Module 3 : Laplace Transforms

Inverse laplace transforms :

$$L[f(t)] = \bar{f}(s) \Longleftrightarrow L^{-1}[\bar{f}(s)] = f(t)$$

Funtion	Inverse transformation
① $\frac{1}{s}$	1
② $\frac{1}{s-a}$	e^{at}
③ $\frac{s}{s^2+a^2}$	$\cos at$
④ $\frac{s}{s^2-a^2}$	$\cosh at$
⑤ $\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
⑥ $\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
⑦ $\frac{1}{s^{n+1}}$, $n > -1$	$\frac{t^n}{\Gamma(n+1)}$
⑧ $\frac{1}{s^{n+1}}$, $n = 1, 2, 3, \ldots$	$\frac{t^n}{n!}$

Convolution

The convolution of two functions $f(t)$ and $g(t)$ usually denoted by $f(t) * g(t)$ is defined in the form of an integral as follows.

$$f(t) * g(t) = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

Verify Convolution theorem :

Working procedure for problems.

We need to verify the theorem in respect of the two given functions $f(t)$ and $g(t)$.

Step 1 - we need $\bar{f}(s) = L[f(t)]$ and $\bar{g}(s) = L[g(t)]$.

Step 2 - we evaluate $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$

Step 3 - we find $L[f(t) * g(t)]$.

Step 4 - If $L[f(t) * g(t)] = \bar{f}(s) \cdot \bar{g}(s)$

The theorem is verified.

Problems : Verify Convolution theorem for the following pairs of function .

Imp

Convolution theorem

If $L^{-1}[\bar{f}(s)] = f(t)$ and $L^{-1}[\bar{g}(s)] = g(t)$ then

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)\, g(t-u)\, du.$$

Proof: we shall show that

$$L\left[\int_{u=0}^{t} f(u)\, g(t-u)\, du\right] = \bar{f}(s)\cdot\bar{g}(s)$$

we have LHS by the definition. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$LHS = \int_{t=0}^{\infty} e^{-st}\left[\int_{u=0}^{t} f(u)\, g(t-u)\, du\right] dt$$

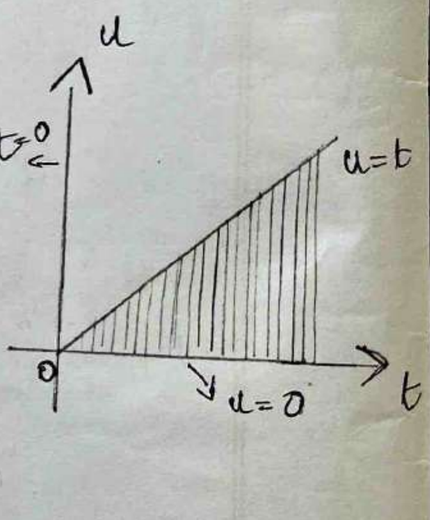
$$LHS = \int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st} f(u)\, g(t-u)\, du\, dt$$

we shall change the order of integration in respect of this double integral,

Existing region.

$t = 0$ to ∞ (Horizontal strip)

$u = 0$ to t (Vertical strip)



On changing the order.

$u = 0$ to ∞ (Vertical strip)

$t = u$ to ∞ (Horizontal strip)

on changing the order of integration ① becomes

$$LHS = \int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st} f(u)\, g(t-u)\, dt\, du$$

Now let us put $t - v = u$ where u is fixed

$\therefore dt = dv$.

X

If $t = u$, $v = 0$

If $t = \infty$, $v = \infty$ and hence eq ② becomes

$$LHS = \int_{u=0}^{\infty} \int_{v=0}^{\infty} e^{-st} f(u)\, g(t-u)\, dt\, du$$

$$LHS = \int_{u=0}^{\infty} \int_{v=0}^{\infty} e^{-s(u+v)} f(u)\, g(v)\, dv\, du$$

$$LHS = \left[\int_{u=0}^{\infty} e^{-su} f(u)\, du\right] \cdot \left[\int_{v=0}^{\infty} e^{-sv} g(v)\, dv\right]$$

$$LHS = \bar{f}(s) \cdot \bar{g}(s) = RHS$$

Hence we proved that

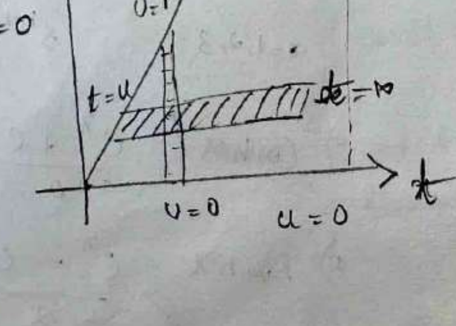
$$L\left[\int_{u=0}^{t} f(u)\, g(t-u)\, du\right] = \bar{f}(s) \cdot \bar{g}(s)$$

$$L^{-1}\left[\bar{f}(s) \cdot \bar{g}(s)\right] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

