Module 5: Linear Algebra

Elementary row transformation of a matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \rightarrow R \quad \downarrow C$$

Example:

Elementary row transformation	Notation	Resultant of matrix A
1) Interchange of first row and second row	$R_1 \leftrightarrow R_2$	$\begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$
2) multiplication of third row by constant K	KR_3	$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ Kc_1 & Kc_2 & Kc_3 \end{bmatrix}$
3) Addition to second row K times the first row	$R_2 \rightarrow KR_1 + R_2$	$\begin{bmatrix} a_1 & a_2 & a_3 \\ (Ka_1+b_1) & (Ka_2+b_2) & (Ka_3+b_3) \\ c_1 & c_2 & c_3 \end{bmatrix}$

Equivalent matrices:

Two matrices A and B of the same order is said to be equivalent matrices.

Echelon form of a matrix:

A non zero matrix A is said to be in the following row echelon form in the following.

a) All the zero rows are below non zero rows.

b) The first non zero entry in any non zero row is 1

Eg: $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Rank of a matrix:

The rank of a matrix A in its echelon form is equal to the number of non zero row. It is denoted by $\rho(A)$

Flow Chart for solving problems - Finding the rank of a matrix.

1. In order to reduce the given matrix (first entry in the first row) non zero, much preferably 1.
2. In case this entry is zero we can interchange with any suitable row.
3. Row echelon form will be achieved first and we can instantly write down the rank.

Find the rank of the following matrices by elementary row transformation:

Dependent = 0

Independent = Non Zero

1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Soln: $R_2 \to -3R_1 + R_2$

$A \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $\quad -3(1) + 3 = -3+3 = 0$

$-3(2) + 6 = -6+6 = 0$

$\rho(A) = 1$

2) $A = \begin{bmatrix} 1 & 2 & -3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

$R_2 \to -2R_1 + R_2$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

$R_2 = -2R_1 + R_2$
$= -2(1) + 2 = 0$
$R_2 = -2(2) + 3$
$= -4 + 3 = -1$
$R_2 = -2(3) + 5$
$= -6 + 5 = -1$
$R_2 = -2(2) + 1$
$= -4 + 1 = -3$

$R_3 \to -R_1 + R_3$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

- Only Row operation
- Reduce to upper triangle matrix
- No of non zero Rows = Rank of matrix

$R_3 \to R_2 + R_3$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \to (-1) R_2$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The matrix A in the row echelon form is having two non zero rows.

Thus $\rho(A) = 2$

Imp

3) $A = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

Sol : $R_1 \leftrightarrow R_2$

$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$R_2 \to -2R_1 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -9 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \to -R_1 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -9 & 1 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & -5 & -9 & 1 \end{bmatrix}$$

$R_3 \to 2R_2 + R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to -\frac{1}{4} R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A in the row echelon form is having three non zero rows

Thus $\rho(A) = 3$.

$-2R_1 + R_2$

$-2(1) + 2 = -2 + 2 = 0$

$-2(2) - 1 = -4 - 1 = -5$

$-2(3) - 3 = -6 - 3 = -9$

$-2(-1) - 1 = 2 - 1 = 1$

$-R_1 + R_3$

$-1 + 1 = 0$

$-2 + 0 = -2$

$-3 + 1 = -2$

$1 + 1 = 2$

$2R_2 + R_3$

$2(0) + 0 = 0$

$2(1) - 2 = 0$

$2(1) - 2 = 0$

$2(-1) + 2 = 0$

$R_3 \to -\frac{1}{4} R_3$

$-\frac{1}{4}(-4) = 1$

$-\frac{1}{4}(-4) = 1$

Dec 2017

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

$R_2 \to R_1$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 0 & 2 & 1 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

$-2R_1 + R_2$
$-2(2) + 4 = 0$
$-2(1) + 0 = -2$
$-2(3) + 2 = -4$
$-2(4) + 1 = -7$

$R_3 \to -R_1 + R_3$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 2 & 1 & 3 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

$-R_1 + R_3$
$-2 + 2 = 0$
$-1 + 3 = 2$
$-3 + 4 = 1$
$-4 + 7 = 3$

$R_4 \to -R_1 + R_4$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

$-R_1 + R_4$
$-2 + 2 = 0$
$-1 + 3 = 2$
$-3 + 1 = -2$
$-4 + 4 = 0$

$R_3 \to R_2 + R_3$, $R_4 \to R_2 + R_4$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & -6 & -7 \end{bmatrix}$$

$R_2 + R_3$
$0 + 0 = 0$
$-2 + 2 = 0$
$-4 + 1 = -3$
$-7 + 3 = -4$

$R_4 \to R_2 + R_4$
$-2 + 2 = 0$
$-4 - 2 = -6$
$-7 + 0 = -7$

$R_4 \to -2R_3 + R_4$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$-2(-3) = 6$
$+6 - 6 = 0$
$-2(-4) + 7$
$+8 - 7 = 1$

$R_1 \to \frac{1}{2} R_1$, $R_2 = -\frac{1}{2} R_2$, $R_3 = -\frac{1}{3} R_3$

$$A \sim \begin{bmatrix} 1 & 1/2 & 3/2 & 2 \\ 0 & 1 & 2 & 7/2 \\ 0 & 0 & 1 & 4/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All the four rows are non zero in the row echelon form A

Thus $\rho(A) = 4$.

5) $$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

June 2018

Sol $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$-3R_1 + R_3$
$-3(1) + 3 = +0$
$-3(0) + 1 = 1$
$-3(1) + 0 = -3$
$-3(1) + 2 = -1$

$R_3 \to -3R_1 + R_3$, $R_4 \to -R_1 + R_4$

$R_4 \to -R_1 + R_4$
$-1+1 = 0$
$-0+1 = 1$
$-1-2 = -3$
$-1+0 = -1$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$
$-0+0 = 0$
$-1+1 = 0$
$+3-3 = 0$
$+1-1 = 0$

$R_3 \to -R_2 + R_3$, $R_4 \to -R_2 + R_4$.

