \cos x - 2 \sin x \, y_1 + \cos x \, y_2 \rightarrow ④$

$y_4 = -\left(y \frac{d}{dx}(\cos x) + \cos x (y_1)\right) - 2\left(\sin x \frac{d}{dx}(y_1) + y_1 \frac{d}{dx}(\sin x)\right)$
$+ \cos x \frac{d}{dx}(y_2) + y_2 \frac{d}{dx}(\cos x)$

$y_4 = -(y(-\sin x) + \cos x \, y_1) - 2(\sin x \, y_2 + y_1 \cos x)$
$+ \cos x (y_3) + y_2(-\sin x)$

$y_4 = +y \sin x - \cos x \, y_1 - 2 \sin x \, y_2 - 2 y_1 \cos x + \cos x \, y_3$
$- y_2 \sin x$

$y_4 = y \sin x - 3 y_1 \cos x - 3 y_2 \sin x + y_3 \cos x$
$\rightarrow ⑤$

From eq ①

$y = e^{\sin x}$

$y(0) = e^{\sin(0)} = e^0 = 1 \rightarrow y(0) = 1$

From eq ②

$y_1 = y \cos x$

$y_1(0) = y\cos(0) = y(1) = 1 \rightarrow y_1(0) = 1$

From eq ③

$y_2 = -y \sin x + \cos x \, y_1$

$y_2(0) = -y \sin(0) + \cos(0)\, y_1(0) \Rightarrow -y(0) + 1(y_1)$

$-1\cdot 0 + 1\cdot 1 \Rightarrow y_2(0) = 1$

From eq ④

$y_3 = -y \cos x - 2 y_1 \sin x + y_2 \cdot \cos x$

$y_3(0) = -y(\cos(0) - 2y_1 \sin(0) + y_2 \cos(0)$

$= -y(1) - 2y_1(0)(0) + y_2(0)\cos(0)$

$= -1\cdot 1 - 2(0) + 1\cdot 1 = -1 + 1 \Rightarrow y_3(0) = 0$

From eq ⑤

$y_4 = y \sin x - 3y_1 \cos x - 3y_2 \sin x + y_3 \cos x$

$y_4(0) = y(0)\sin(0) - 3y_1(0)\cos(0) - 3y_2(0)\sin(0) + y_3(\cos(0)$

$y_4(0) = 1\cdot 0 - 3(1)(1) - 3(1)(0) + 0\cdot 1$

$y_4(0) = 0 - 3\cdot 1 \Rightarrow y_4(0) = -3$

By Maclaurin's theorem.

$$y = y_0 + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y^3(0) + \frac{x^4}{4!}y^4(0) + \cdots$$

$$= 1 + x\cdot 1 + \frac{x^2}{2}(1) + \frac{x^3}{6}(0) + \frac{x^4(-3)}{24} + \cdots$$

$$= 1 + x + \frac{x^2}{2} - \frac{3x^4}{24} + \cdots$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

2) Expand $\log(\sec x)$ upto the term containing x^6 using Maclaurins series. [Dec 2017]

Expand $\log(\sec x)$ in ascending powers of x upto the first three non vanishing terms.

Sol Maclaurins Series

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y^4(0) + \cdots$$

$y = \log(\sec x)$. $\Longrightarrow y(0) = \log(\sec(0)) = \log 1 = 0$

$y_1 = \frac{1(\sec x \cdot \tan x)}{\sec x} \Rightarrow y_1 = \tan x$ $\Rightarrow y_1(0) = \tan(0) = 0$

$y_2 = \sec^2 x$ $\Rightarrow y$

~~$y_3 = 2\sec x(\sec x \tan x)$~~

~~$y_3 = 2\sec^2 x \tan x$~~

~~$y_4 = 2\sec x(\tan x)$~~

Sol: Maclaurins Series:

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y^4(0) + \cdots$$

$y = \log(\sec x)$ $\Longrightarrow y(0) = \log(\sec(0)) = \log 1 = 0$

$y_1 = \frac{1(\sec x \tan x)}{\sec x} = \tan x$ $\Longrightarrow y_1(0) = \tan x = \tan(0) = 0$

$y_2 = \sec^2 x$

$y_2 = 1 + \tan^2 x = 1 + y_1^2$ $\longrightarrow y_2(0) = 1 + y_1^2 = 1 + 0 = 1$

$y_2 = 1 + y_1^2$

$y_3 = 0 + 2 y_1 y_2$

$y_3 = 2 y_1 y_2$ $\Longrightarrow y_3(0) = 2(0)(1) = 0$

$y_4 = 2\left[y_1 \frac{d}{dx}(y_2) + y_2 \frac{d}{dx}(y_1)\right]$

$y_4 = 2(y_1 y_3 + y_2 y_2)$

$y_4 = 2(y_1 y_3 + y_2^2)$ $\Longrightarrow y_4(0) = 2((0)(0)) + (1)^2 = 2$

$$y_4 = 2(y_1 y_3 + y_2^2)$$

$$y_5 = 2\left[y_1 \frac{d}{dx}(y_3) + y_3 \frac{d}{dx}(y_1) + 2y_2 y_3\right]$$

$$y_5 = 2[y_1 y_4 + y_3 y_2 + 2y_2 y_3]$$

$$y_5 = 2y_1 y_4 + 2y_3 y_2 + 4y_2 y_3$$

$$y_5 = 2y_1 y_4 + 6y_2 y_3 \Rightarrow y_5(0) = 2(0)(1) + 6(1)(0) = 0$$

$$y_6 = 2[y_1 y_5 + y_4 y_2] + 6[y_2 y_4 + y_3 y_3]$$

$$y_6 = 2[y_1 y_5 + y_2 y_4] + 6[y_2 y_4 + y_3^2]$$

$$y_6 = 2y_1 y_5 + 2y_2 y_4 + 6y_2 y_4 + y_3^2$$

$$y_6 = 2y_1 y_5 + 8y_2 y_4 + 6y_3^2 \Rightarrow y_6(0) = 2(0)(0) + 8(1)(2) + 0^3 = \underline{\underline{16}}$$

$y(0) = 0$, $y_1(0) = 0$, $y_2(0) = 1$, $y_3(0) = 0$; $y_4(0) = 2$, $y_5(0) = 0$

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0) + \frac{x^5}{5!} y_5(0) + \frac{x^6}{6!} y_6(0)$$

$$\log(\sec x) = 0 + x(0) + \frac{x^2}{2}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{24}(2) + \frac{x^5}{120}(0) + \frac{x^6}{720}(16)$$

$$\log(\sec x) = 0 + 0 + \frac{x^2}{2} + 0 + \frac{x^4}{12} + 0 + \frac{x^6}{45}$$

$$\therefore \log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}$$

3) Using maclaurin's series prove that

$$\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} \cdots$$

Sol:- we have $y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \ldots$

$$y = \sqrt{1+\sin 2x} = \sqrt{\cos^2 x + \sin^2 x + 2\sin x \cos x}$$

$$= \sqrt{(\cos x + \sin x)^2} = \cos x + \sin x$$

$$y = \cos x + \sin x \Rightarrow y(0) = \cos(0) + \sin(0) = 1 + 0 = 1$$

$$y_1 = -\sin x + \cos x \Rightarrow y_1(0) = -\sin(0) + \cos(0) = 0 + 1 = 1$$

$y_1 = -\sin x + \cos x$

$y_2 = -\cos x + (-\sin x) = -\cos x - \sin x = -(\cos x + \sin x)$

$y_2 = -y \Rightarrow y_2(0) = -y \Rightarrow y_2 = -(1) = \underline{\underline{-1}}$

$y_3 = -y_1 \Rightarrow y_3(0) = -y_1 \Rightarrow -(1) = -1$

$y_4 = -y_2 \Rightarrow y_4(0) = -y_2 \Rightarrow -(-1) = 1$

By Maclaurins Series

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0)$$

$$= 1 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(-1) + \frac{x^4}{24}(1)$$

$$\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$$

4) obtain Maclaurins expansion of $\log(1+e^x)$ as for as fourth degree terms ?