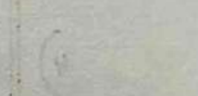
This proves the Convolution theorem

$u = 0$, $u = t$

$x = 0$
$y = 100$



$x - a$



21.1 **Laplace Transforms:**

Definition If $f(t)$ is a real valued function defined for all $t \geq 0$ then the laplace transform of $f(t)$ denoted by $\mathcal{L}[f(t)]$ is defined by

$$\mathcal{L}[f(t)] = \int_{t=0}^{\infty} e^{-st} f(t)\, dt.$$

$\mathcal{L}[f(t)] = \bar{f}(s)$ we can express this in the form

$\mathcal{L}^{-1}[\bar{f}(s)] = f(t)$ and it is called the inverse laplace transform.

Table of Laplace Transforms:

	$f(t)$	$\mathcal{L}[f(t)] = \bar{f}(s)$
1)	a	$\frac{a}{s}$
2)	e^{at}	$\frac{1}{s-a}$
3)	$\cosh at$	$\frac{s}{s^2-a^2}$
4)	$\cos at$	$\frac{s}{s^2+a^2}$
5)	$\sinh at$	$\frac{a}{s^2-a^2}$
6)	$\sin at$	$\frac{a}{s^2+a^2}$
7)	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$
8)	t^n, $n=1,2,3$	$\frac{n!}{s^{n+1}}$

9) $\cosh x = \frac{e^x + e^{-x}}{2}$

10) $\sinh x = \frac{e^x - e^{-x}}{2}$

Problems

1) Find the Laplace transform of the following

$\cosh^2 3t$

$f(t) = \cosh^2 3t$

$f(t) = \left[\dfrac{e^{3t} + e^{-3t}}{2}\right]^2$

$f(t) = \dfrac{1}{4}\left(e^{6t} + e^{-6t} + 2\right)$

$L[f(t)] = \dfrac{1}{4}\left[\dfrac{1}{s-6} + \dfrac{1}{s+6} + \dfrac{2}{s}\right]$

2) $f(t) = e^{-2t} \sinh 4t$

$f(t) = e^{-2t}\left(\dfrac{e^{4t} - e^{-4t}}{2}\right)$

$f(t) = \dfrac{1}{2}\left(e^{2t} - e^{-6t}\right)$

$L[f(t)] = \dfrac{1}{2}\left(\dfrac{1}{s-2} - \dfrac{1}{s+6}\right)$

$L[f(t)] = \dfrac{4}{(s-2)(s+6)}$

3) $f(t) = \sin 5t \cos 2t$

$\sin A \cdot \cos B = \dfrac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$f(t) = \dfrac{1}{2}\left\{\sin(5t+2t) + \sin t(5t-2t)\right\}$

$f(t) = \dfrac{1}{2}\left\{\sin 7t + \sin 3t\right\}$

$L[f(t)] = \dfrac{1}{2}\left[\dfrac{7}{s^2+49} + \dfrac{3}{s^2+9}\right]$

4) $f(t) = \cos t \cos 2t \cos 3t$

$\cos A \cdot \cos B = \dfrac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$f(t) = \cos t \cdot \cos 2t = \dfrac{1}{2}\left\{\cos(t+2t) + \cos(t-2t)\right\}$

$\cos t \cdot \cos 2t = \dfrac{1}{2}\left\{\cos 3t + \cos t\right\}$

$\cos t \cos 2t \cos 3t = \dfrac{1}{2}\left\{\cos 3t \cos 3t + \cos t \cos 3t\right\}$

$f(t) = \dfrac{1}{2}\left\{\dfrac{1}{2}(\cos 6t + \cos 0) + \dfrac{1}{2}(\cos 4t + \cos 2t)\right\}$

$f(t) = \dfrac{1}{4}\left(\cos 6t + 1 + \cos 4t + \cos 2t\right)$

$L[f(t)] = \dfrac{1}{4}\left[\dfrac{s}{s^2+36} + \dfrac{1}{s} + \dfrac{s}{s^2+16} + \dfrac{s}{s^2+4}\right]$

