Integral Calculus.

Multiple Integrals

A repeated process of integration of a function of two and three variables reffered to as.

double integral : $\iint f(x,y)\, dx\, dy$

Triple integral : $\iiint f(x,y)\, dx\, dy\, dz.$

Evaluation of Double integral and triple integrals

1) Evaluate $\int_0^2 \int_1^2 (x^2+y^2)\, dx\, dy.$

$$I = \int_0^2 \left[\int_1^2 (x^2+y^2)\, dx \right] dy.$$

$$I = \int_0^2 \left[\frac{x^3}{3} + y^2 x \right]_1^2 dy.$$

$$I = \int_0^2 \frac{x^3}{3} \Big]_1^2 + y^2 x \Big]_1^2 dy.$$

$$I = \int_0^2 \frac{1}{3}(2^3 - 1^3) + y^2(2-1)\quad dy.$$

$$I = \int_0^2 \frac{7}{3} + y^2\ dy.$$

$$I = \frac{7}{3}y + \frac{y^3}{3} \Big]_0^2$$

$$I = \frac{14}{3} + \frac{8}{3} = \frac{22}{3} \ //$$

2) Evaluate $\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)(1-y^2)}}$.

$\frac{1}{\sqrt{1-x^2}} = \sin^{-1} x$

$$I = \int_{x=0}^{1}\int_{y=0}^{1} \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-y^2}}\, dx\, dy.$$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}} \left(\int_0^1 \frac{1}{\sqrt{1-y^2}}\, dy \right) dx.$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}} (\sin^{-1} y)]_0^1\, dx.$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}} (\sin^{-1}(1) - \sin^{-1}(0)]\, dx.$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}} \left(\frac{\pi}{2}\right) dx.$$

$$= \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx.$$

$$= \frac{\pi}{2} \left[\sin^{-1} x \right]_0^1$$

$$= \frac{\pi}{2} \left[\sin^{-1}(1) - \sin^{-1}(0) \right].$$

$$= \frac{\pi}{2} \left(\frac{\pi}{2} - 0 \right).$$

$$I = \frac{\pi^2}{4}$$

3) Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx \, dy$.

$$I = \int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y \, dx \, dy$$

$$I = \int_{y=0}^{1} \left[\cancel{\frac{x^3}{3}} \; \frac{x^4}{4} y \right]_0^{\sqrt{1-y^2}} dy.$$

$$I = \frac{1}{4} \int_{y=0}^{1} y \left(\sqrt{1-y^2}\right)^4 dy.$$

$$I = \frac{1}{4} \int_0^1 y \left(\left(\sqrt{1-y^2}\right)^{\cancel{2}}\right)^2 dy$$

$$I = \frac{1}{4} \int_0^1 y \,(1-y^2)^2 \, dy.$$

$$I = \frac{1}{4} \int_0^1 y \,(1+y^4-2y^2) \, dy.$$

$$I = \frac{1}{4} \int_0^1 y + y^5 - 2y^3 \, dy.$$

$$I = \frac{1}{4} \left(\frac{y^2}{2} + \frac{y^6}{6} - \frac{2y^4}{4} \right)_0^1$$

$$I = \frac{1}{4} \left[\frac{1}{2}(1) + \frac{1}{6}(1) - \frac{1}{2}(1) \right]$$

$$I = \frac{1}{24}$$

4) ~~Evaluate~~ $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

4) Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

$$I = \int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y+z}\, dz\, dy\, dx$$

5) Evaluate $\int_0^1\int_0^2\int_1^2 x y z^2 \, dx \, dy \, dz$.

$$I = \int_0^1\int_0^2\int_1^2 x y z^2 \, dx \, dy \, dz.$$

$$I = \int_0^1\int_0^2 \frac{x^2 y z^2}{2} \Big]_1^2 \, dy \, dz.$$

$$I = \int_0^1\int_0^2 \frac{y z^2}{2}(2^2 - 1^2) \, dy \, dz.$$

$$I = \int_0^1\int_0^2 \frac{y z^2}{2}(4-1) \, dy \, dz.$$

$$I = \int_0^1\int_0^2 \frac{3yz^2}{2} \, dy \, dz.$$

$$I = \frac{3}{2}\int_0^1 \frac{y^2}{2} z^2 \Big]_0^2 \, dz.$$

$$I = \frac{3}{2}\int_0^1 \frac{1}{2}(z^2(4)) \, dz.$$

$$\cancel{I = \frac{3}{4}\int_0^1 \frac{4z^3}{3}\Big]_0^1}$$

$$I = \frac{3}{2}\cdot\frac{1}{2}\int_0^1 4z^2 \, dz.$$

$$I = \frac{3}{4}\left[\frac{4z^3}{3}\right]_0^1$$

$$\underline{\underline{I = 1}}$$

6) Evaluate $\int_0^a\int_0^a\int_0^a (x^2+y^2+z^2)\, dx\, dy\, dz$

$$I = \int_0^a\int_0^a\int_0^a (x^2+y^2+z^2)\, dx\, dy\, dz.$$

$$I = \int_0^a\int_0^a \left[\frac{x^3}{3} + y^2x + z^2x\right]_0^a dy\, dz$$

$$I = \int_0^a\int_0^a \frac{a^3}{3} + y^2a + z^2a \; dy\, dz.$$

$$I = \int_0^a \left[\frac{a^3y}{3} + \frac{y^3}{3}a + z^2ya\right]_0^a dz.$$

$$I = \int_0^a \frac{a^3.a}{3} + \frac{a^3}{3}.a + z^2aa \; dz.$$

$$I = \int_0^a \frac{a^4}{3} + \frac{a^4}{3} + z^2a^2 \; dz.$$

$$I = \left[\frac{a^4z}{3} + \frac{a^4z}{3} + \frac{z^3}{3}a^2\right]_0^a$$

$$I = \frac{a^4.a}{3} + \frac{a^4.a}{3} + \frac{a^3}{3}.a^2$$

$$I = \frac{a^5}{3} + \frac{a^5}{3} + \frac{a^5}{3}$$

$$I = \frac{3a^5}{3}$$

$$I = a^5$$

QP 7) Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$

$$I=\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz.$$

$$I=\int_{-c}^{c}\int_{-b}^{b}\frac{x^3}{3}+y^2x+z^2x\Big]_{-a}^{a}\,dy\,dz.$$

$$I=\int_{-c}^{c}\int_{-b}^{b}\frac{1}{3}(a^3-(-a)^3+y^2(a-(-a))+z^2(a-(-a))\Big]dy\,dz$$

$$I=\int_{-c}^{c}\int_{-b}^{b}\frac{1}{3}(a^3+a^3)+y^2(a+a)+z^2(a+a)\,dy\,dz.$$

$$I=\int_{-c}^{c}\int_{-b}^{b}\frac{1}{3}(2a^3)+y^2(2a)+z^2(2a)\,dy\,dz.$$

$$I=\int_{-c}^{c}\frac{2}{3}a^3y+\frac{2ay^3}{3}+2az^2y\Big]_{-b}^{b}\,dz.$$

$$I=\int_{-c}^{c}\frac{2}{3}a^3(b-(-b))+\frac{2a}{3}(b^3-(-b)^3)+2az^2(b-(-b))$$

$$I=\int_{-c}^{c}\frac{2a^3}{3}(b+b)+\frac{2a}{3}(b^3+b^3)+2az^2(b+b)\,dz.$$

$$I=\int_{-c}^{c}\frac{2a^3}{3}(2b)+\frac{2a}{3}(2b^3)+2az^2(2b)\,dz.$$

$$I=\frac{4a^3b}{3}z+\frac{4ab^3z}{3}+4ab\frac{z^3}{3}\Big]_{-c}^{c}.$$

$$I=\frac{4a^3b}{3}(c-(-c))+\frac{4ab^3}{3}(c-(-c))+\frac{4ab}{3}(c^3-(-c)^3).$$

$$I=\frac{4a^3b}{3}(2c)+\frac{4ab^3}{3}(2c)+\frac{4ab}{3}(2c^3)$$

$$I = \frac{8a^3bc}{3} + \frac{8ab^3c}{3} + \frac{8abc^3}{3}$$

$$I = \frac{8abc}{3}(a^2+b^2+c^2)$$

QP

8) $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dy\, dx$

$$I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} (x^2+y^2)\, dy\, dx$$

$$I = \int_{x=0}^{1} \left. x^2y + \frac{y^3}{3} \right]_x^{\sqrt{x}} dx$$

$$I = \int_{x=0}^{1} x^2(\sqrt{x}-x) + \frac{1}{3}(\sqrt{x}^3 - x^3)\, dx$$

$$I = \int_{x=0}^{1} x^2(x^{1/2}-x) + \frac{1}{3}(x^{1/2})^3 - \frac{x^3}{3}\, dx$$

$$I = \int_{x=0}^{1} x^{5/2} - x^3 + \frac{1}{3}x^{3/2} - \frac{x^3}{3}\, dx$$

$$I = \int_{x=0}^{1} x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3}\, dx$$

$$I = \left. \frac{x^{5/2+1}}{5/2+1} + \frac{1}{3}\frac{x^{3/2+1}}{3/2+1} - \frac{4}{3}\frac{x^4}{4} \right]_0^1$$

$$I = \left. \frac{x^{7/2}}{7/2} + \frac{1}{3}\frac{x^{5/2}}{5/2} - \frac{x^4}{3} \right]_0^1$$

$$I = \frac{[1-0]}{7/2} + \frac{1}{3}\frac{[1-0]}{5/2} - \frac{1}{3}$$

$$I = \frac{2}{7} + \frac{1}{3}\cdot\frac{2}{5} - \frac{1}{3}$$

$$I = \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{9}{105} = \frac{3}{35}$$