Soln $y = \log(1+e^x) \Rightarrow y(0) = \log(1+e^0) = \log(2) = \log_e 2 \rightarrow ①$

$y_1 = \frac{e^x}{1+e^x} \Rightarrow y_1(0) = \frac{e^0}{1+e^0} = \frac{1}{1+1} = \frac{1}{2} \rightarrow ②$

$(1+e^x) y_1 = e^x$

Diff wrt x

$(1+e^x) y_2 + y_1 e^x = e^x \Rightarrow (1+e^0) y_2 + \frac{1}{2} e^0 = e^0$

$(1+1) y_2 + \frac{1}{2} \cdot 1 = 1$

$2 y_2 = 1 - \frac{1}{2}$

$2 y_2 = \frac{1}{2}$

$y_2 = \frac{1}{4} \Rightarrow y_2(0) = \frac{1}{4} \rightarrow ③$

Diff wrt x

$(1+e^x) y_3 + y_2 (0 + e^x) + y_1 e^x + e^x y_2 = e^x$

$(1+e^x) y_3 + e^x y_2 + y_1 e^x + e^x y_2 = e^x$

$(1+e^x) y_3 + 2e^x y_2 + y_1 e^x = e^x$

$\Rightarrow (1+e^0) y_3 + 2e^0 \left(\frac{1}{4}\right) + \frac{1}{2} e^0 = e^0$

$(1+1) y_3 + 2\left(\frac{1}{4}\right) + \frac{1}{2} = 1$

$2 y_3 + \frac{1}{2} + \frac{1}{2} = 1$

$2 y_3 + \frac{2}{2} = 1$

$2 y_3 = 1 - 1 = 0$

$y_3 = 0/ \ldots$

Diff wrt x

$$(1+e^x)y_4 + y_3(e^x) + 2e^x y_3 + y_2 e^x + y_1 e^x + e^x y_2 = e^x$$

$$(1+e^x)y_4 + y_3 e^x + 2e^x y_3 + 3y_2 e^x + y_1 e^x = e^x$$

$$(1+e^x)y_4 + 3y_3 e^x + 3y_2 e^x + y_1 e^x = e^x \Rightarrow (1+e^0)y_4 + 3(0)e^0 + 3\left(\frac{1}{4}\right)e^0 + \frac{1}{2}e^0 = e^0$$

$$(1+1)y_4 + 3\left(\frac{1}{4}\right)1 + \frac{1}{2}\cdot 1 = 1$$

$$2y_4 + \frac{3}{4} + \frac{1}{2} = 1$$

$$2y_4 + \frac{3+2}{4} = 1$$

$$2y_4 = 1 - \frac{5}{4}$$

$$2y_4 = \frac{4-5}{4}$$

$$2y_4 = -1/4$$

$$y_4 = -1/8 \rightarrow (5)$$

By Maclaurins Series

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0).$$

$$y(x) = \log_e 2 + x\left(\frac{1}{2}\right) + \frac{x^2}{2}\left(\frac{1}{4}\right) + \frac{x^3}{6}(0) + \frac{x^4}{24}\left(-\frac{1}{8}\right)$$

$$y(x) = \log_2 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192}$$

Dec 2016

5) Expand $\frac{e^x}{1+e^x}$ using Maclaurins series upto third order degree.

June 2018

6) Expand $\log(1+\cos x)$ by Maclaurins series upto the term containing x^4

Sol Maclaurins series

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0)$$

~~$y = \log(1+\cos x)$~~

~~$y_1 = \frac{(-\sin x)}{1+\cos x}$~~

$$y = \log(1+\cos x)$$

$$y = \log(2\cos^2(x/2))$$

$$y(0) = \log(2\cos^2(0/2)) = \log(2\cos(0)) = \log_e 2 \rightarrow (1)$$

1) Find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ where $u = x^2+y^2+z^2$, $v = xy+yz+zx$,

$w = x+y+z$

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$$u = x^2+y^2+z^2 \qquad u = x^2+y^2+z^2 \qquad u = x^2+y^2+z^2$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial z} = 2z$$

$$v = xy+yz+zx \qquad v = xy+yz+zx \qquad v = xy+yz+zx$$

$$\frac{\partial v}{\partial x} = y+z \qquad \frac{\partial v}{\partial y} = x+z \qquad \frac{\partial v}{\partial z} = y+x$$

$$w = x+y+z \qquad w = x+y+z \qquad w = x+y+z$$

$$\frac{\partial w}{\partial x} = 1 \qquad \frac{\partial w}{\partial y} = 1 \qquad \frac{\partial w}{\partial z} = 1$$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial x \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial y \end{vmatrix}$$

Substituting for the partial derivatives we have.

$$J = \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$$J = 2x\left[(x+z)-(y+x)\right] - 2y\left[(y+z)-(y+x)\right] + 2z\left[(y+z)-(x+z)\right]$$

$$= 2x\left[x+z-y-x\right] - 2y\left[y+z-y-x\right] + 2z\left[y+z-x-z\right]$$

$$= 2x[z-y] - 2y[z-x] + 2z[y-x]$$

$$= 2(xz-xy) - 2(yz-yx) + 2(zy-zx)$$

$$= 2(xz-xy-yz+yx+zy-zx) = 0$$

Thus $J = 0$

2) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

Show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$ June 2016

Sol: $J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix} \rightarrow (1).$

$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$ $\frac{\partial y}{\partial r} = \sin\theta\cdot\sin\phi$ $\frac{\partial z}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial\theta} = r\cos\theta\cos\phi$ $\frac{\partial y}{\partial\theta} = r\cos\theta\sin\phi$ $\frac{\partial z}{\partial\theta} = -r\sin\theta$

$\frac{\partial x}{\partial\phi} = -r\sin\theta\sin\phi$ $\frac{\partial y}{\partial\phi} = r\sin\theta\cos\phi$ $\frac{\partial z}{\partial\phi} = 0.$

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\cdot\sin\phi & r\cos\theta\cdot\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

On expanding by the last row we get

$= \cos\theta\,[r\cos\theta\cos\phi \cdot r\sin\theta\cos\phi - (r\cos\theta\cdot\sin\phi\cdot -r\sin\theta\sin\phi)]$

$- (-r\sin\theta)\,[r\sin\theta\cos\phi\cdot\sin\theta\cos\phi - \sin\theta\sin\phi(-r\sin\theta\sin\phi)]$

$= \cos\theta\,[r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi]$

$+ r\sin\theta\,[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi]$

$= r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi +$

$r^2\sin^3\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi.$

$= r^2\sin\theta\cos^2\theta\,[\cos^2\phi + \sin^2\phi] + r^2\sin^3[\cos^2\phi + \sin^2\phi]$

$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta$

$= r^2 \sin\theta \, (\cos^2\theta + \sin^2\theta)$

$= r^2 \sin\theta$

Thus $J = \dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta$

3) If $x+y+z = u,\ y+z = v$ and $z = uvw$, Then find the Value of $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ Dec 2015

Sol: $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \rightarrow (1)$

$x+y+z = u \rightarrow (2)$ $\quad y+z = v \rightarrow (3)$ $\quad z = uvw \rightarrow (4)$

using (3) in (2)

$x+y+z = u$

$x + v = u$

$x = u - v$

using (4) in (3)

$y + uvw = v$

$y = v - uvw$

Hence the given data is modified

$x = u - v \quad y = v - uvw \quad z = uvw$

$\dfrac{\partial x}{\partial u} = 1 \quad \dfrac{\partial y}{\partial u} = -vw \quad \dfrac{\partial z}{\partial u} = vw$

$\dfrac{\partial x}{\partial v} = -1 \quad \dfrac{\partial y}{\partial v} = 1 - uw \quad \dfrac{\partial z}{\partial v} = uw$

$\dfrac{\partial x}{\partial w} = 0 \quad \dfrac{\partial y}{\partial w} = -uv \quad \dfrac{\partial z}{\partial w} = uv$

Substitute in eq (1).

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & -1 & 0 \\ -vw & 1-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$= 1\,\cancel{[(1-uw)(uv)] + 1[}$

$= 1[(1-uw)uv - (uw)(-uv)] + 1[(uv)(-vw) - vw(-uv)]$

$= 1[uv - u^2vw + u^2vw] + 1[-uv^2w + uwv^2]$

$= uv - u^2vw + u^2vw - uv^2w + uv^2w$

$\boxed{J = uv}$

HW

4) If $x+y+z = u$, $y+z = uv$, $z = uvw$,

Show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

HW

5) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ show that

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$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

Sol: $u = \frac{yz}{x}$ $\quad v = \frac{zx}{y}$ $\quad w = \frac{xy}{z}$

$\frac{\partial u}{\partial x} = \frac{-yz}{x^2}$ $\quad \frac{\partial v}{\partial x} = \frac{z}{y}$ $\quad \frac{\partial w}{\partial x} = \frac{y}{z}$

$\frac{\partial u}{\partial y} = \frac{z}{x}$ $\quad \frac{\partial v}{\partial y} = \frac{-zx}{y^2}$ $\quad \frac{\partial w}{\partial y} = \frac{x}{z}$