$R_4 \to -R_2 + R_4$
$0+0 = 0$
$-1+1 = 0$
$+3-3 = 0$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the matrix A in the row echelon form has two non zero rows thus $\rho(A) = 2$

i) $A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

$+2R_1 + R_2$
$+2(1) - 2 = 0$
$2(2) - 1 = 3$
$2(3) - 3 = 3$
$2(1) - 1 = 1$

$-R_1 + R_3$
$-1 + 1 = 0$
$-2 + 0 = -2$
$-3 + 1 = -2$
$-1 + 1 = 0$

$R_2 \to 2R_1 + R_2$, $R_3 \to -R_1 + R_3$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

$R_2 \leftrightarrow R_4$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 3 & 3 & 1 \end{bmatrix}$

$+2R_2 + R_3$
$+2(0) + 0 = 0$
$+2(1) - 2 = 0$
$2(1) - 2 = 0$
$2(1) + 0 = 2$

$-3R_2 + R_4$
$-3(0) + 0 = 0$
$-3(1) + 3 = 0$
$-3(1) + 3 = 0$
$-3(1) + 1 = -2$

$R_3 \to 2R_2 + R_3$, $R_4 \to -3R_2 + R_4$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

$\to R_3 + R_4$
$0 + 0 = 0$
$0 + 0 = 0$
$0 + 0 = 0$
$2 - 2 = 0$

$R_4 \to R_3 + R_4$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A matrix A in the row echelon form has threes non zero rows.

$\rho(A) = 3$

Consistency of a system of linear Equations

Working procedure for problems:

Step 1: We first form the augmented matrix $[A:B]$ and we can clearly identify the portions of the coefficient matrix A in it.

Step 2: we reduce the matrix $[A:B]$ to an echelon form by elementary row transformations.

Step 3: The echelon form of $[A:B]$ is converted back to the equation form and the solution will emerge easily.

Problem:

1) Test for Consistency and solve.

$$x+2y+3z=14$$
$$4x+5y+7z=35$$
$$3x+3y+4z=21$$

Soln: $[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 4 & 5 & 7 & : & 35 \\ 3 & 3 & 4 & : & 21 \end{bmatrix}$ is the augmented matrix

$R_2 \to -4R_1+R_2$; $R_3 \to -3R_1+R_3$

$-4R_1+R_2$
$-4(1)+4=0$
$-4(2)+(5)=-3$
$-4(3)+7=-5$
$-4(14)+35=-21$

$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -21 \\ 0 & -3 & -5 & : & -21 \end{bmatrix}$

$R_3 \to -R_2+R_3$

$-R_2+R_3$
$-0+0=0$
$3-3=0$
$5-5=0$

$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & : & 14 \\ 0 & -3 & -5 & : & -21 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$

we have $\rho[A] = 2$, $\rho[A;B] = 2$, That is $r = 2$ also $n = 3$

since $\rho[A] = \rho[A:B] = 2 < 3$ $(r < n)$

the system is Consistent and will have infinite solutions.

Here $n - r = 3 - 2 = 1$

and hence one of the variables can take arbitrary values.

we now have , $x + 2y + 3z = 14 \longrightarrow ①$

$-3y - 5z = -21 \longrightarrow ②$

Let $z = k$ be arbitrary in eq. ②

$-3y - 5k = -21$

$-3y = -21 + 5k$

$3y = 21 - 5k$

$y = \frac{21 - 5k}{3}$

$y = 7 - \frac{5k}{3}$

Thus From ①.

$x + 2y + 3z = 14$

$x + 2\left(7 - \frac{5k}{3}\right) + 3k = 14$

$x + 14 - \frac{10k}{3} + 3k = 14$

$x + 3k - \frac{10k}{3} = 14 - 14$

$x + \frac{3k}{1} - \frac{10k}{3} = 0$

$x + \frac{9k - 10k}{3} = 0$

$x - \frac{k}{3} = 0$

$x = \frac{k}{3}$, $y = 7 - \frac{5k}{3}$, $z = k$

represent infinite solutions, since k is arbitrary

2) Test for Consistency and solve

$x - 4y + 7z = 14$

$3x + 8y - 2z = 13$

$7x - 8y + 26z = 5$

Sol: $[A:B] = \left[\begin{array}{ccc:c} 1 & -4 & 7 & 14 \\ 3 & 8 & -2 & 13 \\ 7 & -8 & 26 & 5 \end{array}\right]$ is the augmented matrix

$R_2 \to -3R_1 + R_2$, $R_3 \to -7R_1 + R_3$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 20 & -23 & -93 \end{array}\right]$

$R_3 \to -R_2 + R_3$

$[A:B] \sim \left[\begin{array}{ccc:c} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 0 & 0 & -64 \end{array}\right]$

$\rho[A] = 2 \quad \rho[A:B] = 3$

Since $\rho[A] \neq \rho[A:B]$ the system is inconsistent (does not posses any solution).

3) Test for Consistency and solve

$5x_1 + x_2 + 3x_3 = 20$

$2x_1 + 5x_2 + 2x_3 = 18$

$3x_1 + 2x_2 + x_3 = 14$

Sol: $[A:B] = \left[\begin{array}{ccc:c} 5 & 1 & 3 & 20 \\ 2 & 5 & 2 & 18 \\ 3 & 2 & 1 & 14 \end{array}\right]$

is the augmented matrix

$R_2 \to -2R_1 + 5R_2$, $R_3 \to -3R_1 + 5R_3$

$$[A:B] \sim \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 7 & -4 & : & 10 \end{bmatrix}$$

$R_3 \to -7R_2 + 23R_3$

$$[A:B] \sim \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 0 & -120 & : & -120 \end{bmatrix}$$

we have $\rho[A] = 3$, $\rho[A:B] = 3$, That is $r = 3$

also $n = 3$.

Since $\rho[A] = \rho[A:B] = 3 \; (r = n = 3)$ the system is Consistent and will have unique solution

$5x_1 + x_2 + 3x_3 = 20 \to (1)$

$23x_2 + 4x_3 = 50 \to (2)$

$-120x_3 = -120 \to (3)$

From eq (3)

$-120x_3 = -120$

$\boxed{x_3 = 1}$

From eq (2)

$23x_2 + 4x_3 = 50$

$23x_2 + 4(1) = 50$

$23x_2 = 50 - 4$

$23x_2 = 46$

$x_2 = \frac{46}{23}$

$\underline{\underline{x_2 = 2}}$

From eq (1)

$5x_1 + x_2 + 3x_3 = 20$

$5x_1 + 2 + 3 = 20$

$5x_1 + 5 = 20$

$5x_1 = 20 - 5$

$5x_1 = 15$

$x_1 = \frac{15}{5}$

$\underline{\underline{x_1 = 3}}$

$x_1 = 3$, $x_2 = 2$, $x_3 = 1$ is unique solution

1) Investigate the values of λ and μ such that the system of equations **Imp**

$x+y+z=6$

$x+2y+3z=10$

$x+2y+\lambda z=\mu$, may have.