QP 9) $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} \, dz \, dy \, dx$

$$I = \int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y} \cdot e^{z} \, dz \, dy \, dx$$

$$I = \int_0^a \int_0^x e^{x+y} \left[e^{z} \right]_0^{x+y} dy \, dx$$

$$I = \int_0^a \int_0^x e^{x+y} \left[e^{x+y} - e^{0} \right] dy \, dx$$

$$I = \int_0^a \int_0^x e^{x+y} \left[e^{x+y} - 1 \right] dy \, dx$$

$$I = \int_0^a \int_0^x e^{2x+2y} - e^{x+y} \, dy \, dx$$

$$I = \int_0^a \int_0^x e^{2x} \cdot e^{2y} - e^{x} \cdot e^{y} \, dy \, dx$$

$$I = \int_0^a e^{2x} \left[\frac{e^{2y}}{2} \right]_0^x - e^{x} \left[e^{y} \right]_0^x dx$$

$$I = \int_0^a \frac{e^{2x}}{2} \left[e^{2x} - e^{2(0)} \right] - e^{x} \left[e^{x} - e^{0} \right] dx$$

$$I = \int_0^a \frac{e^{2x}}{2} \left[e^{2x} - 1 \right] - e^{x} \left[e^{x} - 1 \right] dx$$

$$I = \int_0^a \frac{e^{4x}}{2} - \frac{e^{2x}}{2} - e^{2x} + e^{x} \, dx$$

$$I = \frac{e^{4x}}{2 \cdot 4} - \frac{e^{2x}}{2 \cdot 2} - \frac{e^{2x}}{2} + e^{x} \Big]_0^a$$

$$I = \frac{e^{4x}}{8} - \frac{e^{2x}}{4} - \frac{e^{2x}}{2} + e^{x} \Big]_0^a$$

$$I = \frac{1}{8}\left[e^{4a} - e^{4(0)} \right] - \frac{1}{4}\left[e^{2a} - e^{2(0)} \right] - \frac{1}{2}\left[e^{2a} - e^{2(0)} \right] + \left[e^{a} - e^{0} \right]$$

$$I = \frac{e^{4a}}{8} - \frac{1}{8} - \frac{e^{2a}}{4} + \frac{1}{4} - \frac{e^{2a}}{2} + \frac{1}{2} + e^{a} - 1$$

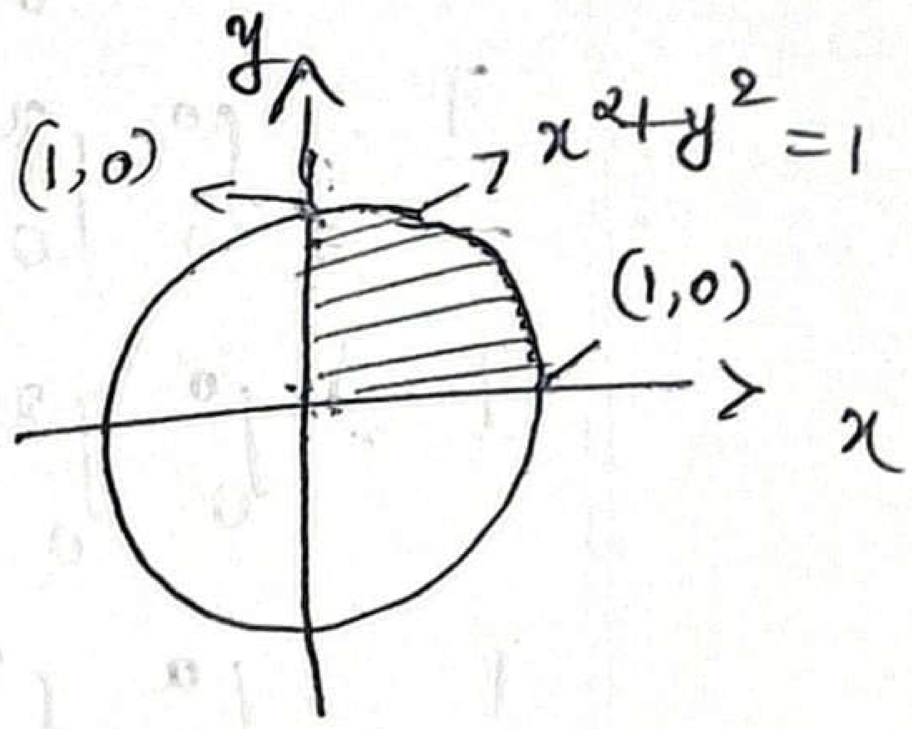
$$I = \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \frac{3}{8}$$

$$I = \frac{1}{8}\left(e^{4a} + 8e^{a} - 6e^{2a} - 3 \right)$$

Evaluation of double integrals by Change of order of integration:

Imp QP

1) Change the order of integration and hence evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dx \, dy$.



$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dx \, dy$$

$$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} y^2 \, dy \, dx$$

$x = 0 - 1 \rightarrow \uparrow$

$y = 0 - \sqrt{1-x^2}$

$y = 0 - 1$

$y = 0 - \sqrt{1-x^2}$

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} y^2 \, dx \, dy$$

$$I = \int_{y=0}^{1} y^2 \left[x\right]_{x=0}^{\sqrt{1-y^2}} dy$$

$$I = \int_{y=0}^{1} y^2 \left[\sqrt{1-y^2}\right) dy.$$

Put $y = \sin\theta$

$dy = \cos\theta \, d\theta$

do and θ varies from 0 to $\pi/2$.

$$I = \int_{\theta=0}^{\pi/2} \sin^2\theta \left(\sqrt{1-\sin^2\theta}\right) d\theta$$

$$I = \int_{\theta=0}^{\pi/2} \sin^2\theta \left(\sqrt{\cos^2\theta}\right) d\theta$$

$$I = \frac{(1)(1)}{(4)(2)} \frac{\pi}{2} = \frac{\pi}{16}$$ by reduction formula

$$I = \frac{\pi}{16}.$$

Evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$ by Changing the order of integration ?

Inf 2) $I = \int_{x=0}^{1} \int_{y=x^2}^{2-x} xy \, dy \, dx$.

To identify the region of intersection R.

$y = x^2$ and $y = 2-x$.

$x^2 = 2-x$.

$x^2+x-2=0$

$(x-1)(x+2)=0$

$x=1, x=-2$.

$\Rightarrow$ $y=x^2$, $y=1^2=1$, $y=-2^2=4$; $y=2-x$, $y=2-1=1$, $y=2+2=4$

$x=(1,-2)$ $y=(1,4)$.

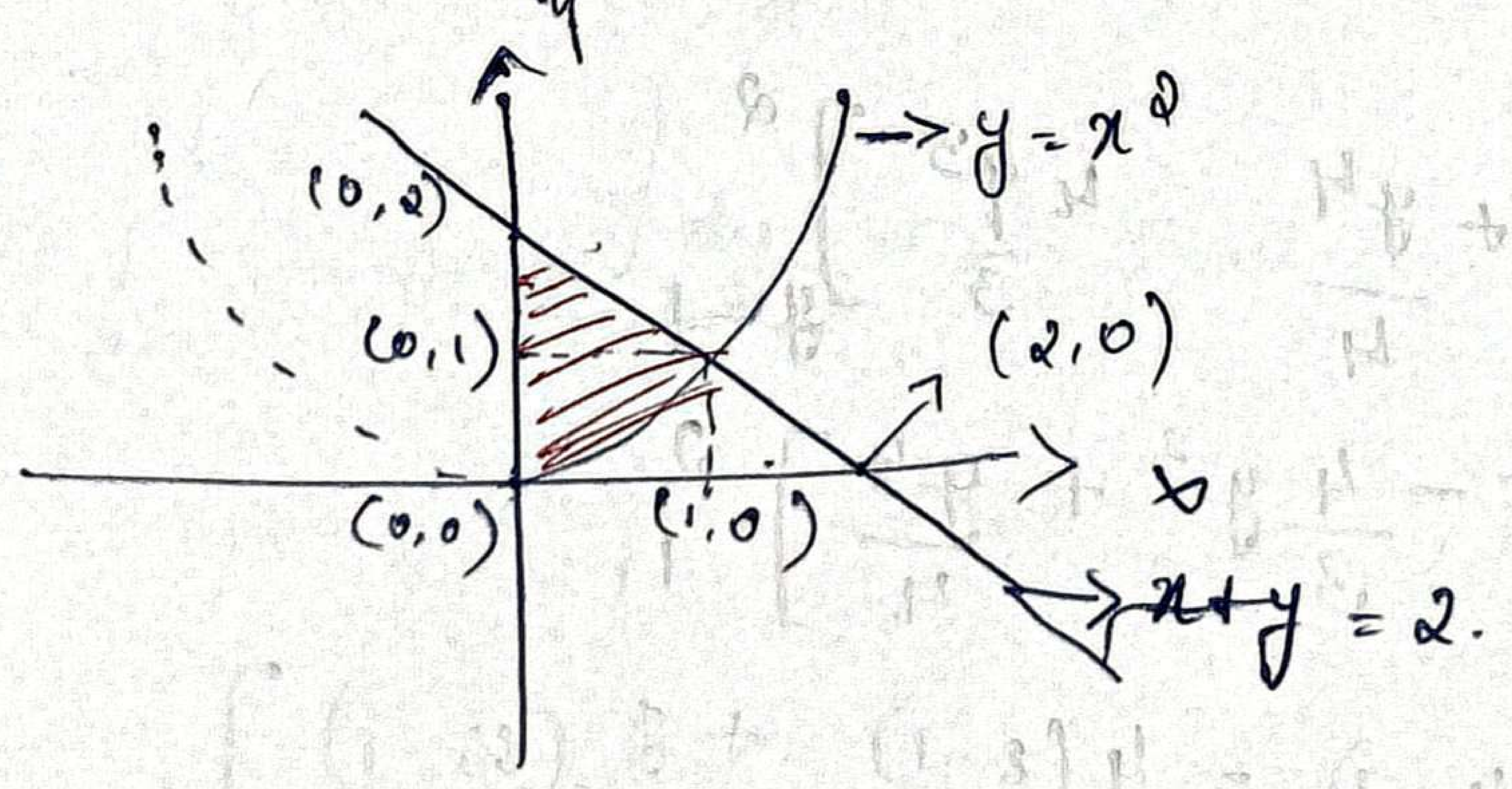
Thus $(1,1)$ and $(-2,4)$ are the points of intersection.

