Module 2 : Calculus

Vector differentiation:

Scalar and Vector fields:

If every point (x, y, z) of a region R in space there corresponds a scalar $\phi(x, y, z)$ then ϕ is called a Scalar point function and a scalar field ϕ is defined in R.

Eg: 1) $\phi = x^2 + y^2 + z^2$ 2) $\phi = xy^2z^3$

If every point (x, y, z) of a region R in space there corresponds a vector $\vec{A}(x, y, z)$ then $\vec{A}$ is called Vector point function and a vector field $\vec{A}$ is defined in R.

Eg: $\vec{A} = x^2 i + y^2 j + z^2 k$ ii) $\vec{A} = xyz i + y^2 j + z k$.

Gradient of a Scalar:

If ϕ is a scalar function, then the gradient of ϕ is a vector defined by.

grad ϕ or $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

Unit normal Vector

If $\phi(x, y, z) = C$ represents a surface then the unit normal vector to the surface is defined by

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

Directional derivative

Directional derivative of ϕ along $\vec{a} = \nabla\phi \cdot \hat{a}$

* Maximum directional derivative $= |\nabla\phi|$

Divergence of a vector:

If $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ then divergence of $\vec{f}$ is a scalar defined by

$$\text{div } \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Solenoidal Vector:

A vector $\vec{f}$ is said to be solenoidal iff $\text{div } \vec{f} = 0$

Curl of Vector

If $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ then the curl of a vector is a vector defined by

$$\text{Curl } \vec{f} \text{ or } \nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{Curl } \vec{f} = i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - j\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + k\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Irrotational Vector

A vector $\vec{f}$ is said to be irrotational iff $\text{curl } \vec{f} = \vec{0}$

Problems

1) If $\phi = x^2y^3z^4$ find $\nabla\phi$, $|\nabla\phi|$ at $(1,-1,1)$

Sol: Given $\phi = x^2y^3z^4$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = \frac{\partial}{\partial x}(x^2y^3z^4)\hat{i} + \frac{\partial}{\partial y}(x^2y^3z^4)\hat{j} + \frac{\partial}{\partial z}(x^2y^3z^4)\hat{k}$$

$$\nabla\phi = 2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}$$

At $(1,-1,1)$

$$\nabla\phi = 2(1)(-1)^3(1)^4\hat{i} + 3(1)^2(-1)^2(1)^4\hat{j} + 4(1)^2(-1)^3(1)^3\hat{k}$$

$$\nabla\phi = -2\hat{i} + 3\hat{j} + 4\hat{k}$$

$$|\nabla\phi| = \sqrt{(-2)^2+3^2+4^2} = \sqrt{4+9+16} = \sqrt{29}$$

2) Find the unit normal vector to the surface $xy + yz + zx = c$ at the point $(-1,2,3)$

Sol: $\nabla\phi = xy + yz + zx - c$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = \frac{\partial}{\partial x}(xy+yz+zx)\hat{i} + \frac{\partial}{\partial y}(xy+yz+zx)\hat{j} + \frac{\partial}{\partial z}(xy+yz+zx)\hat{k}$$

$$\nabla\phi = (y+z)\hat{i} + (x+z)\hat{j} + (y+x)\hat{k}$$

At $(-1,2,3)$.

$$\nabla\phi = (2+3)\hat{i} + (-1+3)\hat{j} + (2-1)\hat{k}$$

$$\nabla\phi = 5\hat{i} + 2\hat{j} + \hat{k}$$

At $(-1,2,3)$

$$\nabla\phi = 5\hat{i} + 2\hat{j} + \hat{k}$$

$$\cancel{|\nabla\phi| = \sqrt{\;^2+\;^2+1}}$$

$$\cancel{|\nabla\phi| = \sqrt{i^2+j^2+k^2}}$$

$$|\nabla\phi| = \sqrt{25+4+1}$$

$$|\nabla\phi| = \sqrt{30}$$

Unit normal vector

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{5\hat{i}+2\hat{j}+\hat{k}}{\sqrt{30}}$$

3) Find the angle between the directions of the normal to surface $x^2yz=1$ at the points $(-1,1,1)$ and $(1,-1,-1)$

Sol Let $\phi = x^2yz-1$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = \frac{\partial}{\partial x}(x^2yz-1)\hat{i} + \frac{\partial}{\partial y}(x^2yz-1)\hat{j} + \frac{\partial}{\partial z}(x^2yz-1)\hat{k}$$

$$\nabla\phi = 2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k} \longrightarrow (1)$$

From eq (1)

At $(-1,1,1)$

$$\nabla\phi = -2i+j+k$$

$$|\nabla\phi| = \sqrt{(-2)^2+1^2+1}$$

$$= \sqrt{4+1+1}$$

$$= \sqrt{6}$$

$$\hat{n}_1 = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2i+j+k}{\sqrt{6}} \rightarrow (2)$$

(ii) At $(1,-1,-1)$.

$$\nabla\phi = 2i - j - k$$

$$|\nabla\phi| = \sqrt{2^2+(-1)^2+(-1)^2}$$

$$|\nabla\phi| = \sqrt{6}.$$

$$\hat{n}_2 = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2i-j-k}{\sqrt{6}.} \rightarrow (3)$$

$$\cos\theta = \hat{n}_1 \cdot \hat{n}_2 = \frac{(-2i+j+k)(2i-j-k)}{6}$$

$$= \frac{-4-1-1}{6} = \frac{-6}{6} = -1$$

$$\cos\theta = -1$$

$$\theta = 180^\circ = \pi$$

4) Find the angle between the surface. $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at $(1,1,1)$

Sol: Let $\phi = x \log z - y^2 + 1$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = \frac{\partial}{\partial x}(x\log z - y^2+1) + \frac{\partial}{\partial y}(x\log z - y^2+1)\hat{j}$$

$$\frac{\partial}{\partial z}(x\log z - y^2+1)\hat{k}$$

$\nabla\phi = \log z\,\hat{i} - 2y\hat{j} + x\hat{k}$

At $(1,1,1)$

$\nabla\phi = -2j + k$

$|\nabla\phi| = \sqrt{(-2)^2 + 1^2}$

$|\nabla\phi| = \sqrt{4+1} = \sqrt{5}$

$\hat{n}_1 = \dfrac{-2j + k}{\sqrt{5}}$

Let $\phi = x^2y + z - 2$

$\nabla\phi = \dfrac{\partial\phi}{\partial x}\hat{i} + \dfrac{\partial\phi}{\partial y}\hat{j} + \dfrac{\partial\phi}{\partial z}\hat{k}$

$\nabla\phi = \dfrac{\partial}{\partial x}(x^2y + z - 2)\hat{i} + \dfrac{\partial}{\partial y}(x^2y + z - 2)\hat{j} + \dfrac{\partial}{\partial z}(x^2y + z - 2)\hat{k}$

$\nabla\phi = 2xy\,\hat{i} + x^2\hat{j} + \hat{k}$

At $(1,1,1)$

$\nabla\phi = 2\hat{i} + \hat{j} + \hat{k}$

$|\nabla\phi| = \sqrt{2^2 + 1^2 + 1} = \sqrt{6}$

$\hat{n}_2 = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$

$\cos\theta = \hat{n}_1 \cdot \hat{n}_2 = \left(\dfrac{-2j + k}{\sqrt{5}} \cdot \dfrac{2i + j + k}{\sqrt{6}}\right)$

$\cos\theta = \dfrac{-2 + 1}{\sqrt{30}} = \dfrac{-1}{\sqrt{30}}$

$\theta = \cos^{-1}\left(-\dfrac{1}{\sqrt{30}}\right)$

6) Find the directional derivative of $\cancel{\phi = x^2yz}$ $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ along $\vec{a} = 2i - j - 2k$.

Sol: Let $\phi = x^2yz + 4xz^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

At $(1,-2,-1)$

$$\nabla\phi = 8i - j - 10k.$$

Directional derivation of ϕ along $\vec{a}$:

$$\vec{a} = \nabla\phi \cdot \hat{a} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (8i - j - 10k)\frac{(2i - j - 2k)}{\sqrt{9}}$$

$$= \frac{16 + 1 + 20}{\sqrt{3}} = \frac{37}{3}$$

7) In which direction the directional derivative of x^2yz^3 is maximum at $(2,1,-1)$ and find the magnitude of this maximum.

Sol Let $\phi = x^2yz^3$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla \phi = 2xyz^3 \hat{i} + x^2 z^3 \hat{j} + 3x^2 y z^2 k$$

At $(2, 1, -1)$

$\nabla \phi = -4i - 4j + 12K$. is the required D.D.