(a) Unique solution (b) Infinite solution (c) No solution

Sol: $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$ is the augmented matrix

$R_2 \to -R_1+R_2$, $R_3 \to -R_1+R_3$

$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix}$

$R_3 \to -R_2+R_3$

$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$

$\rho(A) = \rho(A:B) \neq \rho(A) \neq \rho(A:B)$

$2 \leq 3$ $3 = 3$

a) Unique solution ; We must have $\rho[A] = \rho[A:B] = 3$

If $(\lambda-3) \neq 0$ ~~since the other two entries in the last row of A are zero~~

If $(\lambda-3) \neq 0$ or $\lambda \neq 3$ irrespective the value of μ $\rho[A:B]$ will also be 3

(b) Infinite Solution : Here we have $n=3$

$\rho[A] = \rho[A:B] = 2$ only when the last row ~~and~~ of $[A:B]$ is completely zero.

this is possible if $\lambda-3=0$, $\mu-10=0$.

Hence the system has infinite Solution.
$\lambda = 3$ and $\mu = 10$.

No Solution: we must have $\rho[A] \neq \rho[A:B]$

•) $\rho[A] = 3$ if $\lambda \neq 3$ and hence if $\lambda = 3$ we obtain $\rho[A] = 2$.

If we take $(\mu - 10) \neq 0$ the $\rho[A:B]$ will be 3

Hence the system has no solution is
$\lambda = 3$ and $\mu \neq 10$.

3) Find for what value of K the system of equations posses a solution, solve Completely in each case.

$$x + y + z = 1$$
$$x + 2y + 4z = K$$
$$x + 4y + 10z = K^2$$

Sol: $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & K \\ 1 & 4 & 10 & : & K^2 \end{bmatrix}$ is the augumented matrix.

$R_2 \to -R_1 + R_2$, $R_3 \to -R_1 + R_3$

$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & K-1 \\ 0 & 3 & 9 & : & K^2-1 \end{bmatrix}$

$R_3 \to -3R_2 + R_3$

$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & K-1 \\ 0 & 0 & 0 & : & K^2-3K+2 \end{bmatrix}$

$-3(K-1) + K^2 - 1$
$-3K + 3 + K^2 - 1$
$K^2 - 3K + 2$

$\rho[A] = 2$ and for the system to be Consistent we must have $\rho[A:B]$ also 2.

This is possible if $K^2 - 3K + 2 = 0$, $K = 1$; $K = 2$.

Hence we Conclude that the system possesses a solution if $k=1,2$.

Since $P[A] = [A:B] = 2 < 3$ for the Cases $K=1, 2$ the system will have the infinite solution and the same are as follows.

Case 1: $K=1$. The system of equation are

$$x+y+z=1 \rightarrow ①$$
$$y+3z=0 \rightarrow ②$$

Let $z = k_1$ be arbitrary.

From ② $y+3z=0$

$y+3k_1=0$

$y=-3k_1$

From ① $x+y+z=1$

$x-3k_1+k_1=1$

$x-2k_1=1$

$x=1+2k_1$

Case 2: If $k=2$.

The system of equation are,

$$x+y+z=1 \rightarrow ①$$
$$y+3z=1 \rightarrow ③$$

Let $z=k_2$ be arbitrary

From ③ $y+3z=1$

$y+3k_2=1$

$y=1-3k_2$

From ① $x+y+z=1$

$x+(1-3k_2)+k_2=1$

$x+1-3k_2+k_2=1$

$\cancel{x+1=1}$ $x=2k_2$

Thus $x=1+2k_1$, $y=-3k_1$, $z=k_1$ and

$x=2k_2$, $y=1-3k_2$, $z=k_2$

Eigen Values and Eigen Vectors - Rayle

Solution of a system of linear equations:

→ Gauss elimination method

Problems:

(1) Solve the following systems of equations by Gauss elimination method

Dec 2017

1) $x+y+z=9$

$x-2y+3z=8$

$2x+y-z=3$

Soln: The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 1 & -2 & 3 & 8 \\ 2 & 1 & -1 & 3 \end{array}\right]$$

$R_2 \to -R_1 + R_2$, $R_3 \to -2R_1 + R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{array}\right]$$

$R_3 \to R_2 + (-3) R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & 11 & 44 \end{array}\right]$$

Hence we have $x+y+z=9 \to (1)$

$-3y+2z=-1 \to (2)$

$11z = 44. \to (3)$

$\underline{z = 4}$

From eq ②

$-3y+2z=-1$

$-3y+2(4)=-1$

$-3y+8=-1$

$-3y=-1-8$

$3y=9$

$\underline{\underline{y=3}}$

From eq ①

$x+y+z=9$

$x+3+4=9$

$x+7=9$

$x=9-7$

$\underline{\underline{x=2}}$

Thus $x=2$, $y=3$, $z=4$ is the required solution

2) Solve by Gauss elimination method — June 2018

$2x+y+4z=12$

$4x+11y-z=33$

$8x-3y+2z=20$

Sol: The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc:c} 2 & 1 & 4 & 12 \\ 4 & 11 & -1 & 33 \\ 8 & -3 & 2 & 20 \end{array}\right]$$

$R_2 \to -2R_1+R_2$, $R_3 \to -4R_1+R_3$

$$[A:B]\sim\left[\begin{array}{ccc:c} 2 & 1 & 4 & 12 \\ 0 & 9 & -9 & 9 \\ 0 & -7 & -14 & -28 \end{array}\right]$$

$R_2 \to \frac{1}{9}R_2$, $R_3 \to \frac{-1}{7}R_3$

$$[A:B]\sim\left[\begin{array}{ccc:c} 2 & 1 & 4 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

$R_3 \to -R_2+R_3$

$$A:B \sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 1 & -1 & : & 1 \\ 0 & 0 & 3 & : & 3 \end{bmatrix}$$

Hence we have.

$$2x + y + 4z = 12 \longrightarrow ①$$
$$y - z = 1 \longrightarrow ②$$
$$3z = 3 \longrightarrow ③$$

From ③

$3z = 3$

$z = 1$

From ②

$y - z = 1$

$y - 1 = 1$

$y = 2$

$2x + y + 4z = 12$

$2x + 2 + 4(1) = 12$

$2x + 2 + 4 = 12$

$2x + 6 = 12$

$2x = 6$

$x = 3$

Thus $x = 3$, $y = 2$, $z = 1$ is the required solution.