The region is bounded by $x=0$, $x=1$ the point $(1,-2)$ $(1,4)$ the point $(-2,4)$ will be out of Consideration

$y=x^2$ is a parabola symmetrical about the y axis

and $y=+2-x$ $\Rightarrow x+y=2$ or $\frac{x}{2}+\frac{y}{2}=1$

is a straight line passing through $(2,0)$ and $(0,2)$



$x+y=2$

$x=0 \Rightarrow y=2$

$y=0 \Rightarrow x=2$.

$y=2-x$.

$y+x=2$.

	On changing the order we must have constant limits for y and variable limits for x to cover the same region.
$y=0$, $y=1$ $x=0$, $x=\sqrt{y}$ $y=1$, $y=2$ $x=0$, $x=2-y$	

$$I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$I = I_1 + I_2.$$

$$I_1 = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy.$$

$$I_1 = \int_{y=0}^{1} y\,\frac{x^2}{2}\Big]_{x=0}^{\sqrt{y}} dy = \int_{y=0}^{1} y\,\frac{\sqrt{y}^2}{2}\,dy$$

$$I_1 = \int_0^1 \frac{y^2}{2}\,dy = \frac{y^3}{2\cdot 3}\Big]_0^1 = \frac{1}{6}$$

$$I_2 = \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy.$$

$$I_2 = \int_{y=1}^{2} y\,\frac{x^2}{2}\Big]_{x=0}^{2-y} dy = \int_1^2 y\,\frac{(2-y)^2}{2}\,dy$$

$$I_2 = \frac{1}{2}\int_1^2 y\,(2^2 + y^2 - 2\cdot 2\cdot y)\,dy.$$

$$I_2 = \frac{1}{2}\int_1^2 4y + y^3 - 4y^2\,dy$$

$$I_2 = \frac{1}{2}\left[4\frac{y^2}{2} + \frac{y^4}{4} - \frac{4y^3}{3}\right]_{y=1}^{2}$$

$$I_2 = \frac{1}{2}\left[2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_1^2$$

$$I_2 = \frac{1}{2}\left[2(4-1) - \frac{4}{3}(8-1) + \frac{1}{4}(16-1)\right]$$

$$I_2 = \frac{1}{2}\left[6 - \frac{32}{3} + \frac{4}{3} + \frac{16}{4} - \frac{1}{4}\right].$$

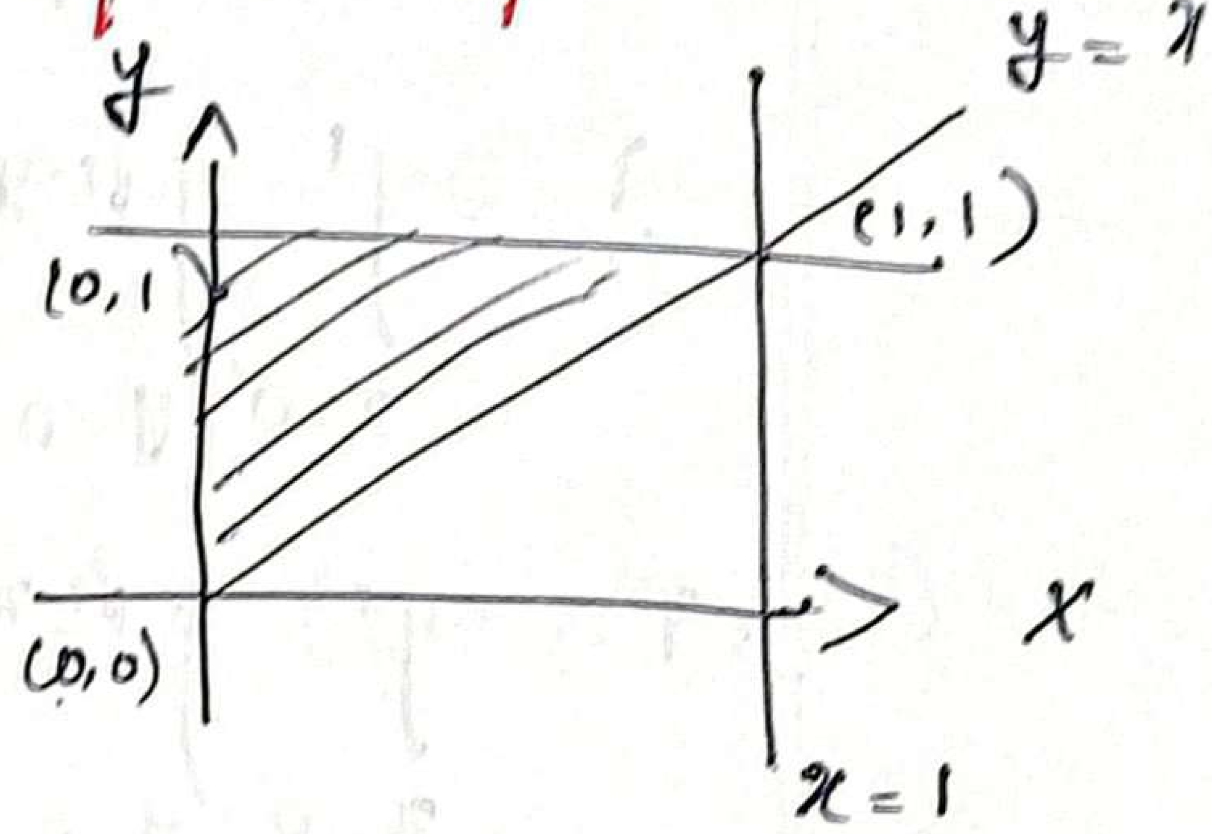
$$I_2 = \frac{5}{24}.$$

$$I = I_1 + I_2.$$

$$I = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

3) Evaluate by Change of order of integration

$$\int_0^1 \int_x^1 \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$



Sol

$$I = \int_{x=0}^{1}\int_{y=x}^{1} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$$

on Changing the order of integration we have.

$$I = \int_{y=0}^{1}\int_{x=0}^{y} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy.$$

$$x^2+y^2 = t$$

$$2x\, dx = dt$$

$$x\, dx = dt/2.$$

we have $\int \frac{x}{\sqrt{x^2+y^2}}\, dx$ reducing to $\frac{1}{2}\int \frac{dt}{\sqrt{t}} = \sqrt{t}$

$$I = \int_{y=0}^{1} \left[\sqrt{x^2+y^2}\,\right]_{x=0}^{y} dy$$

$$I = \int_{y=0}^{1} (\sqrt{2}\, y - y)\, dy = \int_0^1 y(\sqrt{2}-1)\, dy$$

$$I = (\sqrt{2}-1)\left.\frac{y^2}{2}\right]_0^1 = \frac{\sqrt{2}-1}{2}$$

$$I = \frac{\sqrt{2}-1}{2}$$

10) Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx.$$

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\frac{z^2}{2}\Big]_0^{\sqrt{1-x^2-y^2}}\,dy\,dx$$

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \frac{xy}{2}\left(\sqrt{1-x^2-y^2}\right)^2\,dy\,dx.$$

$$I = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \frac{xy}{2}(1-x^2-y^2)\,dy\,dx$$

$$I = \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} xy - x^3y - xy^3\,dy\,dx$$

$$I = \frac{1}{2}\int_0^1 \frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\Big]_0^{\sqrt{1-x^2}}\,dx$$

$$I = \frac{1}{2}\int_0^1 \frac{x}{2}\left(\sqrt{1-x^2}\right)^2 - \frac{x^3}{2}\left(\sqrt{1-x^2}\right)^2 - \frac{x}{4}\left(\sqrt{1-x^2}\right)^4\,dx$$

$$I = \frac{1}{2}\int_0^1 \frac{x}{2}(1-x^2) - \frac{x^3}{2}(1-x^2) - \frac{x}{4}\left((1-x^2)^2\right)\,dx$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{2}(x-x^3) - \frac{1}{2}(x^3-x^5) - \frac{x}{4}(1+x^4-2x^2)\,dx$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{2}(x-x^3) - \frac{1}{2}(x^3-x^5) - \frac{1}{4}(x+x^5-2x^3)\,dx$$

$$I = \frac{1}{2}\left[\frac{1}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right] - \frac{1}{2}\left[\frac{x^4}{4}-\frac{x^6}{6}\right] - \frac{1}{4}\left[\frac{x^2}{2}+\frac{x^6}{6}-\frac{2x^4}{4}\right]\right]_0^1$$

$$I = \frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{4}-\frac{1}{6}\right) - \frac{1}{4}\left(\frac{1}{2}+\frac{1}{6}-\frac{2}{4}\right)\right.$$

$$I = \frac{1}{2}\left[\frac{1}{2}\left(\frac{2-1}{4}\right) - \frac{1}{2}\left(\frac{3-2}{12}\right) - \frac{1}{4}\left(\frac{1}{6}\right)\right]$$

$$I = \frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{12}\right) - \frac{1}{4}\left(\frac{1}{6}\right)\right]$$

$$I = \frac{1}{2}\left[\frac{1}{8} - \frac{1}{24} - \frac{1}{24}\right]$$

$$I = \frac{1}{2}\left[\right.$$

$$I = \cancel{\frac{1}{2}\int_0^1 \frac{x}{2}(x-x^3) - \frac{x}{2}(x^3-x^5) - \frac{1}{4}(}$$

$$I = \frac{1}{2}\int_0^1 \frac{x}{2}(1-x^2) - \frac{x^3}{2}(1-x^2) - \frac{x}{4}((1+x^4-2x^2))\,dx$$

$$I = \frac{1}{2}\cdot\frac{1}{4}\int_0^1 2x(1-x^2) - 2x^3(1-x^2) - x(1+x^4-2x^2)\,dx.$$

$$I = \frac{1}{8}\int_0^1 2x - 2x^3 - 2x^3 + 2x^5 - x - x^5 + 2x^3\,dx$$

$$I = \frac{1}{8}\int_0^1 (x^5 - 2x^3 + x)\,dx.$$

$$I = \frac{1}{8}\left[\frac{x^6}{6} - \frac{2x^4}{4} + \frac{x^2}{2}\right]_0^1$$

$$I = \frac{1}{8}\left[\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right]$$

$$I = 1/48$$

11) $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x+y+z)\,dy\,dx\,dz$

$I = \int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z} (x+y+z)\,dy\,dx\,dz.$