Magnitude $= \sqrt{(-4)^2 + (-4)^2 + (12)^2}$

$= \sqrt{16 + 16 + 144}$

$= \sqrt{176} = \sqrt{\cancel{16 \times 121}}\ \sqrt{11 \times 16} = 4\sqrt{11}$

144
32
176

121
484

Maximum directional derivative $= 4\sqrt{121}$

7) Find the maximum directional derivative of $\phi = xy + yz + zx$ at $(1,1,1)$

Sol : Given $\phi = xy + yz + zx$

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

$$\nabla \phi = (y+z) \hat{i} + (x+z) \hat{j} + (x+y) \hat{k}$$

$$\nabla \phi_{(1,1,1)} = 2i + 2j + 2k$$

Maximum D.D $= \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$

8) Find the divergence of

$f = x^3 z \hat{i} + y^3 x \hat{j} + z^3 y \hat{k}$

504
16
4820
-144
376

$\vec{f} = x^3 z\,\hat{i} + y^3 z\,\hat{j} + z^3 y\,\hat{k}$

$\operatorname{div}\vec{f} = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$

~~$\operatorname{div}\vec{f} = \dfrac{\partial}{\partial x}(x^3$~~

6) Find the directional derivative along $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ along $2i - 3j + 6k$

Sol:

$\phi = 4xz^3 - 3x^2y^2z$

$\nabla\phi = \dfrac{\partial\phi}{\partial x} i + \dfrac{\partial\phi}{\partial y}\hat{j} + \dfrac{\partial\phi}{\partial z}\hat{k}$

$\nabla\phi = \dfrac{\partial(4xz^3)}{\partial x}\hat{i} + \dfrac{\partial}{\partial y}(4xz^3 - 3x^2y^2z)\hat{j} + \dfrac{\partial}{\partial z}(4xz^3 - 3x^2y^2z)\hat{k}$

$\nabla\phi = (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$

$\nabla\phi = (4(2)^3 - 6(2)(-1)^2(2))\hat{i} - 6(2)^2(-1)(2)\hat{j} + 12(2)(2)^2 - 3(2)^2(-1)^2\hat{k}$

$\nabla\phi = 8\hat{i} + 48\hat{j} + 84\hat{k}$

$|\vec{n}| = \sqrt{\text{The unit in the direction of}}\ 2\hat{i} - 3\hat{j} + 6\hat{k}$

$\hat{n} = \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{2^2 + (-3)^2 + 6^2}} = \dfrac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}} = \dfrac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{49}\ (7)}$

The directional derivative $= \nabla\phi\,\hat{n} = \nabla\phi \cdot \dfrac{\vec{n}}{|\vec{n}|}$

$DD = 8\hat{i} + 48\hat{j} + 84\hat{k} \cdot \left(\dfrac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}\right)$

$\nabla\phi \cdot \hat{n} = \dfrac{16 - 144 + 504}{7}$

$\nabla\phi \cdot \hat{n} = \dfrac{376}{7}$

9) Find the divergence of $f = x^3 z \hat{i} + y^3 x \hat{j} + z^3 y \hat{k}$

Soln: $\vec{f} = x^3 z \hat{i} + y^3 x \hat{j} + z^3 y \hat{k}$

$$\text{div } \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\cancel{\text{div } \vec{f} = \frac{\partial}{\partial x}(x^3 z + y^3 x + z^3 y)} + \frac{\partial}{\partial y}(x^3 z + y^3 x + z^3 y) + \frac{\partial}{\partial z}(x^3 z + y^3 x + z^3 y)$$

$$\cancel{\text{div } \vec{f} = 3x^2 z + y^3 + 3y^2 x + z^3 + x^3 + 3z^2 y}$$

$$\text{div } \vec{f}$$

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9) Find div $\vec{F}$ and curl $\vec{F}$ where

$\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$

Soln: $\vec{F} = \text{grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

$$\vec{F} = \frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)\hat{i} + \frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)\hat{j} + \frac{\partial}{\partial z}(\ )\hat{k}$$

$$\vec{F} = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

$$\text{div } \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{(3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}\right.$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$\text{div } \vec{F} = 6x + 6y + 6z$$

$$\text{div } \vec{F} = 6(x+y+z)$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right] - j\left[\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right]$$

$$= i\left[-3x-(-3x)\right] - j\left[(-3y)(+3y)\right] + k\left[-3z-(-3z)\right]$$

$$= i\left[-3x+3x\right] - j\left[3y+3y\right] + k\left[-3z+3z\right].$$

$$= i\,[0] - j\,[0] + k\,[0]$$

$$\text{curl } \vec{F} = 0$$

$$\text{div } \vec{F} = 6(x+y+z)$$

$$\text{Curl } \vec{F} = \vec{0}$$

If $\vec{F} = \nabla(xy^3z^2)$ find div $\vec{F}$ and Curl $\vec{F}$ at the point $(1,-1,1)$

Let $\phi = xy^3z^2$

$$\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\vec{F} = \nabla\phi = \frac{\partial}{\partial x}(xy^3z^2)\,\hat{i} + \frac{\partial}{\partial y}(xy^3z^2)\,\hat{j} + \frac{\partial}{\partial z}(xy^3z^2)\,\hat{k}$$

$$\vec{F} = \nabla\phi = y^3z^2\hat{i} + 3y^2xz^2\hat{j} + 2xy^3z\hat{k}$$

$$\text{div } \vec{F} = \nabla\cdot\vec{F}$$

$$\text{div } \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(y^3z^2\hat{i} + 3y^2xz^2\hat{j} + 2xy^3z\hat{k})$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(y^3z^2) + \frac{\partial}{\partial y}(3y^2xz^2) + \frac{\partial}{\partial z}(2xy^3z)$$

$$\text{div } \vec{F} = 0 + 6xyz^2 + 2xy^3$$

$$= 2xy(3z^2+y^2)$$

div $\vec{F}$ at $(1,-1,1)$

$$\text{div } \vec{F} = 2xy(3z^2+y^2)$$
$$= 2(1)(-1)(3+(-1)^2)$$
$$= -2(3+1) = -2(4) = \underline{-8}$$

Also curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$

$$= i\left[\frac{\partial}{\partial y}(2xy^3z) - \frac{\partial}{\partial z}(3xy^2z^2)\right] - j\left[\frac{\partial}{\partial x}(2xy^3z) - \frac{\partial}{\partial z}(y^3z^2)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(3xy^2z^2) - \frac{\partial}{\partial y}(y^3z^2)\right]$$

$$\text{curl } \vec{F} = i[6xy^2z - 6xy^2z] - j[2y^3z - 2zy^3] + k[3y^2z^2 - 3y^2z^2]$$

$$\text{curl } \vec{F} = i(0) - j(0) + k(0)$$

$$\text{curl } \vec{F} = \underline{\vec{0}}$$

Thus the required div $\vec{F} = -8$

curl $\vec{F} = \vec{0}$

11) If $\vec{F} = (3x^2y - z)\hat{i} + (xz^3 + y^4)\hat{j} - 2x^3z^2 k$

grad (div $\vec{F}$) at $(2,-1,0)$.

$\Rightarrow$ div $\vec{F} = \nabla \cdot \vec{F}$

$$\text{div } \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \{(3x^2y - z)i + (xz^3+y^4)\hat{j} - 2x^3z^2k\}$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(3x^2y - z) + \frac{\partial}{\partial y}(xz^3+y^4) + \frac{\partial}{\partial z}(-2x^3z^2)$$

$$\text{div } \vec{F} = 6xy + 4y^3 - 4x^3z = \phi$$

$$\nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\nabla \phi = \frac{\partial}{\partial x}(6xy + 4y^3 - 4x^3 z)\hat{i} + \frac{\partial}{\partial y}(6xy + 4y^3 - 4x^3 z)\hat{j} + \frac{\partial}{\partial z}(6xy + 4y^3 - 4x^3 z)\hat{k}$$

$$\nabla \phi = (6y - 12x^2 z)\hat{i} + (6x + 12y^2)\hat{j} + (-4x^3)\hat{k}$$

At $(2,-1,0)$

$$\nabla \phi = (6(-1) - 12(2)^2(0))\hat{i} + (6(2) + 12(-1)^2)\hat{j} + (-4(2)^3)\hat{k}$$

$$\nabla \phi = -6\hat{i} + 24\hat{j} - 32\hat{k}$$

12) If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4 k$ find $\nabla \cdot \vec{A}$, $\nabla \times \vec{A}$ and $\nabla \cdot (\nabla \times \vec{A})$. at the point $(1,-1,1)$

Sol : $\nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4 k)$

$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-2x^2yz) + \frac{\partial}{\partial z}(2yz^4)$$

$$\nabla \cdot \vec{A} = z^3 - 2x^2 z + 8yz^3 \quad \text{At } (1-1,1)$$

$$\nabla \cdot \vec{A} = 1 - 2 - 8 = -9 \Rightarrow \text{div } \vec{A}$$

$$\nabla \times \vec{A} \text{ or Curl } \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right] - j\left[\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right] + k\left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]$$

$$= i(2z^4 + 2x^2y) - j(3z^2x) + k(-4xyz)$$

$$= 2(x^2y + z^4)i - (3z^2x)\hat{j} - (4xyz)\hat{k}$$

$\nabla \times \vec{A}$ att $(1,-1,1)$

$$\nabla \times \vec{A} = 3j + 4k$$

$\nabla \cdot (\nabla \times \vec{A})$ or div (curl $\vec{A}$).

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot\left[2(x^2y+z^4)i + 3xz^2 j - 4xyzk\right]$$

$$= \frac{\partial}{\partial x}[2(x^2y+z^4)] + \frac{\partial}{\partial y}[3xz^2] + \frac{\partial}{\partial z}(-4xyz)$$

$$= 4xy - 4xy$$

$$= 0$$

Thus the required $\nabla \cdot \vec{A} = -9$

$$\nabla \times \vec{A} = 3\hat{j} + 4k$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

Use regula falsi method to obtain the eq.

$2x - \log_{10} x = 7$ which lies between 3.5 and 4.