3) Solve by Gauss elimination method

$$x_1 - 2x_2 + 3x_3 = 2$$
$$3x_1 - x_2 + 4x_3 = 4$$
$$2x_1 + x_2 - 2x_3 = 5$$

Sol: The augmented matrix of the system is

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & : & 2 \\ 3 & -1 & 4 & : & 4 \\ 2 & 1 & -2 & : & 5 \end{bmatrix}.$$

$R_2 \to -3R_1 + R_2$, $R_3 \to -2R_1 + R_3$.

$$[A:B] \sim \begin{bmatrix} 1 & -2 & 3 & : & 2 \\ 0 & 5 & -5 & : & -2 \\ 0 & 5 & -8 & : & 1 \end{bmatrix}$$

$R_3 \to -R_2 + R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & -2 & 3 & 2 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & -3 & 3 \end{array}\right]$$

$$x_1 - 2x_2 + 3x_3 = 2 \longrightarrow ①$$
$$5x_2 - 5x_3 = -2 \longrightarrow ②$$
$$-3x_3 = 3 \longrightarrow ③$$

$\therefore x_3 = -1$

$5x_2 - 5x_3 = -2$

$5x_2 - 5(-1) = -2$

$5x_2 + 5 = -2$

$5x_2 = -2 - 5$

$x_2 = \frac{-7}{5} = -1.4$

From eq ①

$x_1 - 2x_2 + 3x_3 = 2$

$x_1 - 2(-1.4) + 3(-1) = 2$

$x_1 + 2.8 - 3 = 2$

$x_1 - 0.2 = 2$

$x_1 = 2 + 0.2$

$x_1 = 2.2$

Thus $x_1 = 2.2$, $x_2 = -1.4$, $x_3 = -1$ is the required solution.

4) Apply Gauss elimination method to solve the system of equation.

$2x + 5y + 7z = 52$

$2x + y - z = 0$

$x + y + z = 9$

Sol: $[A:B] = \left[\begin{array}{ccc:c} 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \\ 1 & 1 & 1 & 9 \end{array}\right]$

The augmented matrix of the system is

As it is convinent to have the leading Coffient as 1 we shall interchange the first and third equation. The augmented matrix will be.

$R_1 \leftrightarrow R_3$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$R_2 \to -2R_1 + R_2$, $R_3 \to -2R_1 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$$

$R_3 \to 3R_2 + R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$$

$x + y + z = 9 \to (1)$

$-y - 3z = -18 \to (2)$

$-4z = -20 \to (3)$

From eq (3)

$-4z = -20$

$\underline{\underline{z = 5}}$

From eq (2).

$-y - 3z = -18$

$-y - 3(5) = -18$

$-y - 15 = -18$

$-y = -18 + 15$

$-y = -3$

$\underline{\underline{y = 3}}$

From eq (1)

$x + y + z = 9$

$x + 3 + 5 = 9$

$x + 8 = 9$

$x = 9 - 8$

$\underline{\underline{x = 1}}$

Thus $x = 1$, $y = 3$, $z = 5$ is the required solution

Gauss Jordan elimination method

Problems:

1) Solve the following system of equations by Gauss Jordan method. Dec 2017

$$x + y + z = 9$$
$$x - 2y + 3z = 8$$
$$2x + y - z = 3$$

Sol: The augumented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 1 & -2 & 3 & 8 \\ 2 & 1 & -1 & 3 \end{array}\right]$$

$R_2 \to -R_1 + R_2, \quad R_3 \to -2R_1 + R_3.$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{array}\right]$$

$R_3 \to R_2 - 3R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & 11 & 44 \end{array}\right]$$

$R_1 \to 3R_1 + R_2$

$$\left[\begin{array}{ccc:c} 3 & 0 & 5 & 26 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{array}\right]$$

∴ ~~Hence we have~~

$R_3 \to 3R_3 - R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 3 & 0 & 5 & 26 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & -11 & -44 \end{array}\right]$$

$R_3 \to -\frac{1}{11} R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 3 & 0 & 5 & 26 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$R_1 \to R_1 - 5R_2$

$$[A:B] \sim \begin{bmatrix} 3 & 0 & 0 & : & 6 \\ 0 & -3 & 2 & : & -1 \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_3$

$$[A:B] \sim \begin{bmatrix} 3 & 0 & 0 & : & 6 \\ 0 & -3 & 0 & : & -9 \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$$

$3x = 6$ $\quad -3y = -9 \quad z = 4$

$x = 2$ $\quad y = 3$

2). $2x + 5y + 7z = 52$

$2x + y - z = 0$

$x + y + z = 9.$

As it is convenient to have leading coefficient as 1 we shall interchange first and third eq.

$$[A:B] = \begin{bmatrix} 2 & 5 & 7 & : & 52 \\ 2 & 1 & -1 & : & 0 \\ 1 & 1 & 1 & : & 9 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 2 & 1 & -1 & : & 0 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 2 & 5 & 7 & : & 52 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$$

$R_1 \to R_2 + R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 3 & 5 & : & 34 \end{bmatrix}$$

$R_3 \to 3R_2 + R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & -2 & -9 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & -4 & -20 \end{array}\right]$$

$R_3 \to -\frac{1}{4} R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & -2 & -9 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$R_1 \to 2R_3 + R_1$, $R_2 \to 3R_3 + R_2$

3(5) - 18
15 - 18

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$x = 1$

$-y = -3 \Rightarrow y = 3$

$z = 5$

is the required solution

3) $2x_1 + x_2 + 3x_3 = 1$

$4x_1 + 4x_2 + 7x_3 = 1$

$2x_1 + 5x_2 + 9x_3 = 3$

Soln $[A:B] = \left[\begin{array}{ccc:c} 2 & 1 & 3 & 1 \\ 4 & 4 & 7 & 1 \\ 2 & 5 & 9 & 3 \end{array}\right]$

$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - R_1$

$$[A:B] \sim \left[\begin{array}{ccc:c} 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 4 & 6 & 2 \end{array}\right]$$

$R_1 \to 2R_1 - R_2$ $R_3 \to R_3 - 2R_2$

$$[A:B] \sim \begin{bmatrix} 4 & 0 & 5 & : & 3 \\ 0 & 2 & 1 & : & -1 \\ 0 & 0 & 4 & : & 4 \end{bmatrix}$$

$R_3 \to \frac{1}{4} R_3$

$$[A:B] \sim \begin{bmatrix} 4 & 0 & 5 & : & 3 \\ 0 & 2 & 1 & : & -1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_3$, $R_1 \to R_1 - 5R_3$

$$[A:B] \sim \begin{bmatrix} 4 & 0 & 0 & : & -2 \\ 0 & 2 & 0 & : & -2 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$4x_1 = -2$

$x_1 = \frac{-2}{4}$

$\underline{\underline{x_1 = -0.5}}$

$2x_2 = -2$

$x_2 = \frac{-2}{2}$

$\underline{\underline{x_2 = -1}}$

$\underline{\underline{x_3 = 1}}$

Gauss - Seidel method

1) Solve the system of equations by Gauss - Seidel method.