$I = \int_{z=-1}^{1}\int_{x=0}^{z} xy + \frac{y^2}{2} + zy \Big]_{x-z}^{x+z} dx\,dz.$

$I = \int_{z=-1}^{1}\int_{x=0}^{z} x(x+z-(x-z)) + \frac{1}{2}(x+z)^2 - (x-z)^2 + z(x+z-(x-z$

$I = \int_{z=-1}^{1}\int_{x=0}^{z} x(x+z-x+z) + \frac{1}{2}(x^2+z^2+2xz-(x^2+z^2-2xz)$

$+ z(x+z-x+z)\,dx\,dz.$

$I = \int_{-1}^{1}\int_{0}^{z} 2xz + \frac{1}{2}4xz + 2z^2\,dx\,dz$

$I = \int_{-1}^{1} 2z\frac{x^2}{2} + 2\frac{x^2}{2}z + 2z^2x\Big]_0^z dz$

$I = \int_{-1}^{1} \frac{2}{2}z(z)^2 + \frac{2}{2}z(z)^2 + 2z^2z\,dz$

$I = \int_{-1}^{1} z^3 + z^3 + 2z^3\,dz$

$I = \int_{-1}^{1} 4z^3\,dz.$

$I = 4\frac{z^4}{4}\Big]_{-1}^{1}$

$I = (1^4 - (-1)^4)$

$I = 1^4 - 1^4$

$I = 0$

12) $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{(a^2-r^2)/a} r\, dr\, d\theta\, dz.$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta}\int_{z=0}^{(a^2-r^2)/a} r\, dz\, dr\, d\theta$$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} r\, z \Big]_0^{(a^2-r^2)/a} dr\, d\theta.$$

dr dz

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} r \cdot \frac{a^2-r^2}{a}\, dr\, d\theta$$

$$I = \frac{1}{a}\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} r a^2 - r^3\, dr\, d\theta.$$

$$I = \frac{1}{a}\int_{\theta=0}^{\pi/2} a^2\frac{r^2}{2} - \frac{r^4}{4}\Big]_{r=0}^{a\sin\theta} d\theta$$

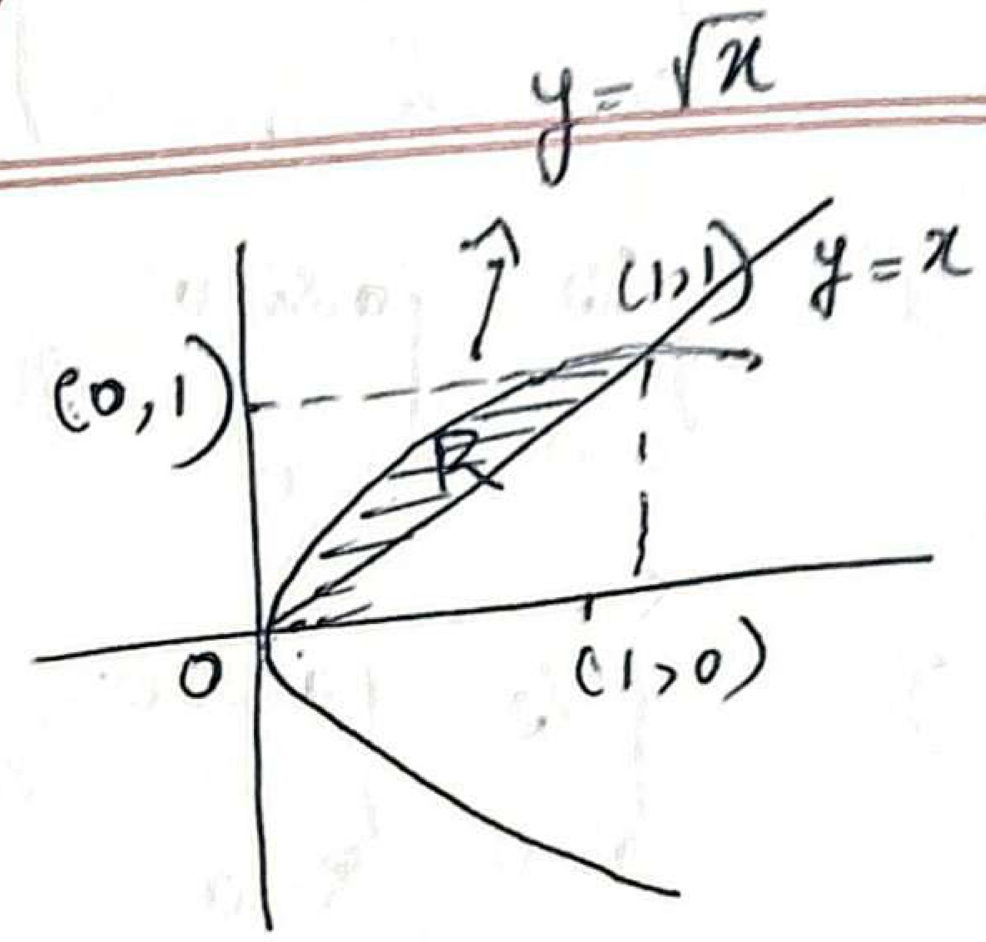
$$I = \frac{1}{a}\int_{\theta=0}^{\pi/2} \frac{a^2}{2}(a^2\sin^2\theta) - \frac{1}{4}(a^4\sin^4\theta)\, d\theta.$$

$$I = \frac{1}{a}\int_{\theta=0}^{\pi/2} \frac{a^4}{2}\sin^2\theta - \frac{a^4}{4}\sin^4\theta\, d\theta$$

$$I = \frac{1}{a}\Big[\frac{a^4}{2}\cdot\frac{1}{2}$$

Q.P 4) Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$ by changing the order of integration?

$$I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} xy \, dy \, dx$$



$y = x$ and $y = \sqrt{x}$

$x = \sqrt{x}$

$x^2 = x$

$x^2 - x = 0 \Rightarrow x(x-1)=0 \Rightarrow x = 0, 1.$

$y = x \Rightarrow y = 0$ $\quad x=0$, $x-1=0$

$y = x \Rightarrow y = 1$

The point of intersection are $(0,0)$ & $(1,1)$

On changing the order of integration

$x = 0 - 1 \longrightarrow y = 0 - 1$

$y = x - \sqrt{x} \quad \uparrow \quad x = y^2 - y$

Change the order. $\uparrow \longrightarrow$

$$I = \int_{y=0}^{1} \int_{x=y^2}^{y} xy \, dx \, dy$$

$y = \sqrt{x}$

$y^2 = x.$

$$I = \int_{y=0}^{1} y \frac{x^2}{2} \Big]_{y^2}^{y} dy$$

$$I = \int_{y=0}^{1} \frac{y}{2}(y^2 - y^4) \, dy$$

$$I = \frac{1}{2} \int_{y=0}^{1} y^3 - y^5 \, dy$$

$$I = \frac{1}{2}\left(\frac{y^4}{4} - \frac{y^6}{6}\right)\Big]_0^1$$

$$I = 1/24$$

5) Change the order of integration and hence evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx$

Sol

$$I = \int_{x=0}^{4a}\int_{y=x^2/4a}^{y=2\sqrt{ax}} xy\,dy\,dx$$

$y = 2\sqrt{ax}$, $y = \frac{x^2}{4a}$

$\frac{x^2}{4a} = 2\sqrt{ax}$.

$\left(\frac{x^2}{4a}\right)^2 = 2^2(\sqrt{ax})^2$.

$\frac{x^4}{16a^2} = 4ax$.

$x^4 = 64a^3x$.