Sol $f(x) = 2x - \log_{10} x - 7$

$f(3.5) = 2(3.5) - \log_{10}(3.5) - 7 = -0.5441$

$f(4) = 2(4) - \log_{10}(4) - 7 = 0.3979$

$a = 3.5 \quad f(a) = -0.5441$

$b = 4 \quad f(b) = 0.3979.$

Ist approximation:

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}.$$

$$x_1 = \frac{3.5(0.3979) - 4(-0.5441)}{0.3979 - (-0.5441)} = \frac{3.5691}{0.9420} = 3.7889$$

IInd approximation

$f(x_1) = 2x - \log_{10} x - 7$

$f(3.7889) = 2(3.7889) - \log(3.7889) - 7$

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

$f(22) = 352$

$f(x) = x^3 - 3x + 4 = 0$ 3 decimal

$f(0) = 4 > 0$

$f(1) = 1 - 3 + 4 = 0 = 2 > 0$

$f(2) = 8 - 6 + 4 = 6 > 0$

$f(3) = 27 - 9 + 4 = 22 > 0$

$f(-1) = 6 > 0$

$f(-2) = 2 > 0$

$f(-3) = -14 < 0$

So the root lies between $(-3, -2)$: $b - a$; $-2 - (-3)$

$-2 + 3 = +1$

$f(2.1) = 1.039 > 0$

$f(-2.2) = -0.048 < 0$

The root lies in $(-2.2, -2.1)$

I iteration $a = -2.2$ $f(a) = -0.048$

$b = -2.1$ $f(b) = 1.039$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_1 = \frac{(-2.2)(1.039) - (2.1)(0.048)}{1.039 + 0.048} = -2.196$$

II iteration $f(x) = x^3 - 3x + 4$

$f(-2.196) = -0.002 < 0$

The root lies in $(-2.196, -2.1)$

$a = -2.196$ $f(a) = -0.002$

$b = -2.1$ $f(b) = 1.039$

$$x_2 = \frac{(-2.196)(1.039) - (2.1)(0.002)}{1.039 + 0.002} = -2.196$$

13) If $\phi = \frac{xz}{x^2+y^2}$ at $(1,-1,1)$ in the direction (along) $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$

Sol $\phi = \frac{xz}{x^2+y^2}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}\left(\frac{xz}{x^2+y^2}\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{xz}{x^2+y^2}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{xz}{x^2+y^2}\right)\hat{k}$$

$$= \left[\frac{(x^2+y^2)(z) - xz(2x)}{(x^2+y^2)^2}\right]\hat{i} + \left[\frac{(x^2+y^2)(0) - xz(2y)}{(x^2+y^2)^2}\right]\hat{j} + \left[\frac{(x^2+y^2)x - \cancel{xz(0)}}{(x^2+y^2)^2}\right]\hat{k}$$

$$= \left[\frac{zx^2 + zy^2 - 2zx^2}{(x^2+y^2)^2}\right]\hat{i} + \left[\frac{-2xyz}{(x^2+y^2)^2}\right]\hat{j} + \left[\frac{(x^2+y^2)x}{(x^2+y^2)^2}\right]\hat{k}$$

$$= \left[\frac{-zx^2 + zy^2}{(x^2+y^2)^2}\right]\hat{i} - \frac{2xyz}{(x^2+y^2)^2}\hat{j} + \frac{x}{(x^2+y^2)^2}\hat{k}$$

$$\nabla\phi = \frac{z(y^2-x^2)}{(x^2+y^2)^2}\hat{i} - \frac{2xyz}{(x^2+y^2)^2}\hat{j} + \frac{x}{(x^2+y^2)^2}\hat{k}$$

At $(1,-1,1)$

$$\nabla\phi = \frac{1(1-1)}{(1+1)^2}\hat{i} - \frac{2(-1)}{4}\hat{j} + \frac{1}{2}\hat{k}$$

$$\nabla\phi = \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$$

$$\nabla\phi = \frac{1}{2}(\hat{j} + \hat{k})$$

The unit normal in the direction of $\vec{a} = \hat{i} - 2\hat{j} + k$.

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{6}}$$

Thus the required directional derivative is

$$\nabla\phi \cdot \hat{a} = \frac{1}{2}(\hat{j} + \hat{k}) \cdot \frac{(\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{6}}$$

$$= \frac{0 - 2 + 1}{2\sqrt{6}} = \frac{-1}{2\sqrt{6}}$$

14) Find curl (curl $\vec{A}$) given that $\vec{A} = xy\hat{i} + y^2z\hat{j} + z^2y\hat{k}$

$$\text{Curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2z & z^2y. \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z^2y) - \frac{\partial}{\partial z}(y^2z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(z^2y) - \frac{\partial}{\partial z}(xy)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y^2z) - \frac{\partial}{\partial y}(xy)\right]$$

$$= \hat{i}\left[z^2 - y^2\right] - \hat{j}\left[0\right] + \hat{k}\left[-x\right]$$

$$\nabla \times \vec{A} = (z^2 - y^2)\hat{i} - x\hat{k}$$

$$\therefore \text{curl } (\text{curl } \vec{A}) = \nabla \times (\nabla \times \vec{A}).$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z^2-y^2) & 0 & -x \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-x) - \frac{\partial}{\partial z}(0)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial z}(z^2-y^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(z^2-y^2)\right]$$

$$= \hat{i}(0) - \hat{j}(-1-2z) + \hat{k}\left[+2y\right]$$

$$\text{curl}(\text{curl } \vec{A}) = (1+2z)\hat{j} + 2y\hat{k}$$

Solenoidal and Irrotational Vectors

1) Show that $\vec{F} = \dfrac{x\hat{i} + y\hat{j}}{x^2+y^2}$ is both solenoidal and irrotational ?

Sol : $\text{div } \vec{F} = \nabla \cdot \vec{F}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left[\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}\right] + \left[\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\right]$$

$$= \frac{1}{(x^2+y^2)^2}\left[x^2+y^2-2x^2+x^2+y^2-2y^2\right]$$

$$= \frac{1}{(x^2+y^2)^2}\left[2x^2+2y^2-2x^2-2y^2\right]$$

$$\text{div } \vec{F} = 0 \Rightarrow \vec{F} \text{ is Solenoidal}$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= i\left[0 - \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2}\right)\right] - j\left[0 - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)\right] + k\left[\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right]$$

$$= k\left[\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right]$$

$$= k\left[\frac{x^2+y^2\frac{d}{dx}(y) - y\frac{d}{dx}(x^2+y^2)}{(x^2+y^2)^2} - \frac{x^2+y^2\frac{d}{dy}(x) - x\frac{d}{dy}(x^2+y^2)}{(x^2+y^2)^2}\right]$$

$$= k\left[\frac{x^2+y^2(0) - y(2x)}{(x^2+y^2)^2}\right] - \left[\frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2}\right]$$

$$= k\left[\frac{-2xy + 2xy}{(x^2+y^2)^2}\right]$$

$$\text{Curl } \vec{F} = \vec{0} \Rightarrow \vec{F} \text{ is irrotational}$$

2) Show that $\vec{F} = (y+z)i + (x+z)j + (x+y)k$ is irrotational. Also find a scalar function ϕ such that $\vec{F} = \nabla\phi$?

Sol: We have to show that Curl $\vec{F} = \vec{0}$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & (z+x) & (x+y) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right] - j\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right] + k\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$$

$$= i[1-1] - j[1-1] + k[1-1]$$

Curl $\vec{F} = \vec{0}$

Thus $\vec{F}$ is irrotational.

$$\nabla \phi = \vec{F}$$

$$\nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = (y+z)i + (z+x)j + (x+y)k$$

$$\int \frac{\partial \phi}{\partial x} = \int y+z$$

$$\phi = \int (y+z)\,dx + f_1(y,z)$$

$$\phi = xy + zx + f_1(y,z) \rightarrow ①$$

$$\frac{\partial \phi}{\partial y} = z+x$$

$$\int \frac{\partial \phi}{\partial y} = \int (z+x)\,dy + f_2(x,z)$$

$$\phi = zy + xy + f_2(x,z) \rightarrow ②$$

$$\int \frac{\partial \phi}{\partial z} = \int (x+y)$$

$$\phi = \int (x+y)\,dz + f_3(x,y)$$

$$\phi = xz + yz + f_3(x,y) \rightarrow ③$$

Let us choose $f_1(y,z) = yz$

$f_2(z,x) = xz$

$f_3(x,y) = xy$ from ① ② and ③

$$\phi = xy + yz + zx$$

3) Show that $\vec{F} = (2xy^2+yz)i + (2x^2y+xz+2yz^2)j + (2y^2z+xy)k$ is a conservative field.

Find its scalar potential. force

Sol: We have to show that curl $\vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2+yz & 2x^2y+xz+2yz^2 & 2y^2z+xy \end{vmatrix}$$

$$= i[4yz+x-x-4yz] - j[y-y] + k[4xy+z-4xy-z]$$

$$\nabla \times \vec{F} = \vec{0}$$

Thus $\vec{F}$ is conservative

Now we have to find ϕ such that

$$\nabla\phi = \vec{F}$$

$$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (2xy^2+yz)i + (2x^2y+xz+2yz^2)j + (2y^2z+xy)k$$

$$\int\frac{\partial\phi}{\partial x} = \int 2xy^2+yz\ dx$$

$$\phi = 2\frac{x^2}{2}y^2 + xyz + f_1(y,z)$$

$$\phi = x^2y^2 + xyz + f_1(y,z) \longrightarrow ①$$

$$\int\frac{\partial\phi}{\partial y} = \int 2x^2y+xz+2yz^2\ dy$$

$$\phi = 2x^2\frac{y^2}{2} + xzy + 2\frac{y^2}{2}z^2 + f_2(x,z)$$

$$\phi = x^2y^2 + xyz + y^2z^2 + f_2(x,z) \longrightarrow ②$$

$$\int\frac{\partial\phi}{\partial z} = \int 2y^2z+xy\ dz$$

$$\phi = 2y^2\frac{z^2}{2} + xyz + f_3(x,y)$$

$$\phi = y^2z^2 + xyz + f_3(x,y) \longrightarrow ③$$

Let us choose.