$\frac{\partial u}{\partial z} = \frac{y}{x}$ $\quad \frac{\partial v}{\partial z} = \frac{x}{y}$ $\quad \frac{\partial w}{\partial z} = \frac{-xy}{z^2}$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{xy}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$J = -\frac{yz}{x^2}\left[\left(\frac{-zx}{y^2}\right)\cdot\left(\frac{-xy}{z^2}\right) - \left(\frac{x}{y}\right)\left(\frac{x}{z}\right)\right] -$$

$$\frac{z}{x}\left[\left(\frac{z}{y}\right)\left(\frac{-xy}{z^2}\right) - \left(\frac{x}{y}\right)\left(\frac{y}{z}\right)\right] + \frac{y}{x}\left[\left(\frac{z}{y}\right)\left(\frac{x}{z}\right) - \frac{y}{z}\left(\frac{-2x}{y^2}\right)\right]$$

$$J = \frac{-yz}{x^2}\left[\frac{zx^2y}{y^2z^2} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[\frac{-xyz}{yz^2} - \frac{xy}{yz}\right] + \frac{y}{x}\left[\frac{xz}{yz} + \frac{yzx}{zy^2}\right]$$

$$J = -\frac{yz}{x^2}\left[\frac{x^2}{yz} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[\frac{-x}{z} - \frac{x}{z}\right) + \frac{y}{x}\left[\frac{x}{y} + \frac{x}{y}\right]$$

$$J = \frac{-yz}{x^2}[0] - \frac{z}{x}\left[\frac{-2x}{z}\right] + \frac{y}{x}\left[\frac{2x}{y}\right]$$

$$J = \frac{z\,2x}{xz} + \frac{y\,2x}{xy}$$

$$J = 2 + 2$$

$$J = 4.$$

Thus $J = \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

6) If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, Then $w = 2z^2 - xy$

find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$. Dec 2017, June 2018

$u = x + 3y^2 - z^3$	$v = 4x^2yz$	$w = 2z^2 - xy$
$\dfrac{\partial u}{\partial x} = 1$	$\dfrac{\partial v}{\partial x} = 8xyz$	$\dfrac{\partial w}{\partial x} = -y$
$\dfrac{\partial u}{\partial y} = 6y$	$\dfrac{\partial v}{\partial y} = 4x^2z$	$\dfrac{\partial w}{\partial y} = -x$
$\dfrac{\partial u}{\partial z} = -3z^2$	$\dfrac{\partial v}{\partial z} = 4x^2y$	$\dfrac{\partial w}{\partial z} = 4z$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} \text{ at } (1,-1,0) = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 6(-1) & -3(0)^2 \\ 8\cdot1\cdot-1(0) & 4(1)(0) & 4(1)(-1) \\ -(-1) & -1 & 4(0) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$$J = 1(0-(4)) + 6(0+4) - 0(0-0)$$

$$J = 1(-4) + 6(4)$$

$$J = -4 + 24$$

$$J_{(1,-1,0)} = 20$$

Indeterminate forms:

→ L'Hospital's rule (Theorem)

Statement : If $f(x)$ and $g(x)$ are two functions such that

(i) $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ or $f(a) = 0 = g(a)$

(ii) $f'(x)$ and $g'(x)$ exist and $g'(a) \neq 0$

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

If $f'(a) = 0$ and $g'(0) = 0$ then we have.

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$ and so on.

Note:
① $\lim_{x\to 0} \frac{\sin x}{x} = 1$ ② $\lim_{x\to 0} \frac{x}{\sin x} = 1$ ③ $\lim_{x\to 0} \frac{\tan x}{x} = 1$ ④ $\lim_{x\to 0} \frac{x}{\tan x} = 1$

Problems:

Indeterminate forms : 0^0, ∞^0, 1^∞, $\left(\frac{0}{0}\right)$, $\infty \times 0$, $\left(\frac{\infty}{\infty}\right)$

Evaluate the following limits $(0^0, 1^\infty, \infty^0)$ June 2018

1) $\lim_{x\to \frac{\pi}{2}} (\sin x)^{\tan x}$

Let $k = \lim_{x\to \frac{\pi}{2}} (\sin x)^{\tan x}$ (1^∞). $\left(\left(\sin \frac{\pi}{2}\right)^{\tan \frac{\pi}{2}} = 1^\infty\right)$

Take log on both the side.

$\log_e k = \log \lim_{x\to \frac{\pi}{2}} (\sin x)^{\tan x}$

$\log_e k = \lim_{x\to \frac{\pi}{2}} \tan x \log(\sin x)$

$= \lim_{x\to \frac{\pi}{2}} \frac{1}{\cot x} \log(\sin x)$ $\left(\frac{0}{0}\right)$. $\left(\frac{\log(\sin(0))}{\cot(0)} = \frac{0}{0}\right)$

$\frac{\log(1)}{\cot(\frac{\pi}{2})} = \frac{0}{0}$

Applying L'Hospitals rule.

$$\log_e k = \lim_{x\to \frac{\pi}{2}} \frac{\frac{1}{\sin x}(\cos x)}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{\frac{-1}{\sin^2 x}}$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\cos x}{\sin} \times \frac{-\sin^2 x}{1} = \cos x \sin x$$

$\cos \frac{\pi}{2} \sin \frac{\pi}{2}$

$0 \cdot 1$

$\log_e k = 0$

$k = e^0 = 1$

1) HW If $x+y+z = u$, $y+z = uv$, $z = uvw$

Show that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

Sol $x+y+z = u \rightarrow (1)$ $\quad y+z = uv \rightarrow (2)$ $\quad z = uvw \rightarrow (3)$

Eq (2) in eq (1) $\qquad$ eq (3) in (2)

$x + uv = u$ $\qquad y + uvw = uv$

$x = u - uv \rightarrow (4)$ $\qquad y = uv - uvw \rightarrow (5)$ $\qquad z = uvw \rightarrow (6)$

$\dfrac{\partial x}{\partial u} = 1 - v$ $\qquad \dfrac{\partial y}{\partial u} = v - vw$ $\qquad \dfrac{\partial z}{\partial u} = vw$

$\dfrac{\partial x}{\partial v} = -u$ $\qquad \dfrac{\partial y}{\partial v} = u - uw$ $\qquad \dfrac{\partial z}{\partial v} = uw$

$\dfrac{\partial x}{\partial w} = 0$ $\qquad \dfrac{\partial y}{\partial w} = -uv$ $\qquad \dfrac{\partial z}{\partial w} = uv$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= 1-v\,[uv(u-uw) - uw(-uv)] + u\,[(v-vw)(uv) - vw(u-uw)] + 0$$

$$= (1-v)[u^2v - u^2vw \cancel{+u^2w} + u^2wv] + u[uv^2 - uv^2w - vwu + vuw^2]$$

$$= u^2v - u^2vw + u^2w + u^2vw - u^2v^2 + u^2v^2w - u^2vw - u^2w^2$$

$$+ u^2v^2 - u^2v^2w - vwu^2 + vu^2w^2 \cancel{+ vu^2w^2}$$

=

$= 1 - v\left(uv(u - uw)\right) - \left(uw(-uv)\right) + u\left(v - vw(uv) - vw(-uv)\right)$

$= 1 - v\left(u^2v - u^2vw + u^2vw\right) + u\left(v^2u - uwv^2 + v^2wu\right)$

$= 1 - v\left(u^2v\right) + u\left(v^2u\right).$

$= u^2v - u^2v^2 + u^2v^2$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2v$$

2). $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan(\pi x/2a)}.$

Sol $\quad K = \lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan(\pi x/2a)}. \quad (1^\infty) \quad \left[\left(2 - \frac{a}{a}\right)^{\tan\left(\frac{\pi a}{2a}\right)} = (2-1)^{\tan\left(\frac{\pi}{2}\right)}\right]$

$\boxed{1^\infty}$

$\log K = \lim_{x \to a} \log\left(2 - \frac{x}{a}\right)^{\tan(\pi x/2a)}$

$\log K = \lim_{x \to a} \log\left(2 - \frac{x}{a}\right)^{\tan(\pi x/2a)}$

$\log K = \lim_{x \to a} \tan(\pi x/2a) \cdot \log\left(2 - \frac{x}{a}\right) \quad (\infty \times 0).$

$\tan\left(\frac{\pi a}{2a}\right) \log\left(2 - \frac{a}{a}\right)$

$\tan\left(\frac{\pi}{2}\right) \log(2-1)$

$\boxed{\infty \times 0}$

$\log K = \lim_{x \to a} \frac{\log\left(2 - \frac{x}{a}\right)}{\cot(\pi x/2a)} \quad \left(\frac{0}{0}\right)$

$\frac{\log\left(2 - \frac{a}{a}\right)}{\cot(\pi \cdot a/2a)} = \frac{\log(1)}{\cot\left(\frac{\pi}{2}\right)} = \frac{0}{0}$

Applying L Hospitals rule.

$$\log K = \lim_{x \to a} \frac{\frac{1}{\left(2 - \frac{x}{a}\right)} \times \frac{-1}{a}}{-\operatorname{cosec}^2\left(\pi x/2a\right) \times \frac{\pi}{2a}}$$

$$\log K = \frac{\frac{1}{\left(2-\frac{a}{a}\right)} \times \frac{-1}{a}}{-\operatorname{cosec}^2\left(\frac{\pi a}{2a}\right)\frac{\pi}{2a}}$$

$$\log K = \frac{\frac{1}{1} \times \frac{-1}{a}}{-\operatorname{cosec}^2\left(\frac{\pi}{2}\right)\frac{\pi}{2a}}$$

$$\log K = \frac{\frac{-1}{a}}{-1\left(\frac{\pi}{2a}\right)}$$

$$\log K = \frac{-1}{a} \times \frac{2a}{\pi}$$

$$\log_e K = \frac{2}{\pi}$$

$$\underline{\underline{K = e^{2/\pi}}}$$

Q3) $\lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$

$$K = \lim_{x \to 0} \left(\frac{a^x+b^x}{2}\right)^{1/x} \quad (1^\infty)$$

$$\left(\frac{a^x+b^x}{2}\right)^{1/x} \qquad \left(\frac{a^0+b^0}{2}\right)^{1/0} = 1^\infty$$

Take log on both side

$$\log K = \lim_{x \to 0} \log\left(\frac{a^x+b^x}{2}\right)^{1/x}$$

$$\log K = \lim_{x \to 0} \frac{\log\left(\frac{a^x+b^x}{2}\right)}{x} \quad \left(\frac{0}{0}\right)$$

$$\frac{\log\left(\frac{a^0+b^0}{2}\right)}{0} \qquad \frac{\log 1}{0} = \frac{0}{0}$$