$x(x^3 - 64a^3) = 0$

$x = 0$ $\quad x^3 - 64a^3 = 0$

$x^3 = 64a^3$

$x^3 = 4^3a^3$

$x = 4a$

$y = \frac{x^2}{4a} = \frac{0}{4a} = 0$

$y = \frac{x^2}{4a} = \frac{4a^2}{4a} = \frac{16a^2}{4a} = 4a$

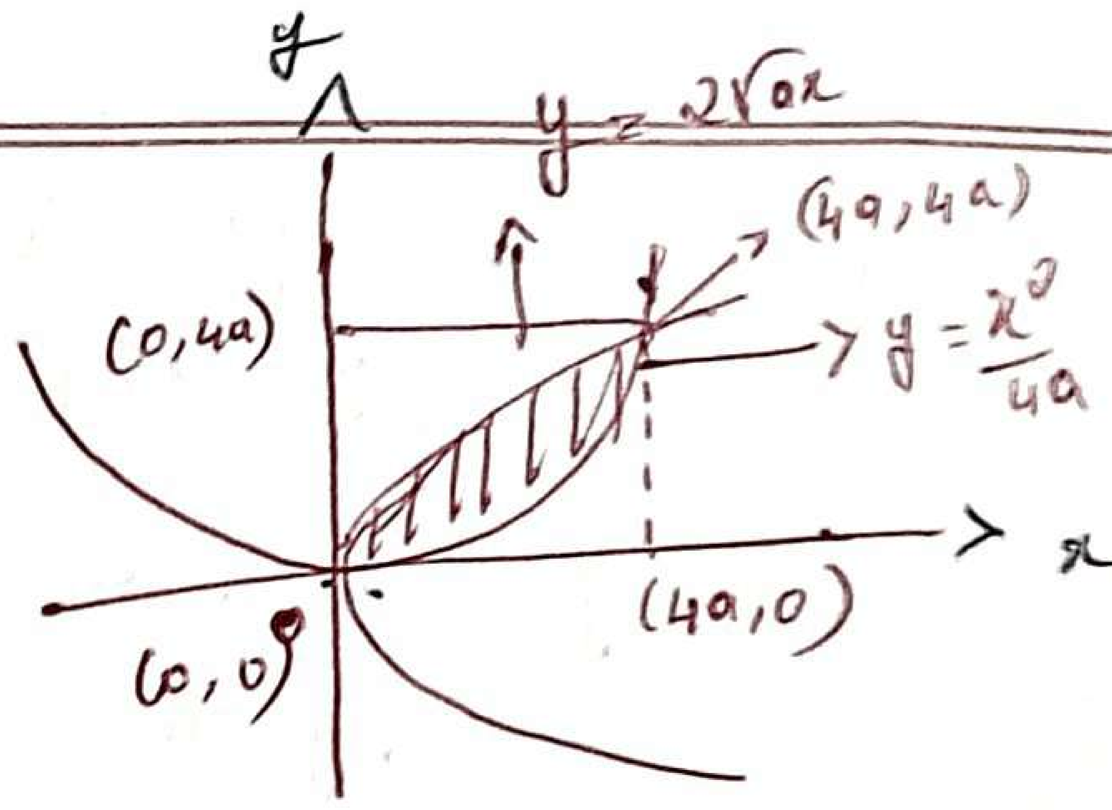
$x = 0$ $\quad x = 4a \rightarrow$

$y = 0$ $\quad y = 4a$

The point of intersection are $(0,4a)$ & $(0,4a)$.

$x = 0 - 4a \rightarrow \uparrow y = 0 - 4a$

$y = \frac{x^2}{4a} - 2\sqrt{ax}$ $\quad x = \frac{y^2}{4a} - 2\sqrt{ay}$



Change the order

$$I = \int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} xy\,dx\,dy$$

$$I = \int_{y=0}^{4a} y\left[\frac{x^2}{2}\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$I = \int_{y=0}^{4a} \frac{y}{2}\left(2^2(\sqrt{ay})^2 - \left(\frac{y^2}{4a}\right)^2\right) dy$$

$$I = \int_{y=0}^{4a} \frac{y}{2}\left(4ay - \frac{y^4}{16a^2}\right) dy$$

$$I = \frac{1}{2}\left[\frac{4ay^3}{3} - \frac{1}{16a^2}\frac{y^6}{6}\right]_0^{4a}$$

$$I = \frac{1}{2}\left[\frac{4a(4a)^3}{3} - \frac{1}{96a^2}(4a)^6\right]$$

$$I = \frac{1}{2}\left[\frac{256a^4}{3} - \frac{4096a^6}{96a^2}\right]$$

$$I = \frac{1}{2}\left[\frac{256a^4}{3} - \frac{128a^4}{3}\right]$$

$$I = \frac{64a^4}{3}$$

6) Change the order of integration in $\int_0^1 \int_{\sqrt{y}}^{2-y} xy \, dx \, dy$ and hence evaluate.

Sol

$$I = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{2-y} xy \, dx \, dy$$

$y = 0$ to 1

$x = \sqrt{y}$ to $2-y$.

$\sqrt{y} = 2-y$

$y = (2-y)^2$

$y^2 - 5y + 4 = 0$

$(y-1)(y-4) = 0$

$y = 1, 4.$

$x = \sqrt{y}$

$x = \pm 1$

$x = \sqrt{y}.$

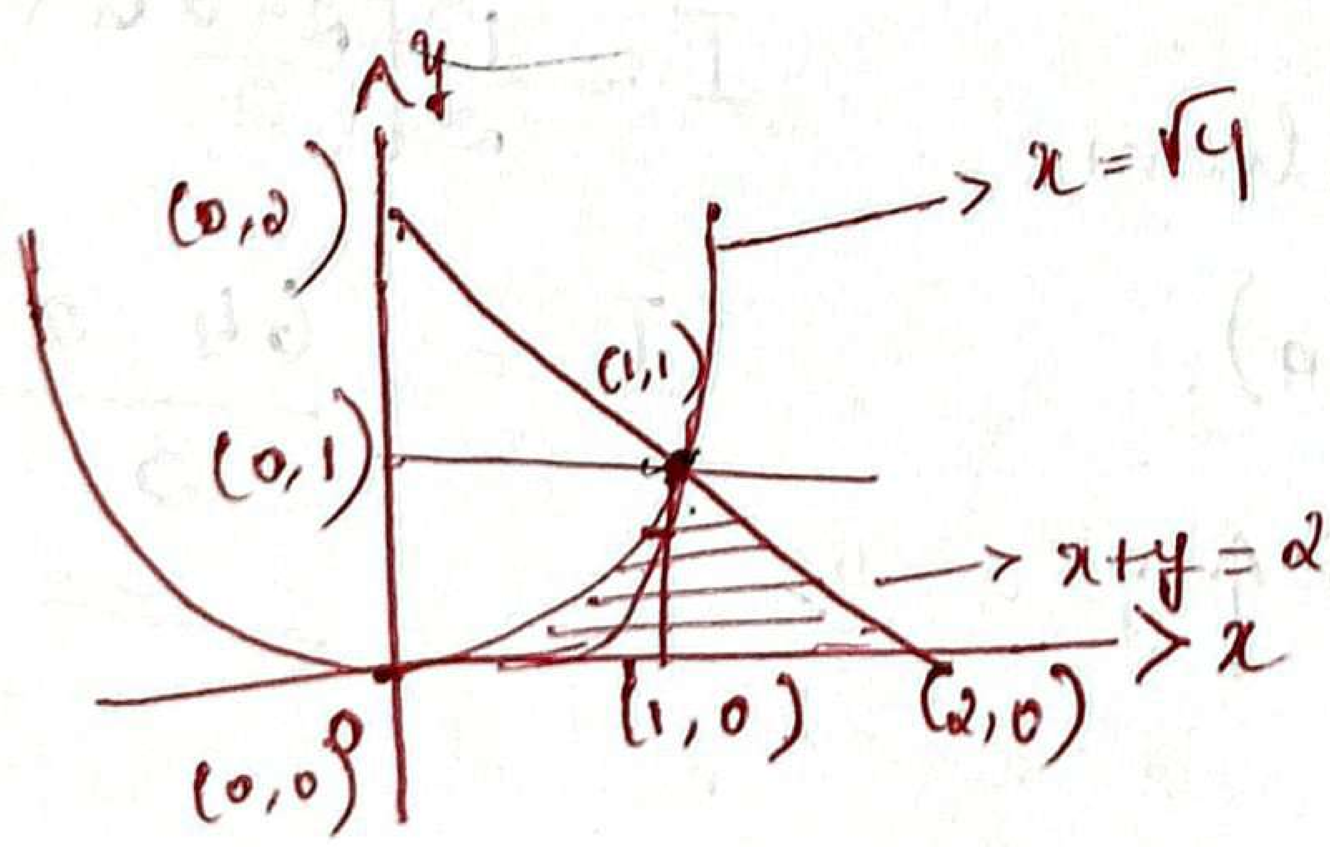
$x = \pm 2$

$x = \pm 1, \pm 2.$

$y = 1 \quad 4.$

The point of intersection are.

$(1,1)\ (-1,1)\ (2,4)\ (-2,4).$



$y = 0$ to 1 →

$x = \sqrt{y}$ to $2-y$. ↑

By Changing the order of integ

$x = 0$ to 1 ↑

$y = 0$ to x^2. →

$x = 1$ to 2. ↑

$y = 1$ to $2-x$. →

$I = I_1 + I_2$

$$I = \int_{x=0}^{1} \int_{y=0}^{x^2} xy \, dy \, dx + \int_{x=1}^{2} \int_{y=0}^{2-x} xy \, dy \, dx$$

$$I_1 = \int_{x=0}^{1} \int_{y=0}^{x^2} xy \, dy \, dx.$$

$$I_1 = \int_{x=0}^{1} x \left. \frac{y^2}{2} \right]_{y=0}^{x^2} dx$$

$$I_1 = \int_0^1 \frac{x^5}{2} \, dx$$

$$I_1 = \left. \frac{x^6}{12} \right]_0^1$$

$$I_1 = \frac{1}{12}$$

$$I_2 = \int_{x=1}^{2} \int_{y=0}^{2-x} xy \, dy \, dx.$$

$$I_2 = \int_1^2 \left. \frac{xy^2}{2} \right]_0^{2-x} dx$$

$$I_2 = \int_1^2 \frac{x}{2} (2-x)^2 \, dx$$

$$I_2 = \int_1^2 \frac{x}{2} (4 + x^2 - 4x)$$

$$I_2 = \frac{1}{2} \int_1^2 4x + x^3 - 4x^2 \, dx$$

$$I_2 = \frac{1}{2} \left[\frac{4x^2}{2} + \frac{x^4}{4} - \frac{4x^3}{3} \right]_1^2$$

$$I_2 = \frac{5}{24}$$

$$I = I_1 + I_2$$

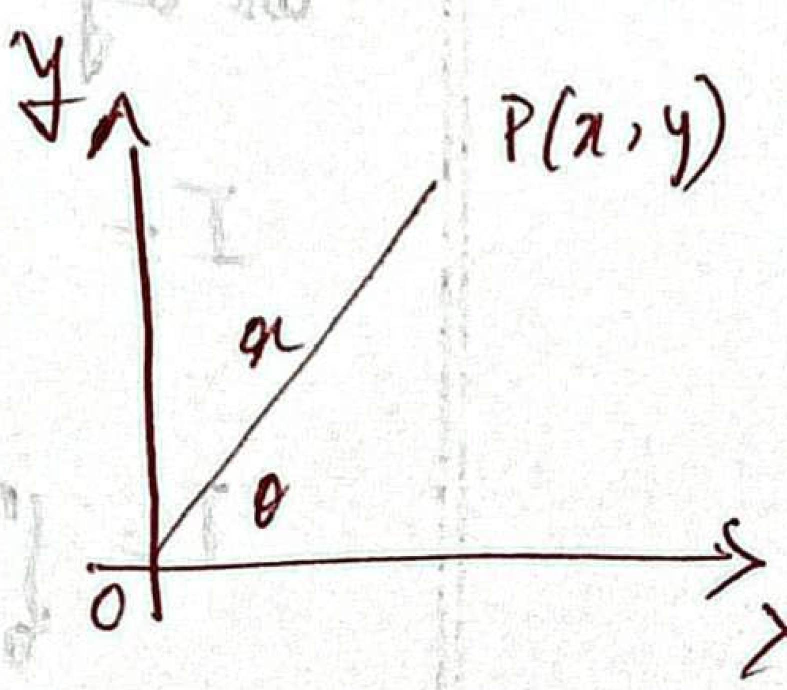
$$I = \frac{1}{12} + \frac{5}{24} = \frac{7}{24}$$

Evaluation by Changing into polar Coordinate form:

1) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$ by changing to polar coordinates ?