$f_1(y,z) = y^2 z^2$

$f_2(x,z) = 0$

$f_3(x,y) = x^2 y^2$ from (1) (2) and (3)

Thus $\phi = x^2y^2 + y^2z^2 + xyz$ is the required

scalar potential.

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15) Find the angle between the surface $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at $(2,-1,2)$

Sol $x^2+y^2+z^2 = 9$ and $z = x^2+y^2-3$

$-x^2-y^2+z = -3$

$x^2+y^2-z = 3$

$\phi_1 = x^2+y^2+z^2$

$\phi_2 = x^2+y^2-z$

$$\nabla\phi_1 = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\nabla\phi_1 = \frac{\partial(x^2+y^2+z^2)}{\partial x} i + \frac{\partial(x^2+y^2+z^2)}{\partial y} j + \frac{\partial(x^2+y^2+z^2)}{\partial z} k$$

$$\nabla\phi_1 = 2x i + 2y j + 2z k$$

$$\nabla\phi_2 = \frac{\partial(x^2+y^2-z)}{\partial x} i + \frac{\partial(x^2+y^2-z)}{\partial y} j + \frac{\partial(x^2+y^2-z)}{\partial z} k$$

$$\nabla\phi_2 = 2x i + 2y \hat{j} - k.$$

~~If θ is~~

$$[\nabla\phi_1]_{(2,-1,2)} = 4i - 2j + 4k = 2(2i - j + 2k)$$

$$[\nabla\phi_2]_{(2,-1,2)} = 4i - 2j - k$$

If θ is the angle between these two normals we have.

$$\cos\theta = \frac{\nabla\phi_1}{|\nabla\phi_1|} \cdot \frac{\nabla\phi_2}{|\nabla\phi_2|}$$

$$\cos\theta = \frac{4i - 2j + 4k}{\sqrt{4^2 + (-2)^2 + 4^2}} \cdot \frac{(4i - 2j - k)}{\sqrt{4^2 + (-2)^2 + (-1)^2}}$$

$$\cos\theta = \frac{4i - 2j + 4k}{\sqrt{36}} \cdot \frac{(4i - 2j - k)}{\sqrt{16 + 4 + 1}}$$

$$\cos\theta = \frac{16 + 4 - 4}{\sqrt{36 \cdot 21}}$$

$$\cos\theta = \frac{16}{\sqrt{756}} = \frac{16}{\sqrt{21 \times 36}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

16) Find the directional derivative of the function xyz along the direction of the normal to the surface $xy^2 + yz^2 + zx^2 = 3$ at the point $(1,1,1)$

Sol $\phi = xyz$ so we have

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$$\nabla\phi = \frac{\partial}{\partial x}(xyz) i + \frac{\partial}{\partial y}(xyz) j + \frac{\partial}{\partial z}(xyz) k$$

$$\nabla\phi = yz\, i + xz\, j + xy\, k$$

$$[\nabla\phi]_{(1,1,1)} = i + j + k$$

$$\psi = xy^2 + yz^2 + zx^2 - 3$$

$$\nabla\psi = \frac{\partial\psi}{\partial x} i + \frac{\partial\psi}{\partial y} j + \frac{\partial\psi}{\partial z} k$$

$$\nabla\psi = (y^2 + 2zx) i + (2yx + z^2) j + (y2z + x^2) k$$

$$[\nabla\psi]_{(1,1,1)} = 3i + 3j + 3k$$

$3(i+j+k)$ is the normal to given surface at $(1,1,1)$

The unit vector along $3(i+j+k)$ is

$$\hat{n} = \frac{3(i+j+k)}{\sqrt{3^2(1)}} \quad \frac{3i+3j+3k}{\sqrt{9+9+9}} = \frac{3(i+j+k)}{\sqrt{27}} = \frac{\cancel{3}(i+j+k)}{\cancel{3}\sqrt{3}}$$

$$\hat{n} = \frac{i+j+k}{\sqrt{3}}$$

The required directional derivative of ϕ along normal to given surface is

$$\nabla\phi \cdot \hat{n} = (i+j+k)\cdot\frac{(i+j+k)}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \frac{\sqrt{3}\times\sqrt{3}}{\sqrt{3}} = \sqrt{3}$$

17) If $\phi = 2x^3y^2z^4$ find div (grad ϕ) and verify that $\nabla\cdot(\nabla\phi) = \nabla^2\phi$.

Sol $\phi = 2x^3y^2z^4$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\nabla\phi = 6x^2y^2z^4 i + 4x^3yz^4 j + 8x^3y^2z^3 k$$

$$\nabla\cdot\nabla\phi = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)(6x^2y^2z^4 i + 4x^3yz^4 j + 8x^3y^2z^3 k)$$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$\nabla\cdot(\nabla\phi) = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$\phi = 2x^3y^2z^4$	$\phi = 2x^3y^2z^4$	$\phi = 2x^3y^2z^4$
$\frac{\partial\phi}{\partial x} = 6x^2y^2z^4$	$\frac{\partial\phi}{\partial y} = 4x^3yz^4$	$\frac{\partial\phi}{\partial z} = 8x^3y^2z^3$
$\frac{\partial^2\phi}{\partial x^2} = 12xy^2z^4$	$\frac{\partial^2\phi}{\partial y^2} = 4x^3z^4$	$\frac{\partial^2\phi}{\partial z^2} = 24x^3y^2z^2$

$\cancel{\nabla(\nabla\phi)} =$

$$\nabla^2\phi = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

$\nabla\cdot\nabla\phi = \nabla^2\phi$ is verified

4) Find the value of constant 'a' such that the vector field, $\vec{F} = (axy - z^3)\,i + (a-2)x^2 j + (1-a)xz^2 k$ is irrotational and hence find a scalar function ϕ such that $\vec{F} = \nabla\phi$

Sol: We have to find a such that $\text{Curl}\,\vec{F} = \vec{0}$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix}$$

$$\nabla\times\vec{F} = i\left[\frac{\partial}{\partial y}(1-a)xz^2 - \frac{\partial}{\partial z}(a-2)x^2\right] - j\left[\frac{\partial}{\partial x}(1-a)z^2x - \frac{\partial}{\partial z}(axy - z^3)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(a-2)x^2 - \frac{\partial}{\partial y}(axy - z^3)\right]$$

$$\nabla\times\vec{F} = i[0] - j[(1-a)z^2 + 3z^2] + k[(a-2)2x - ax]$$

$$\nabla\times\vec{F} = -j[(1-a)z^2 + 3z^2] + k[(a-2)2x - ax]$$

$$\nabla\times\vec{F} = -j[(1-a)z^2 + 3z^2] + k[2ax - 4x - ax]$$

$$\nabla\times\vec{F} = -j[z^2 - az^2 + 3z^2] + k[ax - 4x]$$

$$\nabla\times\vec{F} = j[-z^2 + az^2 - 3z^2] + k[ax - 4x]$$

$$\nabla\times\vec{F} = j[az^2 - 4z^2] + k[ax - 4x]$$

$$\nabla\times\vec{F} = [a-4]z^2 j + [a-4]x k$$

The equation $\nabla\times\vec{F} = \vec{0}$ satisfied when $a = 4$

Now Consider.

$\nabla\phi = (\vec{F})_{a=4}$

$\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z} k = (4xy - z^3) i + 2x^2 j - 3xz^2 k$

$\frac{\partial\phi}{\partial x} = 4xy - z^3 \, dx \Rightarrow \phi = \int(4xy - z^3)\, dx + f_1(y,z)$

$\phi = 2x^2y - xz^3 + f_1(y,z)$

$\frac{\partial\phi}{\partial y} = 2x^2 \Rightarrow \phi = \int 2x^2\, dy + f_2(x,z)$

$\phi = 2x^2y + f_2(x,z)$

$\frac{\partial\phi}{\partial z} = -3xz^2 \Rightarrow \phi = \int -3xz^2\, dz + f_3(x,y)$

$\phi = -xz^3 + f_3(x,y)$

Let us Choose $f_1(y,z) = 0$, $f_2(x,z) = -xz^3$

$f_3(x,y) = 2x^2y$ from ① ② and ③

$\phi = 2x^2y - xz^3$

5) Find Constants a and b such that

$\vec{F} = (axy + z^3) i + (3x^2 - z) j + (bxz^2 - y) k$

Sol We have to find a and b Such that

$\text{Curl } \vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy + z^3 & 3x^2 - z & bxz^2 - y \end{vmatrix}$$

$\nabla \times \vec{F} = i(-1+1) - j(bz^2 - 3z^2) + k(6x - ax)$

$= -j(bz^2 - 3z^2) + k(6x - ax)$

$= a = 6,\ b = 3$

$\nabla \times \vec{F} = \vec{0}$

The above equation is identically satisfied

$b - 3 \Rightarrow 3 - 3 = 0$ $\quad \therefore a = 6,\ b = 3$

$6 - a \Rightarrow 6 - 6 = 0$

$\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = (6xy + z^3) i + (3x^2 - z) j + (3xz^2 - y) k$

$\frac{\partial \phi}{\partial x} = 6xy + z^3 \Rightarrow \phi = \int 6xy + z^3 \, dx + f_1(y,z)$

$\phi = 3x^2 y + xz^3 + f_1(y,z)$

$\frac{\partial \phi}{\partial y} = 3x^2 - z. \Rightarrow \phi = \int 3x^2 - z \, dy + f_2(x,z)$

$\phi = 3x^2 y - zy + f_2(x,z).$

$\frac{\partial \phi}{\partial z} = 3xz^2 - y. \Rightarrow \phi = \int 3z^2 x - y \, dz + f_3(x,y)$

$\phi = xz^3 - yz + f_3(x,y).$

Let us choose $f_1(y,z) = -yz$, $f_2(x,z) = xz^3$

$f_3(x,y) = 3x^2 y$ from ① ② and ③.