June 2016.

$$10x + y + z = 12$$
$$x + 10y + z = 12$$
$$x + y + 10z = 12$$

⇒ The given system of equations are diagonally dominant and the equations are put in the following

$$x = \frac{1}{10}[12 - y - z] \rightarrow ①$$
$$y = \frac{1}{10}[12 - x - z] \rightarrow ②$$
$$z = \frac{1}{10}[12 - x - y] \rightarrow ③.$$

Let us take $x = 0, y = 0, z = 0$

First iteration:

$$x^{(1)} = \frac{1}{10}[12 - 0 - 0] = 1.2$$
$$y^{(1)} = \frac{1}{10}[12 - 1.2 - 0] = 1.08$$
$$z^{(1)} = \frac{1}{10}[12 - 1.2 - 1.08] = 0.972$$

Second iteration:

$$x^{(2)} = \frac{1}{10}[12 - 1.08 - 0.972] = 0.9948$$
$$y^{(2)} = \frac{1}{10}[12 - 0.9948 - 0.972] = 1.00332$$
$$z^{(2)} = \frac{1}{10}[12 - 0.9948 - 1.00332] = 1.000188$$

Third iteration:

$$x^{(3)} = \frac{1}{10}[12 - 1.00332 - 1.000188] = 0.99965$$
$$y^{(3)} = \frac{1}{10}[12 - 0.99965 - 1.000188] = 1.00002$$
$$z^{(3)} = \frac{1}{10}[12 - 0.99965 - 1.00002] = 1.00008$$

Fourth iteration

$$x^{(4)} = \frac{1}{10}[12 - 1.00002 - 1.00003] = 0.999995 \simeq 1$$

$$y^{(4)} = \frac{1}{10}[12 - 1 - 1.00003] = 0.999997 \simeq 1$$

$$z^{(4)} = \frac{1}{10}[12 - 1 - 1] = 1$$

Thus $x = 1,\ y = 1,\ z = 1$

2) Solve the system of equation by Gauss – Seidel method? Dec 2015, 16, 17, 18.

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

The equations are diagonally dominant and hence we first write them in the following form.

$$x = \frac{1}{20}[17 - y + 2z]$$

$$y = \frac{1}{20}[-18 - 3x + z]$$

$$z = \frac{1}{20}[25 - 2x + 3y]$$

we start the trial solution $x = 0, y = 0, z = 0$

First iteration :

$$x^{(1)} = \frac{1}{20}[17 - 0 + 2(0)] = 0.85$$

$$y^{(1)} = \frac{1}{20}[-18 - 3(0.85) + 0] = -1.0275$$

$$z^{(1)} = \frac{1}{20}[25 - 2(0.85) + 3(-1.0275)] = 1.0109$$

Second iteration :

$$x^{(2)} = \frac{1}{20}[17 - (-1.0275) + 2(1.0109)] = 1.0025$$

$$y^{(2)} = \frac{1}{20}[-18 - 3(1.0025) + 1.0109] = -0.9998$$

$$z^{(2)} = \frac{1}{20}[25 - 2(1.0025) + 3(-0.9998)] = 0.9998$$

Third iteration :

$x^{(3)} = \frac{1}{20}[17-(-0.9998)+2(0.9998)] = 0.99997 \approx 1$

$y^{(3)} = \frac{1}{20}[-18-3(0.9997)+0.9998] = -1.0000055 \approx -1$

$z^{(3)} = \frac{1}{20}[25-2(0.99997)+3(-1.0000055)] = 1.0000022 \approx 1$

Thus $x=1, y=-1, z=1$ is the required solution

3) $x+y+54z=110$ Solve the gauss seidel method to
$27x+6y-z=85$ obtain the final solution to
$6x+15y+2z=72$ Correct 3 decimal place ?

⇒ The first equation should require x greater than y and z cofficient.

The second equation should require y greater than x and z Cofficient.

$27x+6y-z=85$

$6x+15y+2z=72$

$x+y+54z=110$

$x = \frac{1}{27}[85-6y+z]$

$y = \frac{1}{15}[72-6x-2z]$

$z = \frac{1}{54}[110-x-y]$

Let we start with the itial solution $x=0, y=0, z=0$

$x^{(1)} = \frac{1}{27}[85-6(0)+0] = 3.14815$

$y^{(1)} = \frac{1}{15}[72-6(3.14815)-0] = 3.54074$

$z^{(1)} = \frac{1}{54}[110-3.14815-3.54074] = 1.91317$

Second iteration

$$x^{(2)} = \frac{1}{27}[85 - 6(3.54074) + 1.91317] = 2.43218$$

$$y^{(2)} = \frac{1}{15}[72 - 6(2.43218) - 2(1.91317)] = 3.57204$$

$$z^{(2)} = \frac{1}{54}[110 - 2.43218 - 3.57204] = 1.92585$$

Third iteration

$$x^{(3)} = \frac{1}{27}[85 - 6(3.57204) + 1.92585] = 2.42569$$

$$y^{(3)} = \frac{1}{15}[72 - 6(2.42569) - 2(1.92585)] = 3.57294$$

$$z^{(3)} = \frac{1}{54}[110 - 2.42569 - 3.57294] = 1.92595$$

Fourth iteration :

$$x^{(4)} = \frac{1}{27}[85 - 6(3.57294) + 1.92595] = 2.42549$$

$$y^{(4)} = \frac{1}{15}[72 - 6(2.42549) - 2(1.92595)] = 3.57301$$

$$z^{(4)} = \frac{1}{54}[110 - 2.42549 - 3.57301] = 1.92595$$

It may be observed that the solution in the third and fourth iteration when approximated to three place of decimal is the same

$x = 2.426$, $y = 3.573$, $z = 1.926$.

4) $5x + 2y + z = 12$

$x + 4y + 2z = 15$

$x + 2y + 5z = 20$

Carry out 4 iteration taking the inital approximation to the solution as $(1, 0, 3)$.