Sol In polars we have $x = r \cos\theta \quad y = r \sin\theta$

$x^2 + y^2 = r^2$ and $dx \, dy = r \, dr \, d\theta.$



x Varies from 0 to ∞

y Varies from 0 to ∞

r Varies from 0 to ∞

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r \, dr \, d\theta$$

$r^2 = t \qquad 2r\,dr = dt \implies r\,dr = \frac{dt}{2}$

t also varies from 0 to ∞.

$$I = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2} \, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \left. -e^{-t} \right]_0^{\infty} d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \left(-e^{-\infty} - (-e^{-0})\right) d\theta$$

$$= \frac{-1}{2} \int_0^{\pi/2} -1 \, d\theta$$

$$= +\frac{1}{2} \left[\theta\right]_0^{\pi/2}$$

$$I = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$ //

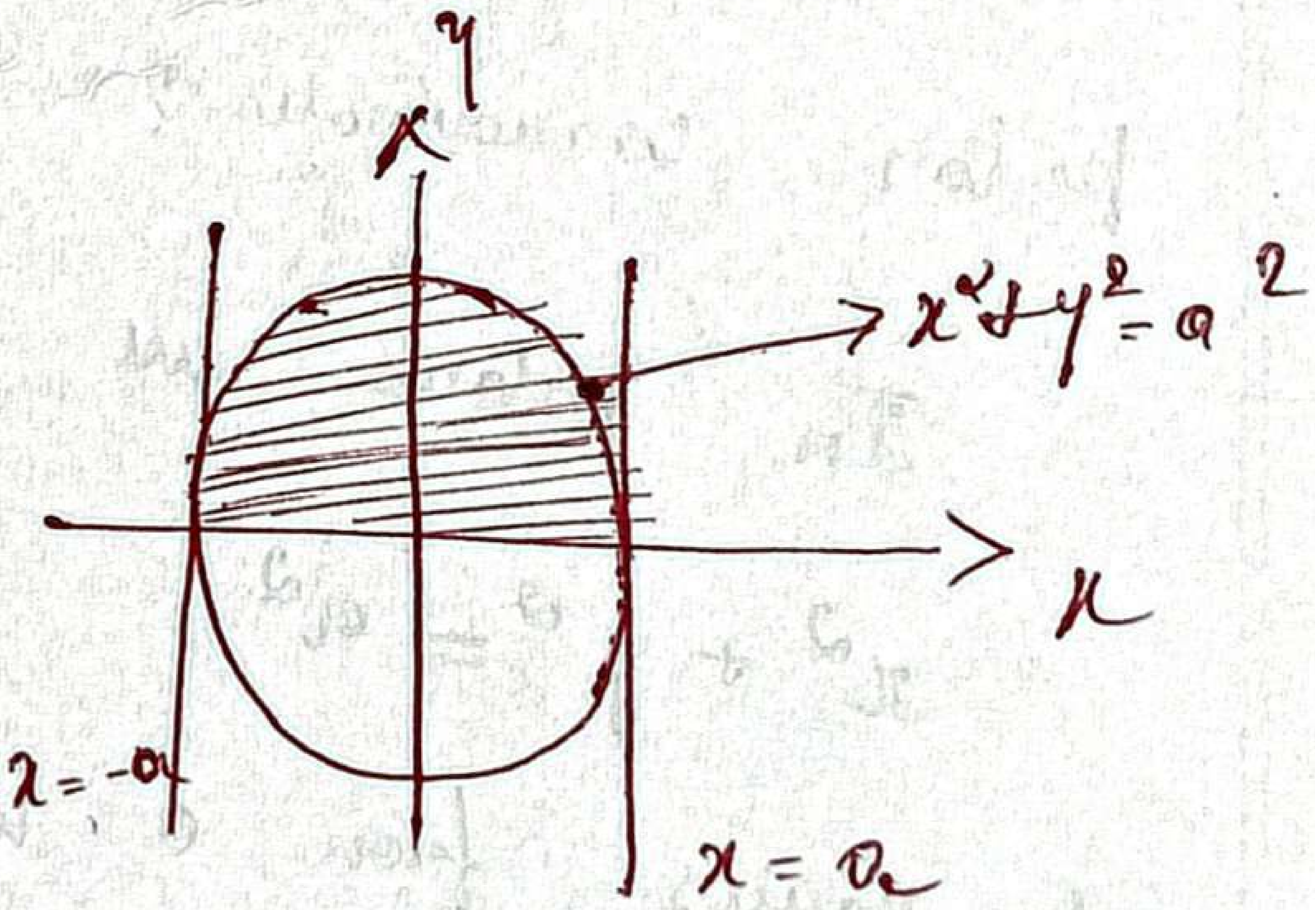
2) Change the integral $\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2} \, dy \, dx$ into polars and hence evaluate the same.

Sol : Clearly θ varies from 0 to π

$x = r\cos\theta \quad y = r\sin\theta \quad , x^2+y^2 = r^2 \quad , a^2 = r^2$

$a = r \implies r = a$

r varies from 0 to a.



$dx\,dy = r\,dr\,d\theta$

$$I = \int_{\theta=0}^{\pi} \int_{r=0}^{a} r \cdot r \, dr \, d\theta$$

$$I = \int_{\theta=0}^{\pi} \left[\frac{r^3}{3}\right]_{r=0}^{a} d\theta$$

$$I = \int_0^{\pi} \frac{a^3}{3} \, d\theta$$

$$= \frac{a^3}{3} (\theta)_0^{\pi}$$

$$I = \frac{a^3 \pi}{3}$$

3) Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$ by Changing into polars.

$$I = \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy.$$

$x = \sqrt{a^2-y^2}$

$x^2 = a^2 - y^2$

$x^2+y^2 = a^2$

$x^2+y^2=a^2$ is a Circle with Center origin and radius a. (First quadrant)

y Varies from 0 to a

$x = r\cos\theta \qquad y = r\sin\theta \qquad x^2+y^2 = r^2 \qquad r = a$

$x = 0, \quad y = 0, \quad r = 0$

r varies from 0 to a.

In the first quadrant θ varies from 0 to $\pi/2$.

$dx\, dy = r\, dr\, d\theta$.

$$I = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r\sin\theta\; r \cdot r\, dr\, d\theta$$

$$I = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r^3 \sin\theta\, dr\, d\theta$$

$$I = \int_{r=0}^{a} r^3 [-\cos\theta]\Big]_0^{\pi/2} dr.$$

$$I = \int_{r=0}^{a} -r^3 (0-1)\, dr$$

$$I = \frac{r^4}{4}\Big]_0^a$$

$$I = \frac{a^4}{4}$$

Applications of Double and Triple integrals

Application to find Area and Volume by double integral:

Application formulae

1) $\iint_R dx\, dy$ = Area of the region R in the Cartesian form

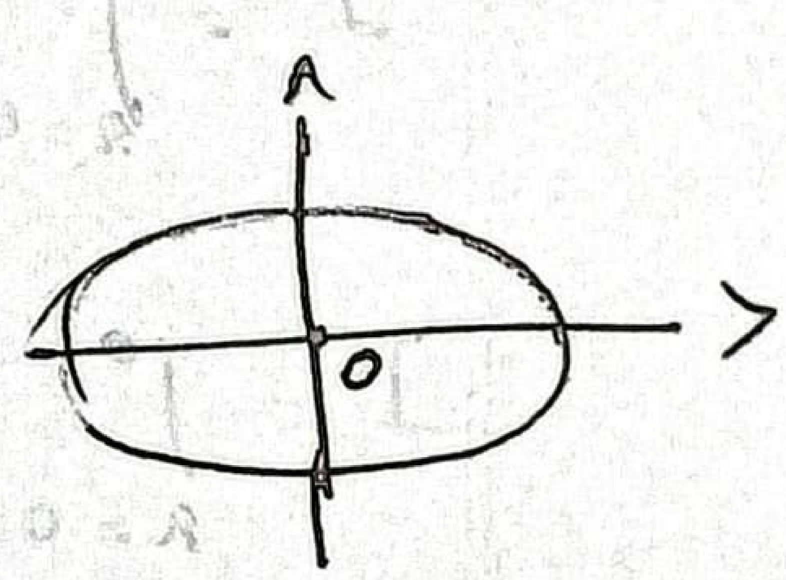
2) $\iint_R r\, dr\, d\theta$ = Area of the region R in the polar form.

3) $\iiint_V dx\, dy\, dz$ = Volume of the solid in the Cartesian form.

4) $\iint_A 2\pi r^2 \sin\theta\, dr\, d\theta$ = Volume of a solid obtained by the revolution of a curve enclosing an area A about the initial line in the polar form.

Problems:

1) Find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ by double integration.