Thus the required $\phi = 3x^2 y + xz^3 - yz.$

6) If $\vec{F} = (x + y + az) i + (bx + 2y - z) j + (x + cy + 2z) k$ and find a, b, c such that curl $\vec{F} = \vec{0}$ then find ϕ such that $\vec{F} = \nabla \phi$

Sol : $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+az) & (bx+2y-z) & (x+cy+2z) \end{vmatrix}$

$\nabla \times \vec{F} = i(c+1) - j(1-a) + k(b-1)$

$= c = -1, \quad a = 1 \quad b = 1$

$\nabla \times \vec{F} = \vec{0}$

$\therefore \ a = 1, \ b = 1, \ c = -1$ are the required values

Now consider $\nabla \phi = \vec{F}$

$$\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k \left[(x+y+z)i + (x+2y-z)j + (x-y+2z)k\right]$$

$$\frac{\partial \phi}{\partial x} = \int x+y+z \, dx \Rightarrow \phi = \int \frac{x^2}{2} + y + z \, dx + f_1(y,z)$$

$$\phi = \frac{x^2}{2} + xy + xz + f_1(y,z) \quad \longrightarrow ①$$

$$\frac{\partial \phi}{\partial y} = x + 2y - z \Rightarrow \phi = \int (x+2y-z)\, dy + f_2(x,z)$$

$$\phi = xy + y^2 - yz + f_2(x,z) \quad \longrightarrow ②$$

$$\frac{\partial \phi}{\partial z} = x - y + 2z \Rightarrow \phi = \int x - y + 2z \, dz + f_3(x,y)$$

$$\phi = xy - yz + z^3 + f_3(x,y). \quad \longrightarrow ③$$

Let us choose $f_1(y,z) = y^2 - yz + z^2$

$f_2(x,z) = \frac{x^2}{2} + xz + z^2$

$f_3(x,y) = \frac{x^2}{2} + xy + y^2$

From ① ② and ③.

$$\phi = \frac{x^2}{2} + xy + xz + y^2 - yz + z^2$$

7) Show that $\vec{F} = (z + \sin y)i + (x\cos y - z)j + (x-y)k$ is irrotational. Hence find a Scalar fuction ϕ such that $\vec{F} = \nabla\phi$

Sol : $\vec{F} = (z + \sin y)i + (x\cos y - z)j + (x-y)k$.

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z+\sin y & x\cos y - z & x-y \end{vmatrix}$$

$$\text{Curl } \vec{F} = i(-1+1) - j(1-1) + k(\cos y - \cos y).$$

$$\text{Curl } \vec{F} = \vec{0}$$

$\therefore \vec{F}$ is irrotational.

To find ϕ such that $\vec{F} = \nabla\phi$

$$\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = (z + \sin y)i + (x\cos y - z)j + (x-y)k.$$

$$\frac{\partial\phi}{\partial x} = z + \sin y$$

$$\phi = \int z + \sin y \, dx$$

$$\phi = xz + x\sin y + f_1(y,z) \longrightarrow ①$$

$$\frac{\partial\phi}{\partial y} = x\cos y - z$$

$$\phi = \int x\cos y - z \, dy$$

$$\phi = x\sin y - zy + f_2(x,z) \longrightarrow ②$$

$$\frac{\partial\phi}{\partial z} = x - y$$

$$\phi = \int x - y \, dz$$

$$\phi = xz - yz + f_3(x,y) ③$$

$$\phi_1(y,z) = -yz$$

$$\phi_2(x,z) = zx$$

$$\phi_3(x,y) = x\sin y.$$

$$\phi = x\sin y - zy + xz$$

Vector line integral

If $\vec{F}$ is the force acted upon by a particle in displacing it along the curve C then $\int_C \vec{F}\cdot\vec{dr}$ represent that total work done by the force. It also represent the circulation of $\vec{F}$ about C where $\vec{F}$ represents the velocity of the fluid

$\vec{F}$ is said to be irrotational if $\oint_C \vec{F}\cdot\vec{dr} = 0$

Problems:

1) If $\vec{F} = xy\,i + yz\,j + zx\,k$, evaluate $\int_C \vec{F}\cdot\vec{dr}$ where C is the curve represented by $x = t,\ y = t^2,\ z = t^3,\ -1 \le t \le 1$

Sol: $\vec{F} = xy\,i + yz\,j + zx\,k$ and $\vec{r} = x\,i + y\,j + z\,k$

$\vec{dr} = dx\,i + dy\,j + dz\,k$

$\vec{F}\cdot\vec{dr} = xy\,i + yz\,j + xz\,k\,(dx\,i + dy\,j + dz\,k)$

$\vec{F}\cdot\vec{dr} = xy\,dx + yz\,dy + xz\,dz.$

Since $x = t,\quad y = t^2\quad z = t^3$

$dx = dt\quad dy = 2t\,dt\quad dz = 3t^2\,dt$

$\vec{F}\cdot\vec{dr} = t^3\,dt + t^5(2t\,dt) + t^4(3t^2)\,dt$

$\vec{F}\cdot\vec{dr} = t^3\,dt + 2t^6\,dt + 3t^6\,dt$

$\vec{F}\cdot\vec{dr} = (t^3 + 2t^6 + 3t^6)\,dt$

$\vec{F}\cdot\vec{dr} = (t^3 + 5t^6)\,dt$

$\int_C \vec{F}\cdot\vec{dr} = \int_{t=-1}^{1} (t^3 + 5t^6)\,dt$

$$= \left[\frac{t^4}{4}\right]_{-1}^{1} + 5\left[\frac{t^7}{7}\right]_{-1}^{1}.$$

$$\int_C \vec{F}\cdot\vec{dr} = \left[\frac{1}{4}-\frac{1}{4}\right] + 5\left[\frac{1}{7}+\frac{1}{7}\right].$$

$$\int_C \vec{F}\cdot\vec{dr} = \frac{10}{7}$$

2) If $\vec{F} = (3x^2+6y)i - 14yz\,j + 20xz^2 k$, evaluate $\int \vec{F}\cdot\vec{dr}$ from $(0,0,0)$ to $(1,1,1)$ along the curve given by $x=t$, $y=t^2$, $z=t^3$

Sol $\vec{F} = (3x^2+6y)i - 14yz\,j + 20xz^2 k$

$\vec{r} = xi + yj + zk$

$\vec{dr} = dx\,i + dy\,j + dz\,k.$

$\vec{F}\cdot\vec{dr} = (3x^2+6y)i - 14yz\,j + 20xz^2 k\,(dx\,i + dy\,j + dz\,k)$

$= (3x^2+6y)dx - 14yz\,dy + 20xz^2\,dz.$

$\vec{F}\cdot\vec{dr} = (3x^2+6y)\,dx - 14yz\,dy$

$x=t \quad y=t^2 \quad z=t^3$, $dx=dt$, $dy=2t\,dt$ $dz=3t^2dt$

$\vec{F}\cdot\vec{dr} = (3t^2+6t^2)\,dt - (14t^5)2t\,dt + (20t^7)3t^2\,dt$

$\vec{F}\cdot\vec{dr} = (9t^2)\,dt - 28t^6\,dt + 60t^9\,dt$

$\vec{F}\cdot\vec{dr} = (9t^2 - 28t + 60t^9)\,dt.$

$0 \le t \le 1$

$$\int_C \vec{F}\cdot\vec{dr} = \int_0^1 (9t^2 - 28t + 60t^9)\,dt$$

$$\int_C \vec{F}\cdot\vec{dr} = \left[9\frac{t^3}{3} - 28\frac{t^2}{2} + 6\,\cancel{0}\,\frac{t^{10}}{10}\right]_{t=0}^{1}$$

$$\int_C \vec{F}\cdot\vec{dr} = 3-4+6$$

$$= 5$$

3) If $\vec{F} = x^2 i + xy\, j$ evaluate $\int_C \vec{F} \cdot \vec{dr}$ from $(0,0)$ to $(1,1)$ along.