Applying L Hospitals rule.

$$\log_e K = \lim_{x \to 0} \frac{\frac{1}{\frac{a^x+b^x}{2}}\left(a^x \log a + b^x \log b\right) \cdot \frac{1}{2}}{x}$$

$$\log_e K = \lim_{x\to 0} \frac{\frac{2}{a^x+b^x}\left(a^x \log a + b^x \log b\right)}{1} \cdot \frac{1}{2}$$

$$\log_e K = \frac{\frac{2}{a^0+b^0}\left(a^0 \log a + b^0 \log b\right)}{1} \cdot \frac{1}{2}$$

$$\log_e K = \frac{2}{1+1}\left(1 \log a + 1 \log b\right) \cdot \frac{1}{2}$$

$$\log_e K = \frac{2}{2}\left(\log a + \log b\right) \cdot \frac{1}{2}$$

$$\log_e K = \frac{1}{2} \log a + \log b$$

$$\log_e K = \frac{1}{2} \log (ab)$$

$$\underline{\log_e K = \log ab^{1/2}} \Rightarrow K = (ab)^{1/2}$$

$$K = \sqrt{ab}$$

4) $\lim_{x\to 1} (1-x^2)^{1/\log(1-x)}$

$$K = \lim_{x\to 1} (1-x^2)^{1/\log(1-x)} \quad (0^0)$$

$$(1-1)^{1/\log(1-1)} \quad \lim (0)^{1/\log 0} \quad \boxed{0^0}$$

log on both side

$$\log_e K = \lim_{x\to 1} \log (1-x^2)^{1/\log(1-x)}$$

$$\log_e K = \lim_{x\to 1} \frac{1}{\log(1-x)} \log(1-x^2)$$

$$\log_e K = \lim_{x\to 1} \frac{\log(1-x^2)}{\log(1-x)} \quad \left(\frac{\infty}{\infty}\right)$$

$$\frac{\log(1-1)}{\log(1-1)} = \frac{\log 0}{\log 0} = \frac{\infty}{\infty}$$

Apply L Hospitals Rule

$$\log_e K = \lim_{x\to 1} \frac{\frac{(-2x)}{1-x^2}}{\frac{-1}{1-x}}$$

$$\log_e K = \lim_{x\to 1} \frac{-2x}{1-x^2} \times \frac{1-x}{-1}$$

$$= \lim_{x\to 1} \frac{2x(1-x)}{(1-x)(1+x)}$$

$\log_e K = \lim_{x\to 1} \frac{2x}{1+x}$

$\log_e K = \frac{2(1)}{1+1} = \frac{2}{2}$

$\log_e K = 1$

$K = e^1 \Rightarrow K = e$

$y = a^x$

$\log y = \log a^x$

$\log y = x \log a$

$\frac{1}{y}\frac{dy}{dx} = \log a$

$\frac{dy}{dx} = y \log a$

$\frac{dy}{dx} = a^x \log a$

Dec 2017

5) $\lim_{x\to 0} \left[\frac{a^x + b^x + c^x + d^x}{4}\right]^{1/x}$

$K = \lim_{x\to 0} \left[\frac{a^x+b^x+c^x+d^x}{4}\right]^{1/x}$ (1^∞)

$\left(\frac{a^0+b^0+c^0+d^0}{4}\right)^{1/0}$

$\left[\frac{4}{4}\right]^\infty = 1^\infty$

Take log on both side.

$\log_e K = \lim_{x\to 0} \log \left[\frac{a^x+b^x+c^x+d^x}{4}\right]^{1/x}$

$\log_e K = \lim_{x\to 0} \frac{1}{x} \log \left[\frac{a^x+b^x+c^x+d^x}{4}\right]$

$\log_e K = \lim_{x\to 0} \frac{1}{x} \frac{\log(a^x+b^x+c^x+d^x) - \log 4}{} \left(\frac{0}{0}\right)$

$\frac{\log(a^0+b^0+c^0+d^0) - \log 4}{0}$

$\frac{\log 4 - \log 4}{0} = \frac{0}{0}$

Apply L hospitals Rule.

$\log_e K = \lim_{x\to 0} \frac{(a^x \log a + b^x \log b + c^x \log c + d^x \log d) - 0}{(a^x + b^x + c^x + d^x)}$

$= \frac{a^0 \log a + b^0 \log b + c^0 \log c + d^0 \log d}{(a^0 + b^0 + c^0 + d^0)}$

$= \frac{\log a + \log b + \log c + \log d}{4}$

$= \frac{\log(ab) + \log(cd)}{4}$

$= \frac{\log(abcd)}{4} = \frac{1}{4}\log(abcd) = \log(abcd)^{1/4}$

$$\log_e K = \log (abcd)^{1/4}$$

$$K = (abcd)^{1/4}$$

June 2018

6)
$$\lim_{x \to 0} \left[\frac{a^x + b^x + c^x}{3}\right]^{1/x}$$

$$K = \lim_{x \to 0} \left[\frac{a^x + b^x + c^x}{3}\right]^{1/x} \quad (1^\infty)$$

$$\left[\frac{a^0 + b^0 + c^0}{3}\right]^{1/0} = \frac{3}{3}^{\infty} = 1^\infty$$

Take log on both side.

$$\log_e K = \lim_{x \to 0} \log \left[\frac{a^x + b^x + c^x}{3}\right]^{1/x}$$

$$\log_e K = \lim_{x \to 0} \frac{1}{x} \log \left[\frac{a^x + b^x + c^x}{3}\right]$$

$$\log_e K = \lim_{x \to 0} \frac{1}{x} \log (a^x + b^x + c^x) - \log 3 \quad \left(\frac{0}{0}\right)$$

$$\frac{\log(a^0 + b^0 + c^0) - \log 3}{0}$$

$$\frac{\log 3 - \log 3}{0} = \frac{0}{0}$$

Apply L Hospitals

$$\log_e K = \lim_{x \to 0} \frac{\frac{a^x \log a + b^x \log b + c^x \log c}{a^x + b^x + c^x} - 0}{1}$$

$$\log_e K = \lim_{x \to 0} \frac{a^0 \log a + b^0 \log b + c^0 \log c}{a^0 + b^0 + c^0}$$

$$\log_e K = \frac{1 \log a + \log b + \log c}{3}$$

$$\log_e K = \frac{\log (ab) + \log (c)}{3}$$

$$\log_e K = \frac{\log (abc)}{3}$$

$$\log_e K = \frac{1}{3} \log (abc) \quad [\log m^n = n \log m]$$

$$\log_e K = \log (abc)^{1/3}$$

$$K = (abc)^{1/3}$$

7) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

Sol $K = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$ (1^∞) $\quad \left(\frac{\sin 0}{0}\right)^{\frac{1}{0^2}} = 0^\infty$ ×

$(1)^{\frac{1}{0^2}} = 1^\infty$

Take log on both Side.

$\log K = \lim_{x\to 0}\log\left(\frac{\sin x}{x}\right)^{1/x^2}$

$\log k = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)$ $(\infty\times 0)$ $\quad \Rightarrow \frac{1}{0^2}\log 1 = \infty\times 0$

$\log k = \lim_{x\to 0}\frac{\log\left(\frac{\sin x}{x}\right)}{x^2}$ $\left(\frac{0}{0}\right)$ $\quad \Rightarrow \frac{\log(1)}{0^2} = \frac{0}{0}$

Apply L Hospital rule.

$\log k = \lim_{x\to 0}\dfrac{\frac{1}{\frac{\sin x}{x}}\times\dfrac{x\frac{d}{dx}(\sin x)-\sin x\frac{d}{dx}(x)}{x^2}}{2x}$

$\log k = \lim_{x\to 0}\dfrac{1/\frac{\sin x}{x}\times\dfrac{x\cos x-\sin x}{x^2}}{2x}$

$= \lim_{x\to 0}\frac{x}{\sin x}\lim_{x\to 0}\frac{x\cos x-\sin x}{x^2}\times\frac{1}{2x}$

$= 1\times\lim_{x\to 0}\frac{x\cos x-\sin x}{2x^3}\left[\frac{0}{0}\right]$ $\quad \left[\frac{0\cos(0)-\sin(0)}{2(0)}=\frac{0}{0}\right]$

$\log k = \lim_{x\to 0}\frac{x(-\sin x)+\cos x-\cos x}{6x^2}$

$\log k = \lim_{x\to 0}\frac{-x\sin x}{6x^2}$

$\log k = -\frac{1}{6}\lim_{x\to 0}\frac{\sin x}{x}$

$\log_e k = \frac{-1}{6}$

$k = e^{-1/6}$ //

8) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$ $(1)^{1/0} = 1^{\infty}$

$K = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x} \cdots (1^{\infty})$ $\left(\frac{\tan(0)}{0}\right)^{1/0} = \left(\frac{0}{0}\right)^{\infty}$