Sol

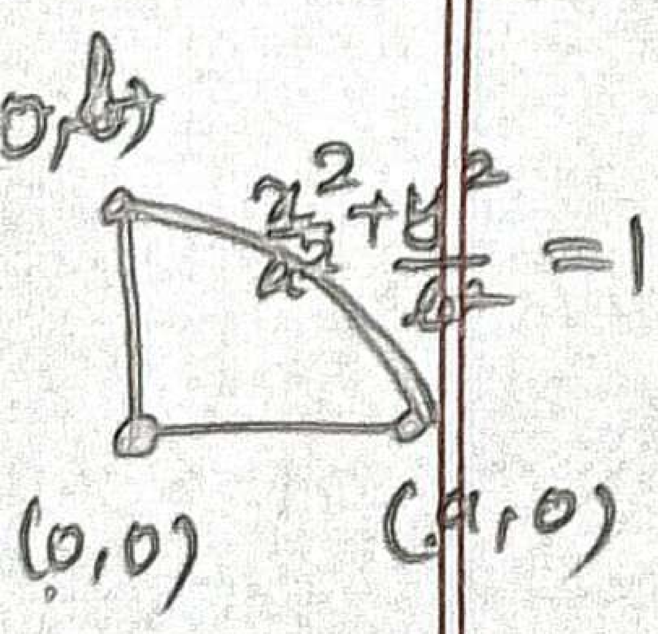
$$\text{Area } (A) = \iint_R dx\, dy$$

$$A = 4A_1 = 4\int_{x=0}^{a}\int_{y=0}^{(b/a)\sqrt{a^2-x^2}} dy\, dx.$$

$dx\, dy - \int_0$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$



$$= 4\int_{x=0}^{a} [y]_0^{(b/a)\sqrt{a^2-x^2}} dx.$$

$$y^2 = b^2\left[1 - \frac{x^2}{a^2}\right]$$

$$= 4\int_{x=0}^{a} \frac{b}{a}\sqrt{a^2-x^2}\, dx$$

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$y = b\sqrt{\frac{a^2-x^2}{a^2}}$$

$\int\sqrt{a^2-x^2}\, dx.$

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

$$= \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_{x=0}^{a}$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\left(\sin^{-1}1 - \sin^{-1}(0)\right)\right] \quad \Big| \; \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2}$$

$$= \frac{4b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \pi ab$$

Thus the required area $(A) = \pi ab$ sq units

2) Find the double integration the area enclosed by the curve $r = a(1+\cos\theta)$ between $\theta = 0$ and $\theta = \pi$? about the inital line.

Sol Area $A = \iint r\,dr\,d\theta$

r varies from 0 to $a(1+\cos\theta)$

θ varies from 0 to π

$$A = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta.$$

$$A = \int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_{r=0}^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$A = \frac{a^2}{2}\int_{\theta=0}^{\pi}\left[2\cos^2(\theta/2)\right]^2 d\theta$$

$$A = 2a^2\int_0^{\pi}\cos^4(\theta/2)\,d\theta.$$

$$\theta/2 = \phi \Rightarrow d\theta = 2\,d\phi$$

ϕ varies from 0 to $\pi/2$.

$$A = 2a^2\int_0^{\pi/2}\cos^4\phi\; 2\,d\phi$$

$$A = 4a^2\int_0^{\pi/2}\cos^4\phi\,d\phi$$

$= 4a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$ by the reduction formula

Thus the required area $(A) = \pi ab$ sq. units

3) ~~Find by double integration the area enclosed~~ bounded

3) Find the volume of the tetrahedron bounded by the planes $x=0,\ y=0,\ z=0$ $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Sol $V = \iiint dx\, dy\, dz.$

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \Rightarrow z = c\left(1-x/a-y/b\right)$

If $z=0$, then $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow y = b\left(1-x/a\right)$

If $z=0$, $y=0$ then $x=a$.

$$V = \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)}\int_{z=0}^{c(1-x/a-y/b)} dz\, dy\, dx.$$

$$= \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)} z\Big]_0^{c(1-x/a-y/b)} dy\, dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\, dx$$

$$= c\int_{x=0}^{a}\int_{y=0}^{b(1-x/a)} \left(1-\frac{x}{a}-\frac{y}{b}\right) dy\, dx.$$

$$= c\int_{x=0}^{a} \cancel{\int_{y=0}^{b(1-x/a)}} \left(1-\frac{x}{a}-\frac{y^2}{2b}\right) \cancel{y}\Big]_{y=0}^{b(1-x/a)} dx$$

$$= c\int_{x=0}^{a} y - \frac{xy}{a} - \frac{y^2}{2b}\Big]_{y=0}^{b(1-x/a)} dx$$

$$= c\int_{x=0}^{a} b\left(1-\frac{x}{a}\right) - \frac{x}{a}\left(b\left(1-\frac{x}{a}\right)\right) - \frac{1}{2b}\left[b^2\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= c\int_{x=0}^{0} b\left(1-\frac{x}{a}\right) - \frac{x}{a}\left(b\left(1-\frac{x}{a}\right)\right) - \frac{1}{2}\left[b\left(1-\frac{x}{a}\right)\right] dx$$

$$= c\int_{x=0}^{a} b\left(1-\frac{x}{a}\right)\left[1 - \frac{x}{a} - \frac{1}{2}\left(\frac{1}{1}-\frac{x}{a}\right)\right] dx$$

$$= bc\int_{x=0}^{a} \cancel{b}\left(1-\frac{x}{a}\right)\left[\cancel{1-\frac{x}{a}} - \frac{1}{2} + \frac{x}{a}\right] dx$$

$$= cb\int_{x=0}^{a} \cdot \left(1-\frac{x}{a}\right)\frac{1}{2}\left[1-\frac{x}{a}\right) dx$$

$$= \frac{bc}{2}\int_{x=0}^{a} \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[-\frac{a}{3}\left(1-\frac{x}{a}\right)^3\right]_{x=0}^{a}$$

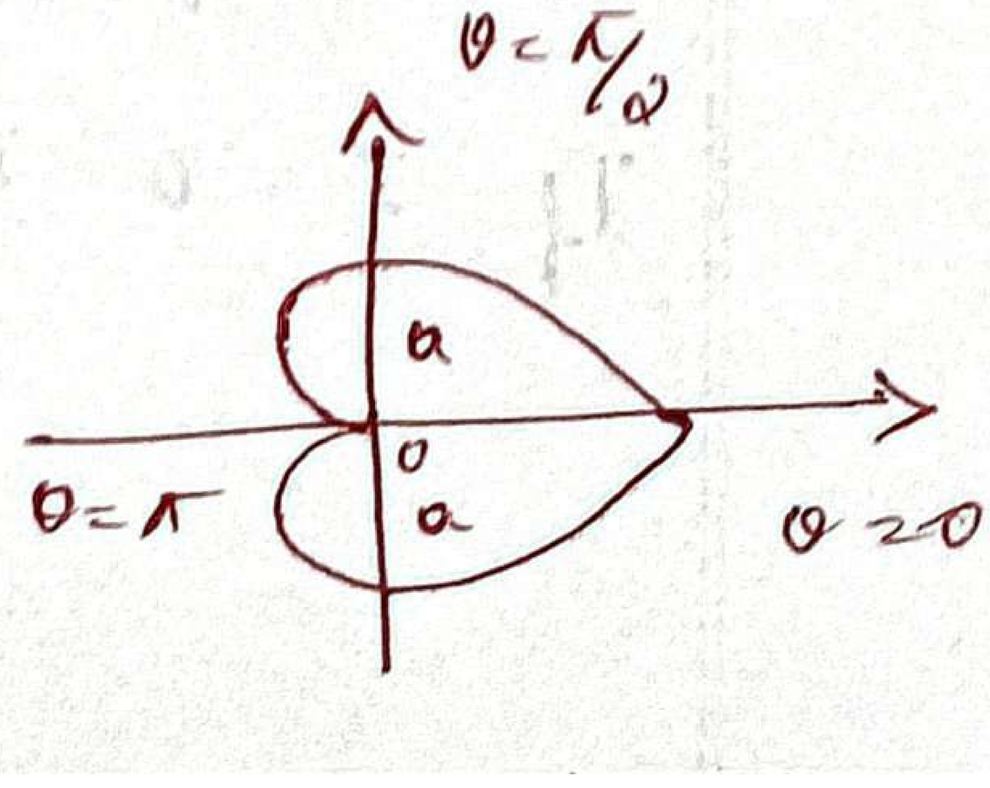
$$V = -\frac{abc}{6}(0-1) = \frac{abc}{6}$$

Thus the required Volume $(V) = abc/6$ cubic units

4) Find the Volume generated by the revolution of the Cardioide $r = a(1+\cos\theta)$ about the inital line,

Sol Volume of the solid of revolution in polars is given by $V = \iint_A 2\pi r^2 \sin\theta \, dr \, d\theta$

$$V = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta.$$



$$= \int_{\theta=0}^{\pi} 2\pi \frac{r^3}{3} \Big]_{r=0}^{a(1+\cos\theta)} \sin\theta \, d\theta.$$

$$= \frac{2\pi}{3} \int_{\theta=0}^{\pi} a^3 (1+\cos\theta)^3 \sin\theta \, d\theta.$$

$= 1+\cos\theta = t$ $\qquad$ $\theta = 0, \; t = 2.$

$-\sin\theta \, d\theta = dt$ $\qquad$ $\theta = \pi, \; t = 0.$

$$V = \frac{2\pi a^3}{3} \int_2^0 t^3 (-dt).$$

$$V = -\frac{2\pi a^3}{3} \int_2^2 t^3 \, dt$$

$$V = -\frac{2\pi a^3}{3} \left(\frac{t^4}{4}\right)_2^0 = \frac{8\pi a^3}{3}$$

Thus the required volume $(V) = \frac{8\pi a^3}{3}$ cubic units

5) A pyramid is bounded by three coordinate planes and the plane $x + 2y + 3z = 6$ compute the volume by double integration.