(i) the line $y = x$ (ii) The parabola $y = \sqrt{x}$

Sol $\vec{F} \cdot \vec{dr} = x^2\, dx + xy\, dy$

① ~~Along~~ $y = x$ we have $0 \le x \le 1$

$y = x$

$dy = dx$

$\cancel{\int_C \vec{F} \cdot \vec{dr} = \int_{x=0}^{1} x^2 dx +}$ $\vec{F} \cdot \vec{dr} = x^2\, dx + x^2\, dx$

$0 \le x \le 1$

$$\int_{x=0}^{1} \vec{F} \cdot \vec{dr} = \int_{x=0}^{1} x^2\, dx + x^2\, dx$$

$$= \left.\frac{x^3}{3}\right]_0^1 + \left.\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{3}$$

$$= \frac{2}{3}$$

② Along $y = \sqrt{x}$; $y^2 = x$ $0 \le y \le 1$

$2y\, dy = dx$

$\cancel{\int_C \vec{F} \cdot \vec{dr} =} \int_0^1$ $\vec{F} \cdot \vec{dr} = x^2\, dx + xy\, dy$

$\vec{F} \cdot \vec{dr} = 2y^5\, dy + y^3\, dy$

$$\int_0^1 \vec{F} \cdot \vec{dr} = \int_0^1 2y^5\, dy + \int_0^1 y^3\, dy$$

$$= \left.\frac{2y^6}{6}\right]_0^1 + \left.\frac{y^4}{4}\right]_0^1$$

$$= \left.\frac{y^6}{3}\right]_0^1 + \left.\frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

4) Evaluate $\int_C \vec{F}\cdot\vec{dr}$ where $\vec{F} = xy\, i + (x^2+y^2)\, j$ along,

(i) the path of the straight line from $(0,0)$ to $(1,0)$ and then to $(1,1)$

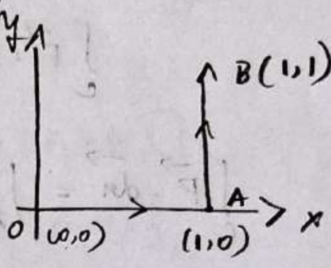
(ii) The straight line joining the origin and $(1,2)$

$\vec{F} = xy\, i + (x^2+y^2)\, j$, $\vec{dr} = dx\, i + dy\, j + dz\, k$

$\vec{F}\cdot\vec{dr} = (xy\, i + (x^2+y^2)\, j)\, dx\, i + dy\, j + dz\, k$

$\int \vec{F}\cdot\vec{dr} = \int_C xy\, dx + (x^2+y^2)\, dy. \rightarrow ①$

$\int_C \vec{F}\cdot\vec{dr} = \int_{OA} \vec{F}\cdot\vec{dr} + \int_{AB} \vec{F}\cdot\vec{dr} \rightarrow ②$



Along OA ; $y=0$

$dy = 0$, $0 \le x \le 1$

From ① $\int_{OA} \vec{F}\cdot\vec{dr} = 0 \rightarrow ③$

Along AB : $x=1$

$dx = 0$, $0 \le y \le 1$

$\int_{AB} \vec{F}\cdot\vec{dr} = \int_{y=0}^{1} x$

$\vec{F}\cdot\vec{dr} = xy\, dx + (x^2+y^2)\, dy$

$\vec{F}\cdot\vec{dr} = (1+y^2)\, dy$.

$\int_{AB} \vec{F}\cdot\vec{dr} = \int_0^1 (1+y^2)\, dy$

$= y + \frac{y^3}{3}\Big]_0^1$

$= 1 + \frac{1}{3} = \frac{4}{3} \rightarrow ④$

Using ③ and ④ in ② we obtain

$\int_C \vec{F}\cdot\vec{dr} = \int_{OA} \vec{F}\cdot\vec{dr} + \int_{AB} \vec{F}\cdot\vec{dr}$

$= 0 + \frac{4}{3}$

$= \frac{4}{3}$

ii) C is the Straight line joining $(0,0)$ and $(1,2)$
$x_1\ y_1$ $x_2\ y_2$

The equation of the line is given by

$$\frac{y-0}{x-0} = \frac{2-0}{1-0} \qquad \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$$

i.e $y = 2x$, $dy = 2\,dx$ and x varies from 0 to 1

Hence from (1) $\int_C \vec{F}\cdot d\vec{r} = \int_C xy\,dx + (x^2+y^2)\,dy$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C x\cdot 2x\,dx + (x^2+4x^2)\,2\,dx$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} 2x^2\,dx + 2x^2 + 8x^2\,dx$$

$$= \int_{x=0}^{1} 12x^2\,dx.$$

$$= 12\left.\frac{x^3}{3}\right]_0^1 = \underline{4}$$

5) Find the total work done by the force represented by $\vec{F} = 3xy\,i - y\,j + 2xz\,k$ in moving particle around the coil $x^2+y^2=4$

Sol : Total work done $W = \int_C \vec{F}\cdot d\vec{r}$

$x^2+y^2=4$ can be represented in the parametric form $x = 2\cos\theta$, $y = 2\sin\theta$ and $z=0$, $0 \le \theta \le 2\pi$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int 3xy\,dx - y\,dy + 2zx\,dz$$

$$W = \int_{\theta=0}^{2\pi} 3(4\cos\theta\sin\theta)(-2\sin\theta)\,d\theta - \int_{\theta=0}^{2\pi}$$

$x = 2\cos\theta$ $\quad\quad$ $y = 2\sin\theta$

$dx = 2(-\sin\theta)\,d\theta$ $\quad\quad$ $dy = 2\cos\theta\,d\theta$

$w = \int_c \vec{F}\cdot\vec{dr} = \int 3\,(2\cos\theta)\,(+2\sin\theta)\,(-2\sin\theta)\,d\theta$

$- 2\sin\theta\; 2\cos\theta\; d\theta$.

$w = \int_c \vec{F}\cdot\vec{dr} = \int_{\theta=0}^{2\pi} 3\,(4\cos\theta\sin\theta)\,(-2\sin\theta)\,d\theta - \int_{\theta=0}^{2\pi} 4\sin\theta\cos\theta\,d\theta$

$= -24\int_0^{2\pi} \sin^2\theta\cos\theta\,d\theta - \int_{\theta=0}^{2\pi} 4\cdot\frac{1}{2}\,(\sin 2\theta)$

$\sin A\cdot\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\sin\theta\cdot\cos\theta = \frac{1}{2}\left[\sin(\theta+\theta) + \sin(\theta-\theta)\right]$

$= \frac{1}{2}\left[\sin 2\theta\right]$

$= -24\int_0^{2\pi} \sin^2\theta\cos\theta\,d\theta - \int_{\theta=0}^{2\pi} 2\sin 2\theta$.

$= -24\int_0^{2\pi} \sin^2\theta\sin\theta\,d\theta - \int_{\theta=0}^{2\pi} 2\sin 2\theta\,d\theta$

$= -24\int_0^{2\pi} \sin^3\theta\,d\theta - 2\int_{\theta=0}^{2\pi} \sin 2\theta\,d\theta$.

$= -24\left[\frac{\sin^3\theta}{3}\right]_0^{2\pi} - 2\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi}$.

$0 + \cos 2\theta\Big]_2^{2\pi}$

$= 0$

$0 + (\cos 2\pi - \cos 0)$.

$0 + [-1 - (-1)]$

$0 + (-1+1)$

$\underline{\underline{0}}$

$\cos\pi = -1$

$\cos 0 = 1$

$\sin 0 = 0$

$\sin\pi = 0$

$\int uv = u\int v - u'\iint v + u''\iiint v$

$= \sin^2\theta \,/\, \sin\theta$ -

6) If $\phi = 2xyz^2$, $\vec{F} = xy\,i - z\,j + x^2 k$ and c is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$ evaluate the following line integrals

(1) $\int_c \phi\, \vec{dr}$ = (2) $\int_c \vec{F} \times dr$

$\vec{r} = xi + yj + zk$ $\quad x = t^2 \quad y = 2t \quad z = t^3$

$\vec{dr} = dx\,i + dy\,j + dz\,k. \quad dx = 2t\,dt \quad dy = 2\,dt \quad dz = 3t^2 dt$

$\vec{dr} = 2t\,dt\,i + 2\,dt\,j + 3t^2 dt\,k.$

$\vec{dr} = (2t\,i + 2j + 3t^2 k)\,dt$

$\phi = 2xyz^2$

$\phi = 2t^2\, 2t \cdot t^6$

$\phi = \underline{4t^9}$

$\phi\,\vec{dr} = (4t^9)(2t\,i + 2j + 3t^2 k)\,dt$

$\phi\,\vec{dr} = (8t^{10} i + 8t^9 j + 12t^{11} k)\,dt$

(1) $\int_c \phi\,\vec{dr} = \int_{t=0}^{1} (8t^{10} i + 8t^9 j + 12t^{11} k)\,dt$

$= 8\frac{t^{11}}{11}\Big]_0^1 i + 8\left[\frac{t^{10}}{10}\right]_0^1 j + 12\frac{t^{12}}{12}\Big]_0^1 k.$

$\int_c \phi\,\vec{dr} = \frac{8}{11} i + \frac{4}{5} j + k.$

(2) $\vec{F} = 2t^3 i - t^3 j + t^4 k.$

$$\int_c \vec{F} \times d\vec{r} = \int_c \vec{F} \times \frac{d\vec{r}}{dt}\, dt$$

$$\vec{F} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} i & j & k \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix}$$

$$= i(-3t^5 - 2t^4) - j(6t^5 - 2t^5) + k(4t^3 + 2t^4).$$

$$\int_c \vec{F} \times d\vec{r} = \int_{t=0}^{1} (-3t^5 - 2t^4)\, i\, dt - \int_{t=0}^{1} 4t^5\, j\, dt + \int_{t=0}^{1} (4t^3 + 2t^4)\, k\, dt$$

$$= \left[-\frac{t^6}{2} - \frac{2t^5}{5}\right]_0^1 i - 4\left[\frac{t^6}{6}\right]_0^1 j + \left[t^4 + \frac{2t^5}{5}\right]_0^1 k.$$

$$\cancel{\int_c \vec{F} \times d\vec{r} = \left[\left[\frac{-1}{2} - \frac{1}{2}\right] - \frac{2}{5}\left[\right.\right.}$$

$$\left[\frac{-1}{2}[(1-0)] - \frac{2}{5}[1-0]\right] i - \frac{2}{3}[1-0]\, j + \left[1 + \frac{2}{5}(1-0)\right] k.$$

$$= \left[-\frac{1}{2} - \frac{2}{5}\right] i - \frac{2}{3} j + \left[1 + \frac{2}{5}\right] k.$$

$$\int_c \vec{F} \times d\vec{r} = \left(-\frac{9}{10}\right) i - \left(\frac{2}{3}\right) j + \left(\frac{7}{5}\right) k.$$