$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)$

$\log K = \lim_{x\to 0} \log\left(\frac{\tan x}{x}\right)^{1/x}$

$\lim_{x\to 0}\left(\frac{\sin x}{x\cos x}\right)$

$\log K = \lim_{x\to 0} \frac{1}{x}\log\left(\frac{\tan x}{x}\right)$

$\lim_{x\to 0}\left\{\left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right)\right\}$

$\log k = \lim_{x\to 0} \frac{\log(\tan x/x)}{x} \cdots \left(\frac{0}{0}\right)$ $\frac{\log(1)}{0} = \frac{0}{0}$

$\lim_{x\to 0}\frac{\sin x}{x} \lim_{x\to 0}\frac{1}{\cos x}$

Apply L Hospitals Rule

$1 \times 1 = 1$

$$\log K = \lim_{x\to 0} \frac{\frac{1}{(\tan x/x)} \cdot \frac{x\sec^2 x - \tan x}{x^2}}{1}$$

$$\log K = \lim_{x\to 0}\frac{x}{\tan x} \cdot \lim_{x\to 0}\frac{x\sec^2 x - \tan x}{x^2}$$

$$\log K = 1 \cdot \lim_{x\to 0}\frac{x\sec^2 x - \tan x}{x^2} \left(\frac{0}{0}\right)$$

$\left(\frac{0\sec^2(0) - \tan(0)}{0^2}\right.$

$\frac{0 \cdot 1 - 0}{0} = \frac{0}{0}$

Apply L Hospitals rule

$$\log K = \lim_{x\to 0}\frac{x\frac{d}{dx}(\sec^2 x) + \sec^2 x \frac{d}{dx}(x) - \frac{d}{dx}(\tan x)}{2x}$$

$$\log K = \lim_{x\to 0}\frac{x \cdot 2\sec x \,\cancel{\sec x}\tan x + \sec^2 x - \sec^2 x}{2x}$$

$\left(\frac{0}{0}\right.$

$$\log_e K = \lim_{x\to 0}\frac{\cancel{x}\,\cancel{2}\sec^2 x \cdot \tan x}{\cancel{2x}}$$

$$\log_e K = \frac{\sec^2(0)\tan(0)}{}$$

$$\log_e K = \frac{(1)(0)}{}$$

$\log_e K = 0$

$K = e^0$

$K = 1$//

5) Exp and $\frac{e^x}{1+e^x}$ using Maclaurins series upto third order degree

Dec 2016

Soln: $y = \frac{e^x}{1+e^x} \rightarrow (1)$ $\Rightarrow y(0) = \frac{e^0}{1+e^0} = \frac{1}{1+1} = \frac{1}{2}$

$\cancel{y(1+e^x) = e^x}$

$y\cancel{(e^x)} + \cancel{(1+e^x)}y_1$

$(1+e^x)y = e^x$

$(1+e^x)y_1 + y(e^x) = e^x$

$(1+e^x)y_1 + ye^x = e^x \rightarrow (2)$ $\Rightarrow y_1(0) = (1+e^0)y_1 + \frac{1}{2}e^0 = e^0$

$y_1(0) = 2y_1 + \frac{1}{2} = 1$

$2y_1 = 1 - \frac{1}{2}$

$2y_1 = \frac{1}{2}$

$y_1 = \frac{1}{4}$

$(1+e^x)y_2 + y_1(e^x) + ye^x + e^x y_1 = e^x$

$(1+e^x)y_2 + y_1e^x + ye^x + y_1e^x = e^x$

$(1+e^x)y_2 + 2y_1e^x + ye^x = e^x \rightarrow (3)$ $\Rightarrow y_2(0) = (1+e^0)y_2 + 2\left(\frac{1}{4}\right)e^0 + \frac{1}{2}e^0 = e^0$

$y_2(0) = (1+1)y_2 + \frac{2}{4} + \frac{1}{2} = 1$

$y_2(0) \Rightarrow 2y_2 + \frac{1}{2} + \frac{1}{2} = 1$

$y_2(0) \Rightarrow 2y_2 + \frac{2}{2} = 1$

$y_2(0) \Rightarrow 2y_2 = 1 - 1$

$y_2(0) \Rightarrow 2y_2 = 0$

$y_2 = 0$

$$(1+e^x)\,y_2 + 2y_1 e^x + y e^x = e^x$$

$$(1+e^x)\,y_3 + y_2(e^x) + 2[y_1 e^x + e^x y_2] + y e^x + e^x y_1 = e^x$$

$$(1+e^x)\,y_3 + y_2 e^x + 2y_1 e^x + 2e^x y_2 + y e^x + e^x y_1 = e^x$$

$$(1+e^x)\,y_3 + 3y_2 e^x + 3y_1 e^x + y e^x = e^x \longrightarrow (4)$$

$$(1+e^0)\,y_3 + 3(0)e^0 + 3\left(\frac{1}{4}\right)e^0 + \frac{1}{2}e^0 = e^0$$

$$(1+1)\,y_3 + 3(0)\cdot 1 + \frac{3}{4} + \frac{1}{2} = 1$$

$$2y_3 + \frac{3+2}{4} = 1$$

$$2y_3 + \frac{5}{4} = 1$$

$$2y_3 = \frac{1}{1} - \frac{5}{4}$$

$$2y_3 = \frac{4-5}{4}$$

$$2y_3 = \frac{-1}{4}$$

$$y_3 = \frac{-1}{8}$$

By Maclaurinse Series :

$$y(x) = y(0) + x\,y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0)$$

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{4}(0) + \frac{x^3}{6}\left(\frac{-1}{8}\right)$$

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48}$$

8) Find the maclaurins series for $\tan x$ upto the term containing x^4.

<u>sol:</u> $y = \tan x \Rightarrow y(0) = \tan(0) = 0$

$y_1 = \sec^2 x$

$y_2 =$

Maxima and Minima for a function of two Variables

Definition: A function $f(x,y)$ is said to have a maximum value at a point of (a,b) if there exists a neighborhood of the point (a,b) say. $(a+h, b+k)$, h and k are small such that $f(a,b) > f(a+h, b+k)$

Similarly if $f(a,b) < f(a+h, b+k)$ then $f(x,y)$ is said to have a minimum value at (a,b) also $f(a,b)$ is said to be an extreme value of $f(x,y)$ if it is a maximum value of a minimum value.

Working procedure for finding extreme values of $f(x,y)$.

Step 1: We have to first find the stationary points (x,y) such that $f_x = 0$ and $f_y = 0$

Step 2: We then find the second order partial derivat $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$. We evaluate these all the stationary points and also compute the corresponding the value of $AC - B^2$.

Step 3: (i) A stationary point (x_0, y_0) is a maximum point if $AC - B^2 > 0$ and $A < 0$. $f(x_0, y_0)$ is a maximum value

(ii) A stationary point (x_1, y_1) is a minimum point if $AC - B^2 > 0$ and $A > 0$, $f(x_1, y_1)$ is a minimum Value.

1) Find the extreme Values of the function.

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

Soln: $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

$f_x = 3x^2 - 3$ $\quad$ $f_y = 3y^2 - 12$

we shall find points (x,y) such that

$f_x = 0$ $\quad$ $f_y = 0$

$3x^2 - 3 = 0$ $\quad$ $3y^2 - 12 = 0$

$3x^2 = 3$ $\quad$ $3y^2 = 12$

$x^2 = \frac{3}{3}$ $\quad$ $y^2 = \frac{12}{3}$

$x^2 = 1$ $\quad$ $y^2 = 4$

$x = \pm 1$ $\quad$ $y = \pm 2$

$\therefore (1,2)\ (1,-2)\ (-1,2)\ (-1,-2)$ are the stationary points

Let $A = f_{xx}$ $\quad$ $B = f_{xy}$ $\quad$ $C = f_{yy}$.

$f_x = 3x^2 - 3$ $\quad$ $f_x = 3x^2 - 3$ $\quad$ $f_y = 3y^2 - 12$

$f_{xx} = 6x$ $\quad$ $f_{xy} = 0$ $\quad$ $f_{yy} = 6y$.

	$(1,2)$	$(1,-2)$	$(-1,2)$	$(-1,-2)$
$A = 6x$	$6 > 0$	6	-6	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	12	-12	12	-12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion	Min pt	Saddle pt	Saddle pt	Max. point

Maximum Value of $f(x,y)$ is

$f(-1,-2) = -1 - 8 + 3 + 24 + 20 = 38$

Minimum Value of $f(x,y)$ is

$f(1,2) = 1 + 8 - 3 - 24 + 20 = 2$

Maximum Value is 38 and minimum Value is 2 //

2) Find the maximum and minimum value of the function. $x^3+3y^2x-15x^2-15y^2+72x$

Sol: $f(x,y) = x^3+3y^2x-15x^2-15y^2+72x$

$f_x = 3x^2+3y^2-30x+72 \rightarrow (1)$, $f_y = 6yx-30y \rightarrow (2)$

We shall find points (x,y) such that

$f_x = 0$ $\qquad$ $f_y = 0$

$3x^2+3y^2-30x+72=0$ $\qquad$ $6yx-30y=0$

$3(x^2+y^2-10x+24)=0$ $\qquad$ $6(xy-5y)=0$

$x^2+y^2-10x+24=0 \rightarrow (3)$ $\qquad$ $yx-5y=0$

$yx=5y \rightarrow (4)$

$x = \frac{5y}{y} = \underline{\underline{5}}$

Put $y=0$ in eq (3) we get

$x^2+y^2-10x+24=0$ $\qquad$ $yx=5y$

$x^2+0-10x+24=0$ $\qquad$ $y(5)=5y$

$x^2-10x+24=0$ $\qquad$ $5y=5y=0$

$(x-4)(x-6)=0$ $\qquad$ $\underline{\underline{y=0}}$

$x=4,6$.