=> The given system of equation are diagonally dominant and we put them in the following

$$x = \frac{1}{5}[12 - 2y - z]$$

$$y = \frac{1}{4}[15 - x - 2z]$$

$$z = \frac{1}{5}[20 - x - 2y]$$

By data $x = 1$, $y = 0$, $z = 3$

First iteration:

$$x^{(1)} = \frac{1}{5}[12 - 2(0) - 3] = 1.8$$

$$y^{(1)} = \frac{1}{4}[15 - 1 - 2(3)] = 1.8$$

$$z^{(1)} = \frac{1}{5}[20 - 1.8 - 2(1.8)] = 2.92.$$

Second iteration:

$$x^{(2)} = \frac{1}{5}[12 - 2(1.8) - 2.92] = 1.096$$

$$y^{(2)} = \frac{1}{4}[15 - 1.096 - 2(2.92)] = 2.016$$

$$z^{(2)} = \frac{1}{5}[20 - 1.096 - 2(2.016)] = 2.9744$$

Third iteration:

$$x^{(3)} = \frac{1}{5}[12 - 2(2.016) - 2.9744] = 0.99872$$

$$y^{(3)} = \frac{1}{4}[15 - 0.99872 - 2(2.9744)] = 2.01312$$

$$z^{(3)} = \frac{1}{5}[20 - 0.99872 - 2(2.01312)] = 2.995$$

Fourth iteration method

$$x^{(4)} = \frac{1}{5}\left[12-2(2.01312) - 2.995\right] = 0.995852$$

$$y^{(4)} = \frac{1}{4}\left[15 - 0.995752 - 2(2.995)\right] = 2.003562$$

$$z^{(4)} = \frac{1}{5}\left[20 - 0.995752 - 2(2.003562)\right] = 2.9994248$$

Thus the solution after four iteration Correct to four decimal place is given by

$$x = 0.9958, \quad y = 2.0036, \quad z = 2.9994$$

Rayleigh's power method

This is an iterative method to determine the numerically largest eigen value (dominant eigen value) and the corresponding eign vector of a square matrix

Flow chart for solving problem

Step 1: Suppose A is the given square matrix, we assume initally an eigen vector (column matrix) X_0 in a simple form like $[1,0,0]'$ or $[0,1,0]'$ or $[0,0,1]'$ or $[1,1,1]'$ and find the matrix product AX_0 which will also be a column matrix.

Step 2: We take out the largest element as the common factor (this technique is called normalization). to obtain $AX_0 = \lambda^{(1)} X^{(1)}$.

Step 3: We then compute $AX^{(1)}$ and gain put in the form $AX^{(1)} = \lambda^{(2)} X^{(2)}$ by normalization.

Step 4: This iterative process is continued till two consecutive iterative values of λ and X are same upto a desired degree of accuracy.

Step 5: The value so obtained are respectively the largest eigen value (λ) and the corresponding eigen vector (X) of the given square matrix

1) Find the largest eigen value and the corresponding eigen vector of the matrix A by the power method given that. June 2016, 18

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Soln: Let $X^{(0)} = [1, 0, 0]'$ be the initial eigen vector

$$AX^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2+0+0.5 \\ 0+0+0 \\ 1+0+1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2+0+0.8 \\ 0+0+0 \\ 1+0+1.6 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$A.X^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \begin{bmatrix} 2+0+0.93 \\ 0+0+0 \\ 1+0+1.86 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 0 \\ 2.86 \end{bmatrix} = 2.93\begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 2+0+0.98 \\ 0+0+0 \\ 1+0+1.96 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 0 \\ 2.96 \end{bmatrix} = 2.98\begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \begin{bmatrix} 2+0+0.99 \\ 0+0+0 \\ 1+0+1.98 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 0 \\ 2.98 \end{bmatrix} = 2.99\begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.997 \end{bmatrix} = \begin{bmatrix} 2+0+0.997 \\ 0+0+0 \\ 1+0+1.94 \end{bmatrix} = \begin{bmatrix} 2.997 \\ 0 \\ 2.94 \end{bmatrix} = 2.997\begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

2) Find the dominant eigen value and the corresponding eigen vector of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$ by power method taking the inital eigen vector as $[1,1,1]'$.

June Dec 2017

Soln: By data $X^{(0)} = [1, 1, 1]'$ is the inital eigen vector.

$$AX^{(0)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 7.34 \\ -2.67 \\ 4.01 \end{bmatrix} = 7.34\begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.36 \\ 0.55 \end{bmatrix} = \begin{bmatrix} 7.82 \\ -3.63 \\ 4.01 \end{bmatrix} = 7.82\begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.46 \\ 0.51 \end{bmatrix} = \begin{bmatrix} 7.94 \\ -3.89 \\ 3.99 \end{bmatrix} = 7.94\begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.49 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 7.98 \\ -3.97 \\ 3.99 \end{bmatrix} = 7.98\begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 4 \end{bmatrix} = 8\begin{bmatrix} 1 \\ -0.5 \\ 0.5 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

Thus the dominant eigen value is 8 and the corresponding eigen vector is $[1, -0.5, 0.5]'$.

3) Using Rayleigh's power method find numerically the largest eigen value and the corresponding eigen vector of the matrix

$$A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

Sol: Let $x^{(0)} = [1, 0, 0]'$ be the inital eigen vector

$$Ax^{(0)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$Ax^{(1)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = 25.2\begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.12 \\ 1.72 \end{bmatrix} = 25.18\begin{bmatrix} 1 \\ 0.04 \\ 0.07 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

we observe that $x^{(2)} = x^{(3)}$

Thus the numerically largest eigen value of A is 25.18 and the corresponding eigen vector is $[1, 0.04, 0.07]'$

4) Find the largest eig en value and the corresponding eigen vector of the matrix A by using the power method by taking inital vector as $[1, 1, 1]^T$ Dec 2015, June 2018

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$\Rightarrow$ By data $x^{(0)} = [1, 1, 1]'$ is the inital eigen vector

$$Ax^{(0)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$Ax^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = 4\begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -3.5 \\ 2.5 \end{bmatrix} = 3.5 \begin{bmatrix} 0.71 \\ -1 \\ 0.71 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.71 \\ -1 \\ 0.71 \end{bmatrix} = \begin{bmatrix} 2.42 \\ -3.42 \\ 2.42 \end{bmatrix} = 3.42 \begin{bmatrix} 0.708 \\ -1 \\ 0.708 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.708 \\ -1 \\ 0.708 \end{bmatrix} = \begin{bmatrix} 2.416 \\ -3.416 \\ 2.416 \end{bmatrix} = 3.416 \begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.7073 \\ -1 \\ 0.7073 \end{bmatrix} = \begin{bmatrix} 2.4146 \\ -3.4146 \\ 2.4146 \end{bmatrix} = 3.4146 \begin{bmatrix} 0.7071 \\ -1 \\ 0.7071 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

Hence the largest eigen value is 3.4146 and the corresponding eigen vector is $[0.7071, -1, -0.7071]'$.