Surface and Volume integral

Volume integral :

If V is the Volume bounded by a surface and if $F(x, y, z)$ is a Single Value function defined over V then the volume integral of $F(x, y, z)$ over V is given by $\iiint_V F\, dv$

Green theorem in a plane

Statement ; If R is a closed region of the x-y plane bounded by a Simple closed curve C, and if M and N are two continous functions of x, y. having continous first order partial derivative in the region R then.

$$\oint_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Stokes theorem

Statement : If S is a Surface bounded by a simple closed curve C and if $\vec{F}$ is any continously differtiable vector function then

$$\oint_c \vec{F} \cdot \vec{dr} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds.$$

Gauss divergence theorem

Statement ; If V is the volume bounded by a surface S and $\vec{F}$ is a continously differentiabl vector function then

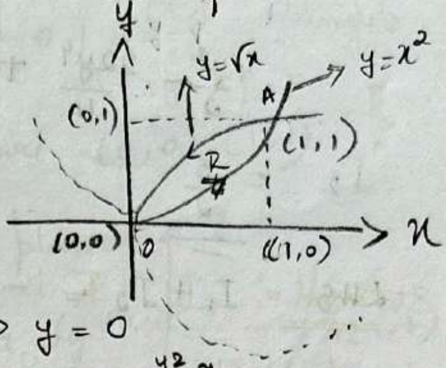
$$\iiint_V \text{div } \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{n}\, ds$$

where $\hat{n}$ is the positive unit vector outward drawn normal to S

Problems

1) Verify Green's theorem in a plane for $\oint_c (3x^2-8y^2)\,dx + (4y-6xy)\,dy$, where C is the boundary of the region enclosed by $y=\sqrt{x}$ and $y=x^2$.

Sol: We shall find the point of intersection of the parabolas $y=\sqrt{x}$ and $y=x^2$.



$\sqrt{x} = x^2$ $y^2 = x$ $\Rightarrow x = x^4 \Rightarrow x^4 = x$

$x^4 - x = 0$

$x(x^3-1)=0$

$x=0, \quad x^3-1=0$ $y=\sqrt{x} \Rightarrow y=0$

$x=0, \quad x=1$ $y = x^2 \Rightarrow y=1$

$\therefore x = 0,1$ and hence $y=0,1$

The points of intersection are $(0,0)$ $(1,1)$.

$M = 3x^2 - 8y^2$ $N = 4x - 6xy$

$\frac{\partial M}{\partial y} = -16y$ $\frac{\partial N}{\partial x} = 4-6y$

We have Greens theorem

$$\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$LHS = \oint_c M\,dx + N\,dy.$

$$= \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy = I_1 + I_2$$

Along OA: $y = x^2$

$dy = 2x\,dx$ x varies from 0 to 1

$$I_1 = \int_{x=0}^{1} (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)(2x\,dx)$$

$$= \int_{x=0}^{1} (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= \left.\frac{3x^3}{3} + \frac{8x^4}{4} - \frac{20x^5}{5}\right]_0^1 = \underline{\underline{-1}}$$

Along AO : $y = \sqrt{x} \Rightarrow x = y^2 \Rightarrow dx = 2y\,dy$

y varies from 1 to 0.

$$I_2 = \int_{y=1}^{0} (3y^4 - 8y^2)(2y)\,dy + (4y - 6y^3)\,dy$$

$$= \int_1^0 (4y - 22y^3 + 6y^5)\,dy$$

$$= \left.\frac{4y^2}{2} - \frac{22y^4}{4} + \frac{6y^6}{6}\right]_1^0$$

$$I_2 = \frac{5}{2}$$

$$LHS = I_1 + I_2 = -1 + 5/2 = 3/2$$

Also $RHS = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (-6y + 16y)\,dy\,dx$$

$$= \int_{x=0}^{1} 10\left[\frac{y^2}{2}\right]_{y=x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (\sqrt{x}^2 - x^4)\,dx$$

$$= 5\int_{x=0}^{1} (x - x^4)\,dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left(\frac{1}{2} - \frac{1}{5}\right) = 3/2$$

$\therefore$ Thus we have $LHS = RHS = 3/2$

2) Verify Greens theorem for $\oint_C (xy + y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$

Sol : We shall find the point of intersection

$y = x$ and $y = x^2$

$x = x^2$

$x - x^2 = 0 \Rightarrow x(1-x) = 0 \quad , x = 0 \quad , 1 - x = 0 \quad , x = 1$

$x = 0, 1$ Take $x = 0, 1$

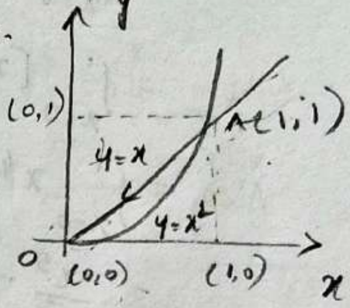
$y = x \Rightarrow y = 0$ $\quad y = 0 \; 1$

$y = x^2 \Rightarrow y = 1$

The point of intersection are $(0,0)$ $(1,1)$

We have Greens theorem in a plane.

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy$$



LHS $\Rightarrow \int_C M dx + N dy$

$$= \int_{OA} M dx + N dy + \int_{AO} M dx + N dy = I_1 + I_2$$

$(0,0) \; (1,1) \quad (0,1)$

$x_1 \; x_2 \; y_1 \; y_1$

Along OA : $y = x^2$

$dy = 2x \, dx$ $\quad$ x varies from 0 to 1

$$I_1 = \int_{x=0}^{1} (xy + y^2) dx + x^2 dy$$

$$= \int_{x=0}^{1} x^3 + x^4 dx + (x^2) 2x \, dx$$

$$= \int_{x=0}^{1} 3x^3 + x^4 \; dx$$

$$= \frac{3x^4}{4} + \frac{x^5}{5} \Big]_0^1$$

$$I_1 = \frac{19}{20}$$

Along AO : $y = x$

$dy = dx$ $\quad$ x varies from 1 to 0

$$I_2 = \int_1^0 3x^2 dx$$

$$= \frac{3x^3}{3} \Big]_1^0$$

$$= -1$$

$$LHS = I_1 + I_2$$

$$= \frac{19}{20} - 1$$

$$= \frac{-1}{20}$$

To evaluate RHS.

$M = xy + y^2 \qquad N = x^2$

$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = 2x$

$$RHS = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\, dy\, dx$$

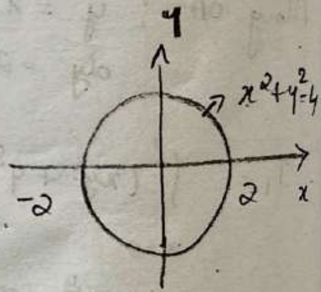
$$= \int_{x=0}^{1} \left[xy - \frac{2y^2}{2}\right]_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} [x^2 - x^2] - [x^3 - x^4]\, dx$$

$$= \int_{x=0}^{1} x^4 - x^3\, dx$$

$$= \left.\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \underline{\underline{\frac{-1}{20}}}$$

3) Verify Greens theorem in the plane for $\int_C (x^2+y^2)\, dx + 3x^2y\, dy$ where C is the circle $x^2+y^2=4$ traced in the positive sense.



Sol $\quad \int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$$LHS = \int_C M\, dx + N\, dy$$

$= \int_C (x^2+y^2)\, dx + 3x^2y\, dy$ and parametric eq of the given circle $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$

$dx = -2\sin\theta\, d\theta \qquad dy = 2\cos\theta\, d\theta$

$$LHS = \int_0^{2\pi} (2\cos\theta)^2 + (2\sin\theta)^2 (-2\sin\theta\, d\theta) +$$

$$\int_0^{2\pi} 3 (2\cos\theta)^2 (2\sin\theta)(2\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} (4\cos^2\theta + 4\sin^2\theta)(-2\sin\theta)\, d\theta +$$

$$\int_0^{2\pi} 3 \cdot 4\cos^2\theta \cdot 2\sin\theta \cdot 2\cos\theta\, d\theta$$

$$= 4\int_0^{2\pi} (-2\sin\theta)\, d\theta + \int_0^{2\pi} 48 \cos^3\theta \sin\theta \, d\theta$$

$$= +8 (\cos\theta)\Big]_0^{2\pi} + 48 \left[\cos^3\theta \cdot (-\cos\theta)\right]_0^{2\pi}$$

$$= 8 [\cos 2\theta]_0^{2\pi} - 48 \left[\frac{\cos^4\theta}{4}\right]_0^{2\pi}$$

$$= 8(\cos 2\pi - \cos 0) - \frac{48}{4}\left[\cos^4 2\pi - \cos 0\right]$$

$$= 8(1-1) - \frac{48}{4}[1-1]$$

$$= 0$$

$x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \sqrt{4 - x^2}$

$M = x^2 + y^2$, $N = 3x^2 y$.

$\frac{\partial M}{\partial y} = 2y$ $\qquad$ $\frac{\partial N}{\partial x} = 6xy$.

$$RHS = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2y(3x-1)\, dy\, dx$$

$$= \int_{x=-2}^{2} (3x-1) \left[\frac{2y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{x=-2}^{2} (3x-1) \left[(4-x^2) - (4-x^2)\right]$$

$$= 0$$

The theoreme is verified.