$(4,0)$ $(6,0)$ are stationary points

Put $x=5$ in eq (3) we get

$x^2+y^2-10x+24=0$

$5^2+y^2-10(5)+24=0$

$25+y^2-50+24=0$

$y^2-1=0$

$y=+1,-1$

$(5,1)$ $(5,-1)$ are the stationary points let $A = f_{xx}$ $\qquad$ $B = f_{xy}$

$f_x = 3x^2 + 3y^2 - 30x + 72$; $\quad f_{xy} = 3x^2 + 3y^2 - 30x + 72$

$f_{xx} = 6x - 30$ $\qquad$ $f_{xy} = 6y$

$C = f_{yy}$

$f_y = 6yx - 30y$

$f_{yy} = 6x - 30$

	(4,0)	(6,0)	(5,1)	(5,-1)
$A = 6x - 30$	$-6 < 0$	$6 > 0$	0	0
$B = 6y$	0	0	6	-6
$C = 6x - 30$	-6	6	0	0
$AC - B^2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 < 0$
Conclusion	Max point	Min. point	Saddle point	Saddle point

$(4,0) \Rightarrow A = 6x - 30 \Rightarrow 6(4) - 30 \Rightarrow 24 - 30 = -6$

$(6,0) \Rightarrow A = 6x - 30 \Rightarrow 6(6) - 30 \Rightarrow 36 - 30 = 6$

$(5,1) \Rightarrow A = 6x - 30 \Rightarrow 6(5) - 30 \Rightarrow 30 - 30 = 0$

$(5,-1) \Rightarrow A = 6x - 30 \Rightarrow 6(5) - 30 \Rightarrow 30 - 30 = 0$

$AC - B^2 \Rightarrow -6 \times -6 - 0^2 \Rightarrow 36$

$AC - B^2 \Rightarrow 6 \times 6 - 0^2 \Rightarrow 36$

$AC - B^2 \Rightarrow 0.0 - 6^2 \Rightarrow -36$

$AC - B^2 \Rightarrow 0.0 - (-6)^2 \Rightarrow -36$

Maximum value of $f(x,y)$ is

$f(4,0) \Rightarrow x^3 + 3y^2x - 15x^2 - 15y^2 + 72x$

$\Rightarrow 4^3 + 3(0)^2(4) - 15(4)^2 - 15(0) + 72(4)$

$\Rightarrow 64 + 0 - 15(16) + 288$

$\Rightarrow 112$

$f(6,0)$ minimum value of $f(x,y)$ is

$f(6,0) = x^3 + 3y^2x - 15x^2 - 15y^2 + 72x$

$6^3 + 3(0)6 - 15(6)^2 - 15(0) + 72(6)$

$72 - 540 + 432 = 108$

3) Find the extreme values of $f(x, y) = x^3y^2(1-x-y)$

Sol $f(x, y) = x^3y^2(1-x-y)$.

$f(x, y) = x^3y^2 - x^4y^2 - x^3y^3$

we shall find point (x, y) such that

$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 \rightarrow (1)$

$f_y = 2x^3y - 2yx^4 - 3y^2x^3 \rightarrow (2)$

$f_x = 0$

$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$x^2y^2(3-4x-3y) = 0$

$3-4x-3y = 0$

$-4x-3y = -3$

$4x+3y = 3 \rightarrow (3)$

$f_y = 0$

$2x^3y - 2yx^4 - 3y^2x^3 = 0$

$x^3y(2-2x-3y) = 0$

$2-2x-3y = 0$

$-2x-3y = -2$

$2x+3y = 2 \rightarrow (4)$

on solving (3) and (4) we get

$4x+3y = 3$

$2x+3y = 2 \times 2$

$4x+3y = 3$

$4x+6y = 4$

(-) (-) (-)

$-3y = -1$

$y = \frac{1}{3}$

$4x+3y = 3$

$4x+3\left(\frac{1}{3}\right) = 3$

$4x+1 = 3$

$4x = 2$

$x = 1/2$

$x = \frac{1}{2}$ and $y = \frac{1}{3}$

$(0,0)$ $(\frac{1}{2}, \frac{1}{3})$ are the stationary points

$A = f_{xx}$

$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3 \Rightarrow 6xy^2(1 - 2x - y)$

$B = f_{xy}$

$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$f_{xy} = 6x^2y - 8x^3y - 9x^2y^2 \Rightarrow x^2y(6 - 8x - 9y)$

$C = f_{yy}$

$f_y = 2x^3y - 2yx^4 - 3y^2x^3$

$f_{yy} = 2x^3 - 2x^4 - 6yx^3 \Rightarrow 2x^3(1 - x - 3y)$

(i) Points $(0,0)$

$A = 6xy^2(1-2x-y) = 6(0)(0)(1-2(0)-0) = 0$

$B = x^2y(6-8x-9y) = 0(0)(6-8(0)-9(0)) = 0$

$C = 2x^3(1-x-3y) = 2(0)(1-0-3(0)) = 0.$

$AC - B^2 = 0$

(ii) points $(\frac{1}{2}, \frac{1}{3})$

$A = 6xy^2(1-2x-y) \Rightarrow 6(\frac{1}{2})(\frac{1}{3})^2(1-2(\frac{1}{2})-\frac{1}{3}) = 6(\frac{1}{18})(1-1-\frac{1}{3})$

$\Rightarrow \frac{6}{18}(\frac{-1}{3}) = \underline{\underline{\frac{-1}{9}}}$

$B = x^2y(6-8x-9y) \Rightarrow (\frac{1}{2})^2(\frac{1}{3})(6-8(\frac{1}{2})-9(\frac{1}{3}))$

$\Rightarrow (\frac{1}{4})(\frac{1}{3})(6-4-3) = \frac{1}{12}(6-7) = \frac{-1}{12}$

$C = 2x^3(1-x-3y) = 2(\frac{1}{2})^3(1-\frac{1}{2}-3(\frac{1}{3}))$

$= \frac{2}{8}(1-\frac{1}{2}-1) = \frac{1}{4}(\frac{-1}{2}) = \frac{-1}{8}$

$AC - B^2 \Rightarrow \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2$

$$\frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$$

$$A = -\frac{1}{9} < 0$$

Maximum value of $f(x,y)$ is

$$f\left(\frac{1}{2}, \frac{1}{3}\right) = x^3 y^2 (1 - x - y)$$

$$= \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right)$$

$$= \left(\frac{1}{8}\right)\left(\frac{1}{9}\right)\left(1 - \frac{3-2}{6}\right)$$

$$= \frac{1}{72}\left(\frac{1}{1} - \frac{5}{6}\right)$$

$$= \frac{1}{72}\left(\frac{6-5}{6}\right)$$

$$= \frac{1}{432}$$

The maximum of $f(x,y)$ is $f(1/2, 1/3) = 1/432$

4) Examine the function $\sin x + \sin y + \sin(x+y)$ for extreme values $f(x,y) = \sin x + \sin y + \sin(x+y)$

soln $f(x,y) = \sin x + \sin y + \sin(x+y)$

$\sin x \cos y + \cos x \sin y$

$f_x = \cos x + \cos(x+y)(1+0) \rightarrow$ ① $\quad \cos x - \sin x$

$f_y = \cos y + \cos(x+y) \rightarrow$ ②

Now we shall find point (x,y) such that

$f_x = 0$ $\qquad\qquad$ $f_y = 0$

$\cos x + \cos(x+y) = 0$ $\qquad\qquad$ $\cos y + \cos(x+y) = 0$

$\cos(x+y) = -\cos x \rightarrow ③$ $\qquad\qquad$ $\cos(x+y) = -\cos y \rightarrow ④$

$-\cos x = -\cos y$ Comparing eq ③ and ④

$\cos x = \cos y$

$\underline{\underline{x = y}}$

Put $y = x$ in eq ①.