5) Find the numerically largest eigen value of ~~A is 25.18~~ and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ by taking the initial approximation to eigen vector as $[1, 0.8, -0.8]^T$. Perform 5 iteration

<u>Sol</u>: By data $X^{(0)} = [1, 0.8, -0.8]^T$ is the initial eigen vector

$$AX^{(0)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 5.2 \\ -5.2 \end{bmatrix} = 5.6 \begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \begin{bmatrix} 5.86 \\ 5.72 \\ -5.72 \end{bmatrix} = 5.86 \begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \begin{bmatrix} 5.96 \\ 5.92 \\ -5.92 \end{bmatrix} = 5.96 \begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 5.98 \\ 5.98 \\ -5.98 \end{bmatrix} \simeq 5.98 \begin{bmatrix} 1 \\ 0.997 \\ -0.997 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix}4 & 1 & -1\\ 2 & 3 & -1\\ -2 & 1 & 5\end{bmatrix}\begin{bmatrix}1\\ 0.997\\ -0.997\end{bmatrix} = \begin{bmatrix}5.994\\ 5.988\\ -5.988\end{bmatrix} = 5.994\begin{bmatrix}1\\ 0.999\\ -0.999\end{bmatrix} = \lambda^{(5)} X^{(5)}$$

Thus after 5 iterations the numerically largest eigen value is 5.994 and the corresponding eigen vector is $[1, 0.999, -0.999]^T$

Question paper Problems:

June 2011

1) Find the rank of the matrix $A = \begin{bmatrix}1 & 2 & 4 & 3\\ 2 & 4 & 6 & 8\\ 4 & 8 & 12 & 16\\ 1 & 2 & 3 & 4\end{bmatrix}$

Sol: $R_2 \to R_2 - 2R_1$, $R_3 \to -4R_1 + R_3$

$R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix}1 & 2 & 4 & 3\\ 0 & 0 & -2 & 2\\ 0 & 0 & -4 & 4\\ 0 & 0 & -1 & 1\end{bmatrix}$$

$R_3 \to R_3 - 2R_2$, $R_4 \to$

$$A \sim \begin{bmatrix}1 & 2 & 4 & 3\\ 0 & 0 & -2 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 1\end{bmatrix}$$

$R_2 \to -\frac{1}{2} R_2$

$$A \sim \begin{bmatrix}1 & 2 & 4 & 3\\ 0 & 0 & +1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 1\end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix}1 & 2 & 4 & 3\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & -2 & 0\end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A \sim \begin{bmatrix}1 & 2 & 4 & 3\\ 0 & 0 & -2 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 1\end{bmatrix}$$

$R_4 \to 2R_4 - R_2$

$$A \sim \begin{bmatrix}1 & 2 & 4 & 3\\ 0 & 0 & -2 & 2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$R_2 \to \frac{-1}{2} R_2$

$$A \sim \begin{bmatrix}1 & 2 & 4 & 3\\ 0 & 0 & +1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Row echelon form has two non zero rows

$\rho(A) = 2$

2) Find the rank of the matrix $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Dec 2017

Sol: $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 4R_1$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 - 3R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to -\frac{1}{5} R_2$.

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 1 & 7/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are two non zero rows in the row echelon form of A

Thus $\rho(A) = 2$

M.Q

3) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - \overset{3R_1}{\cancel{4R_2}}$, $R_4 \to R_4 - 6R_1$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 1 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 6 & 12 & 17 \end{bmatrix}.$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 1 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 6 & 12 & 17 \end{bmatrix}$$

$R_3 \to 5R_3 - 4R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 1 & 7 \\ 0 & 0 & 44 & 22 \\ 0 & 6 & 12 & 17 \end{bmatrix}$$

$R_3 \to \frac{1}{22} R_3$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 1 & 7 \\ 0 & 0 & 22 & 1 \\ 0 & 6 & 12 & 17 \end{bmatrix}$$

$R_4 \to 5R_4 - 6R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 1 & 7 \\ 0 & 0 & 22 & 1 \\ 0 & 0 & 54 & 43 \end{bmatrix}$$

Gauss elimination method.

June 2017.

(1) $x+2y+z=3$

$2x+3y+2z=5$

$3x-5y+5z=2$

Soln: The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 \\ 3 & -5 & 5 & 2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -11 & 2 & -7 \end{array}\right]$$

$R_3 \to -11R_2 + R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$x+2y+z=3 \to (1)$

$-1y=-1 \to (2)$

$2z=4 \to (3)$

$2z=4$

$\underline{\underline{z=2}}$

$-1y=-1$

$\underline{\underline{y=1}}$

$x+2y+z=3$

$x+2+2=3$

$x+4=3$

$\underline{\underline{x=-1}}$

Thus $x=-1$, $y=1$, $z=2$ is the required solution

(2) $3x+y+2z=3$

$2x-3y-z=-3$

$x+2y+z=4$

Soln: The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 3 & 1 & 2 & 3 \\ 2 & -1 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 2 & -1 & -1 & : & -3 \\ 3 & 1 & 2 & : & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 0 & -7 & -3 & : & -11 \\ 0 & -5 & -1 & : & -9 \end{bmatrix}$$

$R_3 \to 7R_3 - 5R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 0 & -7 & -3 & : & -11 \\ 0 & 0 & 8 & : & -8 \end{bmatrix}$$

$$x + 2y + z = 4$$
$$-7y - 3z = -11$$
$$8z = -8$$

$8z = -8$

$z = -1$

$-7y - 3z = -11$

$-7y - 3(-1) = -11$

$-7y + 3 = -11$

$-7y = -14$

$y = 2$

$x + 2y + z = 4$

$x + 2(2) - 1 = 4$

$x + 4 - 1 = 4$

$x + 3 = 4$

$x = 1$

Thus $x = 1,\ y = 2,\ z = -1$ is the required solution

Gauss - Jordan method

Dec 2016

1) $2x + y + z = 10$

$3x + 2y + 3z = 18$

$x + 4y + 9y = 16$

Sol: The augmented matrix of the given system is

$$[A:B] = \begin{bmatrix} 2 & 1 & 1 & : & 10 \\ 3 & 2 & 3 & : & 18 \\ 1 & 4 & 9 & : & 16 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 3 & 2 & 3 & : & 18 \\ 2 & 1 & 1 & : & 10 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 4 & 9 & : & 16 \\ 0 & -10 & -24 & : & -30 \\ 0 & -7 & -17 & : & -22 \end{bmatrix}$$

$R_1 \to 10R_1 + 4R_2$, $R_3 \to 7R_2 - 10R_3$

$$[A:B] \sim \begin{bmatrix} 10 & 0 & -6 & : & 40 \\ 0 & -10 & -24 & : & -30 \\ 0 & 0 & 2 & : & 10 \end{bmatrix}$$

$R_1 \to R_1 + 3R_3$, $R_2 \to 12R_3 + R_2$.

$$[A:B] \sim \begin{bmatrix} 10 & 0 & 0 & : & 70 \\ 0 & -10 & 0 & : & 90 \\ 0 & 0 & 2 & : & 10 \end{bmatrix}$$