$$V = \iint z \, dx \, dy.$$

$$x + 2y + 3z = 6.$$

$$\frac{x}{6} + \frac{y}{3} + \frac{z}{2} = 1$$

$$z = 2\left[1 - (x/6) - (y/3)\right]$$

If $z = 0$, $\frac{z}{2} = \left[1 - \left(\frac{x}{6}\right) - \left(y/3\right)\right]$

$$0 = 1 - \left(\frac{x}{6}\right) - (y/3)$$

$$1 = \frac{x}{6} + \frac{y}{3} \Rightarrow y = 3\left[1 - (x/6)\right]$$

If $z=0$, $y=0$ then $x=6$.

$$V = \int_{x=0}^{6} \int_{y=0}^{3[1-(x/6)]} 2[1-(x/6)-(y/3)]\, dy\, dx$$ (6)

$$V = 2\int_0^6 \left[y - \frac{xy}{6} - \frac{y^2}{6} \right]_0^{3[1-(x/6)]}$$

$$V = 2\int_0^6 3\left[1-\frac{x}{6}\right] - \frac{x}{6}\left[3\left(1-\frac{x}{6}\right)\right] - \frac{1}{6}\left(3^2\left(1-\frac{x}{6}\right)^2\right) dx$$

$$V = 2\int_0^6 3\left(1-\frac{x}{6}\right)\left[1 - \frac{x}{6} - \frac{1}{6}\left(3\left(1-\frac{x}{6}\right)\right)\right] dx.$$

$$V = 2\int_0^6 3\left(1-\frac{x}{6}\right)\left[1 - \frac{x}{6} - \frac{1}{2}\left(1-\frac{x}{6}\right)\right] dx.$$

$$V = 2\int_0^6 3\left(1-\frac{x}{6}\right)\left[1 - \frac{x}{6} - \frac{1}{2} + \frac{x}{12}\right) dx$$

$\frac{1}{1}-\frac{1}{2}$

$\frac{2-1}{2}$ 1

$\frac{x}{12}-\frac{x}{6}$

$\frac{x-2x}{12}$

$$V = 2\int_0^6 3\left(1-\frac{x}{6}\right)\left(\frac{1}{2} - \frac{x}{12}\right) dx$$

$$V = 2\int_0^6 3\left(1-\frac{x}{6}\right)\frac{1}{2}\left(1-\frac{x}{6}\right) dx$$

$$V = \frac{6}{2}\int_0^6 \left(1-\frac{x}{6}\right)^2 dx$$

$$V = \left[\frac{72+1-12}{72}\right]$$

$$V = 3\int_0^6 1^2 + \frac{x^2}{36} - \frac{2x}{6}\, dx$$

$$V = \frac{61}{72}$$

$$V = 3\left[x + \frac{x^3}{3\cdot 36} - \frac{2x^2}{2\cdot 6} \right]_0^1$$

X

$$V = 3\left[1 + \frac{1}{72} - \frac{2}{12}\right]$$

$$V = 3\left[1 + \frac{1}{72} - \frac{1}{6}\right]$$

Beta and Gamma functions

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad , (m,n>0)$$

is called the Beta function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \quad (n>0)$$

is called the Gamma function

$$B(m,n) = 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta.$$

$$\Gamma(n) = 2\int_0^\infty e^{-t^2} t^{2n-1}\,dt$$

Properties of Beta and Gamma functions

1) $B(m,n) = B(n,m)$

Proof: we have $B(m,n) = \int_{x=0}^1 x^{m-1}(1-x)^{n-1}\,dx.$

$1-x = y \Rightarrow x = 1-y.$

$dx = -dy$

when $x=0$, $y=1$

$x=1$, $y=0$

$$B(m,n) = \int_{y=1}^0 (1-y)^{m-1} y^{n-1}(-dy).$$

$$B(m,n) = \int_{y=0}^1 y^{n-1}(1-y)^{m-1}\,dy.$$

$$B(m,n) = B(n,m)$$

2) ① $\Gamma(n+1) = n\Gamma(n)$ ② $\Gamma(n+1) = n!$ for a positive integer n.

Proof

By definition

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-1}\, dx$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n}\, dx$$

$$\Gamma(n+1) = \int_0^\infty x^{n} e^{-x}\, dx$$

Integrating by parts we get

$$\cancel{\Gamma(n+1) = x^n(-e^{-x}) - \int n x^{n-1}}$$

$$\Gamma(n+1) = \left[x^n(-e^{-x})\right]_0^\infty - \int_0^\infty (-e^{-x})\, n x^{n-1}\, dx.$$

$x^n/e^x \to 0$ as $x \to \infty$ by L'Hospital rule.

$$\Gamma(n+1) = (0-0) + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

②

$\Gamma(n) = (n-1)\Gamma(n-1)$, $\Gamma(n-1) = (n-2)\Gamma(n-2)$ ....

$\Gamma(3) = 2\Gamma(2)$, $\Gamma(2) = 1\Gamma(1)$

Now we have

$$\Gamma(n+1) = n\{(n-1)\cdot\Gamma(n-1)\} = n(n-1)\{(n-2)\Gamma(n-2)\} \text{ etc}$$

$$\Gamma(n+1) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\,\Gamma(1) = n!\,\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty e^{-x} x^0\, dx = -\left[e^{-x}\right]_0^\infty = -(0-1) = 1$$

$\Gamma(n+1) = n!$ for a positive integer n.

Imp

Relationship between Beta and Gamma function

$$B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

$x \to n$

Proof We have by the definition of Beta and Gamma functions.

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad \to (1) \text{ (Polar form)}$$

$$\Gamma(n) = 2\int_0^{\infty} e^{-x^2} x^{2n-1}\, dx \quad \to (2)$$

x, y

$$\Gamma(m) = 2\int_0^{\infty} e^{-y^2} y^{2m-1}\, dy \quad \to (3)$$

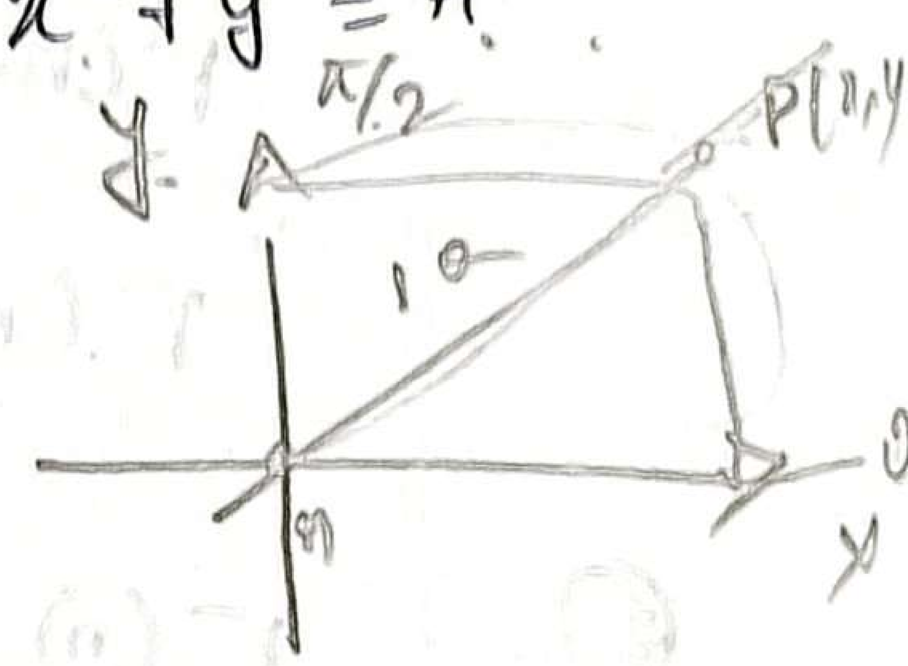
r

$$\Gamma(m+n) = 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1}\, dr \quad \to (4)$$

$$\Gamma(m)\cdot\Gamma(n) = 4\int_{y=0}^{\infty}\int_{x=0}^{\infty} e^{-(x^2+y^2)} x^{2n-1} y^{2m-1}\, dx\, dy \quad \to (5)$$

Let us evaluate RHS by changing into polars

Putting $x = r\cos\theta$, $y = r\sin\theta$ $\quad x^2+y^2 = r^2$.



$dx\,dy = r\,dr\,d\theta$.

r varies from 0 to ∞.

θ varies from 0 to $\pi/2$.

We $\Gamma(m)\cdot\Gamma(n) = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}(r\cos\theta)^{2n-1}(r\sin\theta)^{2m-1}\, r\, dr\, d\theta$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r^{2m+2n-1} \sin^{2m-1}\theta \cos^{2n-1}\theta \, dr\, d\theta$$

$$= \left[2\int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1}\, dr\right]\left[2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta\right]$$

$\Gamma(m)\cdot\Gamma(n) = \Gamma(m+n)\cdot B(m,n)$ by using (1) and (4)

$$B(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

Corollary : To show that $\Gamma(1/2) = \sqrt{\pi}$

Putting $n = \frac{1}{2}$ in this result

$$B(1/2, 1/2) = \frac{\Gamma(1/2) \cdot \Gamma(1/2)}{\Gamma(1)} \quad \text{but } \Gamma(1) = 1$$

$$B(1/2, 1/2) = \{\Gamma(1/2)\}^2 \longrightarrow (6).$$

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

$$B(1/2, 1/2) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta.$$

$$= 2\int_0^{\pi/2} 1 \, d\theta.$$

$$= 2[\theta]_0^{\pi/2}$$

$$B(1/2, 1/2) = \pi$$

From (6) we have.

$$B(1/2, 1/2) = \{\Gamma(1/2)\}^2$$

$$B\cancel{(m,n)} = 2\cancel{\int_0^{\pi/2} \sin^{2m-1}}$$

$$\pi = \{\Gamma(1/2)\}^2$$

$$\Gamma(1/2) = \sqrt{\pi}$$

Duplication formula

$$\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\Gamma(m)\,\Gamma(m + 1/2)$$

$$B(m, 1/2) = 2^{2m-1} B(m, m)$$

$$\Gamma(1/4)\,\Gamma(3/4) = \pi\sqrt{2}$$