$\cos x + \cos(x+y) = 0$

$\cos x + \cos(x+x) = 0$

$\cos x + \cos 2x = 0$

$\cos x + 2\cos^2 x - 1 = 0$

$2\cos x - \cos x + 2\cos^2 x - 1 = 0$

$2\cos^2 x + 2\cos x - \cos x - 1 = 0$

$2\cos x(\cos x + 1) - 1(\cos x + 1) = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$2\cos x - 1 = 0$ $\qquad$ $\cos x + 1 = 0$

$2\cos x = 1$ $\qquad$ $\cos x = -1$

$\cos x = \frac{1}{2}$ $\qquad$ $x = \cos^{-1}(-1)$

$x = \cos^{-1}\left(\frac{1}{2}\right)$ $\qquad$ $x = \pi$

$x = \frac{\pi}{3}$

$\left(\frac{\pi}{3}, y\right)$ (π, y)

$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ (π, π) are the stationary points

$A = f_{xx}$

$f_x = \cos x + \cos(x+y)$

$f_{xx} = -\sin x - \sin(x+y)$

$B = f_{xy}$

$f_x = \cos x + \cos(x+y)$

$f_{xy} = -\sin(x+y)$

$C = f_{yy}$

$f_y = \cos y + \cos(x+y)$

$f_{yy} = -\sin y - \sin(x+y).$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$A = -\sin x - \sin(x+y) = -\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} - \sin\frac{2\pi}{3}$

$= -\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right) = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$

$B = -\sin(x+y) = -\sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right) = -\sin\left(\frac{2\pi}{3}\right) = -\left(\frac{\sqrt{3}}{2}\right).$

$C = -\sin y - \sin(x+y) = -\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) - \sin\frac{2\pi}{3}$

$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\frac{2\sqrt{3}}{2} = -\sqrt{3}$

$AC - B^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2$

$= \sqrt{9} - \frac{\sqrt{9}}{4}$

$= \frac{3}{1} - \frac{3}{4} \Rightarrow \frac{12-3}{4} = \frac{9}{4}$

$AC - B^2 = \frac{9}{4} > 0$

$A = -\sqrt{3} < 0$

At (π, π)

$A = -\sin x - \sin(x+y) = -\sin\pi - \sin(\pi+\pi)$

$= -\sin\pi - \sin(2\pi) = \underline{\underline{0}}$

$B = -\sin(x+y) = -\sin(\pi+\pi) = -\sin(2\pi) = 0$

$C = -\sin y - \sin(x+y) = -\sin\pi - \sin(\pi+\pi) = 0$

$A = 0, \quad B = 0, \quad C = 0$

Thus: Maximum Value of $f(x,y)$ is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$f(x,y) = \sin x + \sin y + \sin(x+y)$

$f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \sin\frac{2\pi}{3}$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

The maxim. of $f(x,y)$ is $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$.

Partial differentiation:

Total derivative - Differentiation of Composite function:

If $u = f(x,y)$ then the total differential of u is defined as.

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \rightarrow (1).$$

Type (I): Total derivative rule :

If $u = f(x, y)$, $x = x(t)$ and $y = y(t)$ then u is a composite function of the single variable t. Therefore differentiate u with respect to t.

$$u \to (x, y) \to t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

This is called the total derivative of u.

Type (II) : Chain rule

If $u = f(x, y)$ where $x = x(r, s)$ and $y = y(r, s)$ then u is a composite function of two independent variable r, s.

Therefore differentiate u wrt r and also wrt s partially. Thus we have the following chain rule for two partial derivatives.

$$u \to (x, y) \to (r, s) \Rightarrow u \to (r, s) \left\langle \begin{array}{l} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \end{array} \right.$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \; ; \; \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}.$$

Problems:

Find the total derivative of the following function and also verify the result by direct substitution

1) $z = xy^2 + x^2y$ where $x = at$; $y = 2at$.

Sol: $z = xy^2 + x^2y$ $\quad x = at$, $y = 2at$

$z \to (x, y) \to t \quad \Rightarrow z \to t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{\partial}{\partial x}(xy^2 + x^2y) \cdot \frac{d}{dt}(at) + \frac{\partial}{\partial y}(xy^2 + x^2y) \cdot \frac{d}{dt}(2at)$$

$$\frac{dz}{dt} = (y^2 + 2xy)(a) + (2xy + x^2)\, 2a$$

$$(2at \cdot 2at + a^2t^2)\, 2a$$

$$\frac{dz}{dt} = (2^2a^2t^2 + 2 \cdot at \cdot 2at)(a) + (4a^2t^2 + a^2t^2)\, 2a$$

$$\frac{dz}{dt} = (4a^2t^2 + 4a^2t^2)\, a + (5a^2t^2)\, 2a$$

$$\frac{dz}{dt} = (8a^2t^2)\, a$$

$$\frac{dz}{dt} = 8a^3t^2 + 10a^3t^2$$

$$\frac{dz}{dt} = 18a^3t^2 \to (1) \Rightarrow \text{Total derivat}$$

Now by direct substitution

$z = xy^2 + x^2y$

$z = (at)(2at)^2 + (at)^2(2at)$

$z = (at)(4a^2t^2) + a^2t^2(2at)$

$z = 4a^3t^3 + 2a^3t^3 = 6a^3t^3$

$$\frac{dz}{dt} = 6a^3 \cdot 3t^2 = 18a^3t^2 \to (2)$$

Thus from eq (1) and (2) the result is Verified

2) $u = xy + yz + zx$; $x = t\cos t$, $y = t\sin t$, $z = t$

at $t = \frac{\pi}{4}$

Sol: $u \to (x, y, z) \to t \Rightarrow u \to t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$\cancel{\frac{du}{dt} = \frac{\partial(xy+yz+zx)}{\partial x} + \frac{\partial(xy+yz+zx)}{\partial y}}$$

$$\frac{du}{dt} = \frac{\partial}{\partial x}(xy+yz+zx)\frac{d}{dt}(t\cos t) + \frac{\partial}{\partial y}(xy+yz+zx)\frac{d}{dt}(t\sin t)$$
$$+ \frac{\partial}{\partial z}(xy+yz+zx)\frac{d}{dt}(t).$$

$$\frac{du}{dt} = (y+z)(t(-\sin t) + \cos t) + (x+z)(t\cos t + \sin t)$$
$$+ (y+x)$$

$$\frac{du}{dt} = (y+z)(-t\sin t + \cos t) + (x+z)(t\cos t + \sin t) + (y+x)$$

$$\frac{du}{dt} = (t\sin t + t)(-t\sin t + \cos t) + (t\cos t + t)(t\cos t + \sin t)$$
$$+ (t\sin t + t\cos t)$$

$$\frac{du}{dt} = (-t^2\sin^2 t + t\sin t\cos t - t^2\sin t + t\cos t) +$$
$$(t^2\cos^2 t + t\cos t\sin t + t^2\cos t + t\sin t) + (t\sin t + t\cos t)$$

$$\frac{du}{dt} = t^2\cos^2 t - t^2\sin^2 t + 2t\sin t\cos t + t^2\cos t - t^2\sin t +$$
$$2t\sin t + 2t\cos t$$

$$\frac{du}{dt} = t^2(\cos^2 t - \sin^2 t) + 2t\sin t\cos t + t^2(\cos t - \sin t)$$
$$+ 2t(\sin t + \cos t)$$

$$\frac{du}{dt} = \left(\frac{\pi}{4}\right)^2\left(\cos^2\left(\frac{\pi}{4}\right) - \sin^2\left(\frac{\pi}{4}\right)\right) + 2\frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)\cos\frac{\pi}{4} + \left(\frac{\pi}{4}\right)^2\left(\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$$
$$+ 2\frac{\pi}{4}\left(\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\right).$$

$$\frac{du}{dt} = \left(\frac{\pi}{4}\right)^2\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + 2\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\pi}{4}\right)^2\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + 2\frac{\pi}{4}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right).$$

$$\left(\frac{du}{dt}\right)_{t=\frac{\pi}{4}} = 0 + \frac{2\pi}{4}\left(\frac{1}{2}\right) + \frac{2\pi}{4}\left(\frac{2}{\sqrt{2}}\right)$$

$$\left(\frac{du}{dt}\right) = \frac{\pi}{4} + \frac{\pi}{\sqrt{2}} \longrightarrow (1).$$

Now by direct substitution we have.

$$u = xy + yz + zx$$

$$u = t\cos t \cdot t\sin t + t\sin t \cdot t + t \cdot t\cos t$$

$$u = t^2\cos t \cdot \sin t + t^2 \sin t + t^2 \cos t$$

$$u = t^2(\cos t \cdot \sin t + \sin t + \cos t)$$

$$\cos A \cdot \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

diff wrt t we have.

$$\cos t \cdot \sin t = \frac{1}{2}\left[\sin(t+t) - \sin(t-t)\right]$$

$$\cos t \cdot \sin t = \frac{1}{2}\left[\sin 2t\right]$$

$$u = t^2\left(\frac{1}{2}\cdot \sin 2t + \sin t + \cos t\right)$$

Differentiating wrt 't' we have

$$\frac{du}{dt} = t^2(\cos 2t + \cos t - \sin t) + \left(\frac{1}{2}\cdot\sin 2t + \sin t + \cos t\right)(2t)$$

$$\cos 2t = 2\cos^2 t - 1$$
$$= 2\cos^2\left(\frac{\pi}{4}\right) - 1$$
$$= 2\left(\frac{1}{2}\right) - 1$$
$$1 - 1 = 0$$

$$\left(\frac{du}{dt}\right)_{t=\frac{\pi}{4}} = \left(\frac{\pi}{4}\right)^2\left(\cos 2\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right) + \left(\frac{1}{2}\sin 2\frac{\pi}{4} + \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\right)\left(2\left(\frac{\pi}{4}\right)\right)$$

$$\left(\frac{du}{dt}\right)_{t=\frac{\pi}{4}} = \left(\frac{\pi^2}{16}\right)\left(0 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\cdot 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\left(\frac{\pi}{2}\right)$$

$$\left(\frac{du}{dt}\right)_{t=\frac{\pi}{2}} = \frac{\pi}{2}\left(\frac{1}{2} + \frac{2}{\sqrt{2}}\right) = \frac{\pi}{4} + \frac{2\pi}{2\sqrt{2}} = \frac{\pi}{4} + \frac{\pi}{\sqrt{2}}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)$$
$$= 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$
$$= 1$$

$$\left(\frac{du}{dt}\right)_{t=\frac{\pi}{2}} = \frac{\pi}{4} + \frac{\pi}{\sqrt{2}}$$

3) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ Prove that $x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

[June 2016, 2017, 2018]

Sol: Here we need to convert the given function u into a composite function.