$10x = 70 \quad -10y = 90 \quad 2z = 10$

$\underline{\underline{x = 7}} \quad \underline{\underline{y = -9}} \quad \underline{\underline{z = 5}}$

Thus $(x, y, z) = (7, -9, 5)$ is the required solution

2) $x + 2y + z = 3$, $2x + 3y + 3z = 10$, $3x - y + 2z = 13$

Sol: The augmented matrix of system <u>Dec 2017</u>

$$[A:B] = \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 2 & 3 & 3 & : & 10 \\ 3 & -1 & 2 & : & 13 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & : & 3 \\ 0 & -1 & 1 & : & 4 \\ 0 & -7 & -1 & : & 4 \end{bmatrix}$$

$R_1 \to R_1 + 2R_2$, $R_3 \to R_3 - 7R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 3 & : & 11 \\ 0 & -1 & 1 & : & 4 \\ 0 & 0 & -8 & : & -24 \end{bmatrix}$$

$R_3 \to \frac{-1}{8} R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 3 & : & 11 \\ 0 & -1 & 1 & : & 4 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$R_1 \to R_1 - 3R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & -1 & 1 & : & 4 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & -1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$x = 2$, $-1y = 1$ $\quad Z = 3$

$y = -1$

$\therefore x = 2, \; y = -1, \; Z = 3$ is the required Solution

3) $x + y + Z = 8$, $-x - y + 2Z = -4$, $3x + 5y - 7 = 14$

Sol: The augumented matrix of the system

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ -1 & -1 & 2 & : & -4 \\ 3 & 5 & -7 & : & 14 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & 0 & 3 & : & 4 \\ 3 & 5 & -7 & : & 14 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & 0 & 3 & : & 4 \\ 0 & 2 & -10 & : & -10 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 0 & 2 & -10 & -10 \\ 0 & 0 & 3 & 4 \end{array}\right]$$

$R_1 \to 2R_1 - R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 2 & 0 & 12 & 26 \\ 0 & 2 & -10 & -10 \\ 0 & 0 & 3 & 4 \end{array}\right]$$

$R_1 \to R_1 - 4R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 2 & 0 & 0 & 38 \\ 0 & 2 & -10 & -38 \\ 0 & 0 & 3 & 4 \end{array}\right]$$

$R_2 \to 3R_2 + 10R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 2 & 0 & 0 & 38 \\ 0 & 6 & 0 & -74 \\ 0 & 0 & 3 & 4 \end{array}\right]$$

$2x = 38$ $\quad$ $6y = -74$ $\quad$ $3z = 4$

$x = \frac{38}{2}$ $\quad$ $y = -\frac{74}{6}$ $\quad$ $z = \frac{4}{3}$

$x = 19$ $\quad$ $y = -\frac{37}{3}$

$R_1 \to \frac{R_1}{2}$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & 6 & 13 \\ 0 & 2 & -10 & -10 \\ 0 & 0 & 3 & 4 \end{array}\right]$$

$R_1 \to R_1 - 2R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 5 \\ 0 & 2 & -10 & -10 \\ 0 & 0 & 3 & 4 \end{array}\right]$$

$R_2 \to 3R_2 + 10R_3$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 5 \\ 0 & 6 & 0 & 10 \\ 0 & 0 & 3 & 4 \end{array}\right]$$

$x = 5$, $6y = 10$ $\quad$ $3z = 4$

$y = \frac{10}{6}$ $\quad$ $z = \frac{4}{3}$

$y = \frac{5}{3}$

Thus $x = 19$, $y = -\frac{37}{3}$, $z = \frac{4}{3}$ is the required solution

4) $x + y + z = 9$

$2x + y - z = 0$

$2x + 5y + 7z = 52$

Sol: $$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 2 & 1 & -1 & 0 \\ 2 & 5 & 7 & 52 \end{array}\right]$$ The augmented form is

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 3 & 5 & 34 \end{array}\right]$$

$R_1 \to R_1 + R_2$, $R_3 \to$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & -4 & : & -20 \end{bmatrix}$$

$R_3 \to \frac{-1}{4} R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -2 & : & -9 \\ 0 & -1 & -3 & : & -18 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

$R_1 \to R_1 + 2R_3$, $R_2 \to R_2 + 3R_3$

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & -1 & 0 & : & -3 \\ 0 & 0 & 1 & : & 5 \end{bmatrix}$$

Thus. $x = 1$, $y = 3$, $z = 5$ is the required solution

Gauss Seidel method.

June 2017

1) Solve the following system of equation by Gauss seidel method.

① $10x + 2y + z = 9$, $x + 10y - z = -22$, $-2x + 3y + 10z = 22$

The given system of equation is diagonally dominat and we write the system of equation in following form

$$x = \frac{1}{10}[9 - 2y - z]$$

$$y = \frac{1}{10}[-22 - x + z]$$

$$z = \frac{1}{10}[22 + 2x - 3y]$$

we start with the trial solution $x = 0$, $y = 0$, $z = 0$

First iteration

$$x^{(1)} = \frac{1}{10}[9 - 2(0) - 0] = 0.9$$

$$y^{(1)} = \frac{1}{10}[-22 - 0.9 - 0] = -2.29$$

$$z^{(1)} = \frac{1}{10}[22 + 2(0.9) - 3(-2.29)] = 3.067$$

Second iteration:

$x^{(2)} = \frac{1}{10}[9-2(-2.29)-3.067] = 1.0513$

$y^{(2)} = \frac{1}{10}[-22-1.0513+3.067] = -1.9984$

$z^{(2)} = \frac{1}{10}[22+2(1.0513)-3(-1.9984)] = 3.0098$

Third iteration

$x^{(3)} = \frac{1}{10}[9-2(-1.9984)-3.0098] = 0.9987 \approx 1 = 1$

$y^{(3)} = \frac{1}{10}[-22-0.9987+3.0098] = -1.999 \approx -2$

$z^{(3)} = \frac{1}{10}[22+2(0.9987)-3(-1.999)] = 2.99 \approx 3.$

Fourth iteration

$x^{(4)} = \frac{1}{10}[9-2(-1.999)-2.99] = 1.0000$

$y^{(4)} = \frac{1}{10}[-22-1+2.99] = -2.0010$

$z^{(4)} = \frac{1}{10}[22+2(1)-3(-2)] = 3.003$

~~Thus $(x,y,z) = (1,-2,3)$ is the required solution.~~

Rayleighs Power method

~~1) using power method, find~~

$x^{5} = \frac{1}{10}[9-2(-2)-3] = 1$

$y^{(5)} = \frac{1}{10}[-22-1+3.003] =$

Thus $(x, y, z) = (1, -2, 3)$ is the required solution.