Stokes theorem: $\hat{n}\, ds = dy\,dz\, i + dz\,dx\, j + dx\,dy\, k$

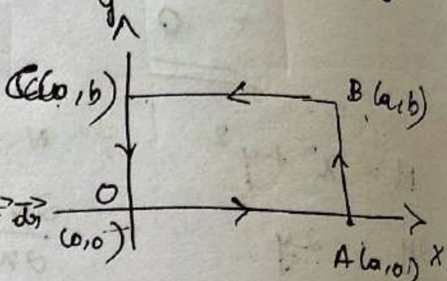
1) Verify Stokes theorem for the vector $\vec{F} = (x^2+y^2)\,i - 2xy\, j$ taken around the rectangle bounded by $x=0$, $x=a$, $y=0$, $y=b$.

Sol: we have Stokes theorem

$\vec{r} = xi + yj + zk$

$\vec{dr} = dx\,i + dy\,j + dz\,k$

$$\oint_S \vec{F}\cdot\vec{dr} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds$$



$$\vec{F}\cdot\vec{dr} = (x^2+y^2)\,dx - 2xy\,dy.$$

$$\oint_C \vec{F}\cdot\vec{dr} = \int_{OA}\vec{F}\cdot\vec{dr} + \int_{AB}\vec{F}\cdot\vec{dr} + \int_{BC}\vec{F}\,dr + \int_{CO}\vec{F}\,\vec{dr}$$

$$\oint_C \vec{F}\cdot\vec{dr} = I_1 + I_2 + I_3 + I_4.$$

① Along OA : $y=0$, $dy = 0$ and $0 \le x \le a$

$$I_1 = \int_{x=0}^{a} x^2\,dx = \left.\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

② Along AB : $x = a$, $dx = 0$ and $0 \le y \le b$

$$I_2 = \int_{y=0}^{b} (a^2+y^2)(0) - 2ay\,dy.$$

$$I_2 = \int_0^b -2ay\,dy$$

$$I_2 = -2a\left.\frac{y^2}{2}\right]_0^b.$$

$$I_2 - a\left[b^2 - 0^2\right] = -ab^2.$$

(3) Along BC : $y = b$, $dy = 0$ and $a \le x \le 0$.

$$I_3 = \int_a^0 (x^2+b^2)\,dx - 2xb\,(0)$$

$$I_3 = \int_a^0 (x^2+b^2)\,dx$$

$$I_3 = \left.\frac{x^3}{3} + b^2x\right]_a^0$$

$$I_3 = \frac{-a^3}{3} - b^2a.$$

(4) Along CO : $x = 0$, $dx = 0$ and $b \le y \le 0$.

$$I_4 = \int_b^0 (y^2)\,0\,dy.$$

$$I_4 = 0.$$

$$\oint_c \vec{F}\cdot\vec{dr} = I_1 + I_2 + I_3 + I_4$$

$$= \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - b^2a + 0$$

$$= -2ab^2 \rightarrow (1)$$

Now
$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= k(-2y-2y) = -4yk.$$

$$(\nabla\times\vec{F})\,\hat{n}\,ds = (-4yk)(dy\,dz\,i + dz\,dx\,j + dx\,dy\,k)$$

$$= -4y\,dx\,dy$$

$x = (0,0)\ (a,0)$

$y = (0,0)\ (0,b)$

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_R -4y\,dx\,dy$$

$$= -4\int_{x=0}^{a}\int_{y=0}^{b} y\,dy\,dx$$

$$= -4\int_{x=0}^{a} \left.\frac{y^2}{2}\right]_0^b dx$$

$$= -2b^2\,dx$$

$$= -2b^2x\Big]_0^a$$

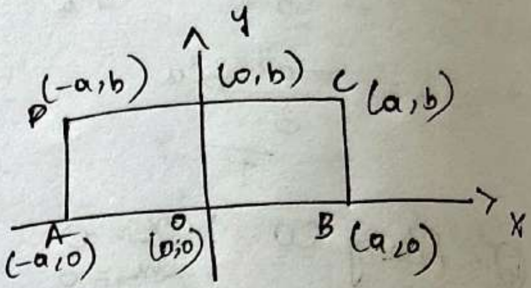
$$= -2b^2a \rightarrow (2)$$

Thus from (1) and (2) theorem is verified

2) Verify Stokes theorem for $\vec{F} = (x^2+y^2)\,i - 2xy\,j$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$ and $y = b$

Sol : $\int_C \vec{F}\cdot\vec{dr} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$

$x = (a, -a)$
$y = (0, b)$



$\int_C \vec{F}\cdot\vec{dr} = \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CD} \vec{F}\cdot\vec{dr} + \int_{DA} \vec{F}\cdot\vec{dr}$

$\int_C \vec{F}\cdot\vec{dr} = I_1 + I_2 + I_3 + I_4.$

$\vec{F}\cdot\vec{dr} = (x^2+y^2)\,i - 2xy\,j\,(dx\,i + dy\,j)$

$\vec{F}\cdot\vec{dr} = (x^2+y^2)\,dx - 2xy\,dy$

① Along AB : $y = 0$, $dy = 0$ $\quad -a \le x \le a$

$I_1 = \int_{-a}^{a} x^2\,dx$

$= \left.\frac{x^3}{3}\right]_{-a}^{a} = \frac{a^3}{3} - \frac{(-a)^3}{3} = \frac{a^3}{3} + \frac{a^3}{3} = \underline{\underline{\frac{2a^3}{3}}}$

② Along BC : $x = a$, $dx = 0$ $\quad 0 \le y \le b$

$I_2 = \int_0^b (a^2+y^2)\,0 - 2ay\,dy.$

$I_2 = -2a \left.\frac{y^2}{2}\right]_0^b$

$= -ay^2\Big]_0^b$

$\underline{I_2 = -ab^2}$

(3) Along CD : $y = b$ $\quad +a \le x \le -a$ $\quad (a,b)\ (-a,b)$

$dy = 0$

$$I_3 = \int_a^{-a} (x^2 + b^2)\, dx$$

$$I_3 = \frac{x^3}{3}\Big]_a^{-a} + b^2 x\Big]_a^{-a}$$

$$I_3 = \frac{1}{3}\left[(-a)^3 - a^3\right] + b^2\left[-a - a\right]$$

$$I_3 = \frac{1}{3}\left[-a^3 - a^3\right] + b^2\left[-2a\right] \Rightarrow -\frac{2a^3}{3} - 2ab^2$$

(4) Along DA : $x = -a$ $\quad b \le y \le 0.$

$dx = 0$

$$I_4 = \int_b^0 ((-a)^2 + y^2)\, 0 - 2(-a) y\, dy$$

$$I_4 = \int_b^0 2ay\, dy.$$

$$I_4 = 2a \frac{y^2}{2}\Big]_b^0 = ay^2\Big]_b^0 = a[0 - b^2] = -ab^2$$

$$\int_C \vec{F}\, \vec{dr} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2$$

$$\int_C \vec{F} \cdot \vec{dr} = -4ab^2$$

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$\nabla \times \vec{F} = (-4y\, k)(dy\, dz\, i + dz\, dx\, j + dx\, dy\, k).$$

$$= -4y\, dx\, dy.$$

$$\iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds = \int_{-a}^{a}\int_0^b -4y\, dx\, dy$$

$$= -4\int_{-a}^{a} \left[\frac{y^2}{2}\right]_0^b dx$$

$$= -2b^2 x\Big]_{-a}^{a}$$

$$= -2b^2(a + a) = -4ab^2$$

3) ~~Verify Stokes theorem~~

2) Find the work done in moving a particle in the force field $\vec{F} = 3x^2 i + (2xz - y) j + z k$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Sol $\int_C \vec{F} \cdot \vec{dr}$

$\vec{r} = xi + yj + zk.$

$\vec{dr} = dx\, i + dy\, j + dz\, k.$

$\vec{F} \cdot \vec{dr} = (3x^2 i + (2xz - y) j + zk)(dx\, i + dy\, j + dz\, k)$

$\vec{F} \cdot \vec{dr} = 3x^2\, dx + (2xz - y)\, dy + z\, dz.$

Line integral. $(0,0,0)$ $(2,1,3)$
x_1, y_1, z_1 $x_2\ y_2\ z_2.$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

$$\frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$\frac{x}{2} = t \Rightarrow x = 2t \Rightarrow dx = 2\,dt$

$y = t \Rightarrow dy = dt$

$\frac{z}{3} = t \Rightarrow z = 3t \Rightarrow dz = 3\,dt$

Put $x = 0 \Rightarrow t = 0$ $x = 2 \Rightarrow t = 1$ $0 \le t \le 1$

$y = 0 \Rightarrow t = 0$ $y = 1 \Rightarrow t = 1$

$z = 0 \Rightarrow t = 0$ $z = 3 \Rightarrow t = 1$

$$\int_{t=0}^{1} \vec{F}\cdot dr = \int_0^1 3(2t)^2\, 2dt + (2(2t)(3t) - t)\, dt + (3t)(3dt).$$

$$\int_0^1 (24t^2 + 12t^2 - t + 9t)\, dt$$

$$\int_0^1 (36t^2 + 8t)\, dt = 36\frac{t^3}{3}\Big]_0^1 + 8\frac{t^2}{2}\Big]_0^1$$

$$= 12t^3 + 4t^2\Big]_0^1$$

$$= 12 + 4$$

$$= 16$$

3) Verify Stokes theorem $\vec{F} = (2x - y)i - yz^2 j - y^2 z k$ where S is the upper surface of the sphere $x^2 + y^2 + z^2 = 1$, C is its boundary.

$$\Rightarrow \int_C \vec{F}\cdot dr = \int$$