Let $u = f(p, q, r)$ where $p = \frac{x}{y}$, $q = \frac{y}{z}$, $r = \frac{z}{x}$

i.e $\{u \rightarrow (p, q, r) \rightarrow (x, y, z)\} \Rightarrow u \rightarrow x, y, z$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{\partial u}{\partial q}\frac{\partial}{\partial x}\left(\frac{y}{z}\right) + \frac{\partial u}{\partial r}\frac{\partial}{\partial x}\left(\frac{z}{x}\right).$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{1}{y} + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}\left(\frac{-z}{x^2}\right)$$ x'ly by x

$$x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial p} + \frac{\partial u}{\partial r}\left(\frac{-zx}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial p} - \frac{z}{x}\frac{\partial u}{\partial r} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \frac{\partial u}{\partial q}\frac{\partial}{\partial y}\left(\frac{y}{z}\right) + \frac{\partial u}{\partial z}\frac{\partial}{\partial y}\left(\frac{z}{x}\right).$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial q}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial z}(0)$$ x'ly by y

$$y\frac{\partial u}{\partial y} = -\frac{x\cdot y}{y^2}\left(\frac{\partial u}{\partial p}\right) + \frac{\partial u}{\partial q}\left(\frac{y}{z}\right)$$

$$y\frac{\partial v}{\partial y} = -\frac{x}{y}\left(\frac{\partial v}{\partial p}\right) + \frac{y}{z}\left(\frac{\partial v}{\partial q}\right)$$

$$y\frac{\partial v}{\partial y} = \frac{y}{z}\frac{\partial v}{\partial q} - \frac{x}{y}\frac{\partial v}{\partial p} \rightarrow ③$$

Similarly by symmetry we can write

$$z\frac{\partial v}{\partial z} = \frac{z}{x}\frac{\partial v}{\partial r} - \frac{y}{z}\frac{\partial v}{\partial q} \rightarrow ②$$

Thus by adding ① ② and ③ we get

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 0$$

$$\frac{x}{y}\cancel{\frac{\partial v}{\partial p}} - \frac{z}{x}\cancel{\frac{\partial v}{\partial r}} + \frac{y}{z}\cancel{\frac{\partial v}{\partial q}} - \frac{x}{y}\cancel{\frac{\partial v}{\partial p}} + \frac{z}{x}\cancel{\frac{\partial v}{\partial r}} - \frac{y}{z}\cancel{\frac{\partial v}{\partial q}}$$

$$\therefore x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 0$$

4) If $u = f(x-y, y-z, z-x)$ show that

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} = 0.$$

Soln: Let $u = f(p, q, r)$ where $p = x-y$, $q = y-z$, $r = z-x$

i.e $u \rightarrow (p, q, r) \rightarrow (x, y, z) \Rightarrow u \rightarrow x, y, z$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial v}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial p}\cdot\frac{\partial(x-y)}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial(y-z)}{\partial x} + \frac{\partial v}{\partial r}\frac{\partial(z-x)}{\partial x}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial p}\cdot 1 + \frac{\partial v}{\partial q}(0) + \frac{\partial v}{\partial r}(-1)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial p} - \frac{\partial v}{\partial r} \rightarrow ①$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial}{\partial y}(x-y) + \frac{\partial u}{\partial q}\cdot\frac{\partial}{\partial y}(y-z) + \frac{\partial u}{\partial r}\frac{\partial}{\partial y}(z-x)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0)$$

$$\frac{\partial u}{\partial y} = \frac{-\partial u}{\partial p} + \frac{\partial u}{\partial q}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial (x-y)}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial}{\partial z}(y-z) + \frac{\partial u}{\partial r}\frac{\partial}{\partial z}(z-x)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \longrightarrow (3)$$

Add eq (1) and (2) and (3).

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

5) If $z = f(x, y)$ and $x = e^u \sin v$, $y = e^u \cos v$ prove that $\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}$

Sol: $z \longrightarrow (x,y) \longrightarrow (u,v) \Longrightarrow z \longrightarrow (u,v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial (e^u \sin v)}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial}{\partial u} (e^u \cos v)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} (e^u \sin v) + \frac{\partial z}{\partial y} (e^u \cos v) \longrightarrow (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial (e^u \sin v)}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial}{\partial v} (e^u \cos v)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} (e^u \cos v) + \frac{\partial z}{\partial y} (e^u (-\sin v))$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} (e^u \cos v) - \frac{\partial z}{\partial y} e^u \sin v \longrightarrow (2)$$

squaring and adding (1) and (2)

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 e^{2u} \sin^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u} \cos^2 v$$

$$+ \left(\frac{\partial z}{\partial x}\right)^2 e^{2u} \cos^2 v \quad \left(\frac{\partial z}{\partial y}\right)^2 e^{2v} \sin^2 v$$

$$= e^{2u} \left(\frac{\partial z}{\partial x}\right)^2 (\sin^2 v + \cos^2 v) + e^{2u} \left(\frac{\partial z}{\partial y}\right)^2 (\sin^2 v + \sin^2 v$$

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x} (e^u \sin v) + \frac{\partial z}{\partial y} (e^u \cos v)\right)^2 +$$

$$\left(\frac{\partial z}{\partial x} (e^u \cos v) - \frac{\partial z}{\partial y} (e^u \sin v)\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 e^{2u} \sin^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u} \cos^2 v + 2 \cdot \frac{\partial z}{\partial x} (e^u \sin v) \cdot \frac{\partial z}{\partial y} (e^u \cos$$

$$+ \left(\frac{\partial z}{\partial x}\right)^2 e^{2u} \cos^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u} \sin^2 v - 2 \cdot \frac{\partial z}{\partial x} e^u \cos v \cdot \frac{\partial z}{\partial y} e^u \sin y$$

$$= e^{2u}\left(\frac{\partial z}{\partial x}\right)^2 [\sin^2 v + \cos^2 v] + e^{2u}\left(\frac{\partial z}{\partial y}\right)^2 (\cos^2 v + \sin^2 v)$$

$$+ 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} e^{2u}\sin v\cos v - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} e^{2u}\cos v\sin v$$

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left(\frac{\partial z}{\partial x}\right)^2 + e^{2u}\left(\frac{\partial z}{\partial y}\right)^2$$

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

6) If $z = f(x, y)$ where $x = r\cos\theta$, $y = r\sin\theta$

show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

Sol: $z \to (x, y) \to (r, \theta) \Rightarrow z \to (r, \theta)$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial (r\cos\theta)}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial (r\sin\theta)}{\partial r}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \longrightarrow ①$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial (r\cos\theta)}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial (r\sin\theta)}{\partial \theta}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\, {-r\sin\theta} + \frac{\partial z}{\partial y}\, r\cos\theta$$

$$\frac{\partial z}{\partial \theta} = r\left(-\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta\right)$$

$$\frac{1}{r}\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta \longrightarrow ②$$

Squaring and adding ① and ②

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 + \left(\frac{\partial z}{\partial y}\cos\theta - \frac{\partial z}{\partial x}\sin\theta\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta + 2\cdot\frac{\partial z}{\partial x}\cos\theta\,\frac{\partial z}{\partial y}\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta$$

$$+ \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta - 2\cdot\frac{\partial z}{\partial y}\cos\theta\cdot\frac{\partial z}{\partial x}\sin\theta$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

7) If $z = f(x,y)$ where $x = e^u + e^{-v}$, $y = e^{-u} - e^v$

prove that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$.

Sol: $\{z \to (x,y) \to (u,v)\} \Rightarrow z \to (u,v)$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial}{\partial u}(e^u + e^{-v}) + \frac{\partial z}{\partial y}\frac{\partial}{\partial u}(e^{-u} - e^v)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u + \frac{\partial z}{\partial y}(-e^{-u}) \longrightarrow ①$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial}{\partial v}(e^u + e^{-v}) + \frac{\partial z}{\partial y}\frac{\partial}{\partial v}(e^{-u} - e^{+v})$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v) \longrightarrow ②$$

Equation ① − ②

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot e^{u} + \frac{\partial z}{\partial y}(-e^{-u}) \rightarrow ①$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}) \rightarrow ②$$

$$\frac{\partial z}{\partial x} e^{u} - \frac{\partial z}{\partial y} e^{-u} \div \left(-\frac{\partial z}{\partial x} e^{-v} - \frac{\partial z}{\partial y} e^{v}\right)$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} e^{u} - \frac{\partial z}{\partial y} e^{-u} + \frac{\partial z}{\partial x} e^{-v} + \frac{\partial z}{\partial y} e^{v}$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^{u} + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^{-v})$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$$

Linear algebra

Image Recognition -