5) $f(t) = \sin^2(2t+1)$

$f(t) = \frac{1}{2}\{1-\cos 2(2t+1)\}$

$f(t) = \frac{1}{2}\{1-\cos(4t+2)\}$ $\cos(A+B) = \cos A\cdot\cos B - \sin A\cdot\sin B$

$f(t) = \frac{1}{2}[1-\cos 4t\cdot\cos 2 + \sin 4t \sin 2\}$

$L[f(t)] = \frac{1}{2}\left[\frac{1}{s} - \frac{s\cos 2}{s^2+4^2} + \frac{4\sin 2}{s^2+4^2}\right\}$.

$L[f(t)] = \frac{1}{2}\left[\frac{1}{s} - \frac{s\cos 2}{s^2+16} + \frac{4\sin 2}{s^2+16}\right]$

21.4 Properties of Laplace transforms

Property 1 If $L[f(t)] = \bar{f}(s)$ then $L[e^{at}f(t)] = \bar{f}(s-a)$

Proof: we have by the definition

$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$

$L[e^{at}f(t)] = \int_0^\infty e^{-st}e^{at} f(t)\,dt = \int_0^\infty e^{-st+at} f(t)\,dt$

$L[e^{at}f(t)] = \int_0^\infty e^{-(s-a)t} f(t)\,dt$

Comparing this integral with the integral in (1) being denoted by $\bar{f}(s)$ we observe that $(s-a)$ has replaced s.

$L[e^{at}f(t)] = \bar{f}(s-a)$.

$L[e^{-at}f(t)] = \bar{f}(s+a)$

This property known as Shifting property.

21.7 Transform of Derivatives

① If $f'(t)$ be continous and $\mathcal{L}\{f(t)\} = \bar{f}(s)$

then $\mathcal{L}\{f'(t)\} = s\bar{f}(s) - f(0)$.

$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$

$$= e^{-st} f(t)\Big]_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\, dt.$$

Now assuming $f(t)$ to be such that $\lim_{t \to \infty} e^{-st} f(t) = 0$.

when this condition is satisfied,

$f(t)$ is said to be exponential order s.

Thus $\mathcal{L}[f'(t)] = f(0) + s\int_0^\infty e^{-st} f(t)\, dt$

② If $f'(t)$ and its first $(n-1)$ derivatives

be continous then

$\mathcal{L}[f^n(t)] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$.

using the general rule of integration by

parts.

$\mathcal{L}[f^n(t)] = \int_0^\infty e^{-st} f^n(t)\, dt$

$$= e^{-st} f^{n-1}(t) \Big] (-s) e^{-st} f^{n-2}(t) + (-s)^2 e^{-st} f^{n-3}(t) - \dots$$

$$+ (-1)^{n-1} (-s)^{n-1} e^{-st} f(t)\Big]_0^\infty + (-1)^n (-s)^n \int_0^\infty e^{-st} f(t)\, dt$$

$$= -f^{n-1}(0) - s f^{n-2}(0) - s^2 f^{n-3}(0) - \dots - s^{n-1} f(0) + s^n \int_0^\infty e^{-st} f(t)\, dt$$

$\lim_{t \to \infty} e^{-st} f^m(t) = 0$ for $m = 0, 1, 2, \dots, n-1$

This proves the required result.

21.8 Transforms of Integrals

If $\mathcal{L}[f(t)] = \bar{f}(s)$ then $\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$

$\phi(t) = \int_0^t f(u)\,du$ then $\phi'(t) = f(t)$ and
$\phi(0) = 0$.

$$\mathcal{L}[\phi'(t)] = s\bar{\phi}(s) - \phi(0).$$

$$\therefore \bar{\phi}(s) = \frac{1}{s}\mathcal{L}[\phi'(t)].$$

$$\mathcal{L}\left(\int_0^t f(u)\,du\right) = \frac{1}{s}\bar{f}(s)$$

21.9 If $\mathcal{L}[f(t)] = \bar{f}(s)$ then $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$ where n is a positive integer.

Proof: we establish the result by the principal of mathematical induction

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Diff wrt s on both sides.

$$\frac{d}{ds}[\bar{f}(s)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}] f(t)\,dt.$$

In the RHS we shall apply Leibnitz rule for differentiation under the integral sign.

$$\frac{d}{ds}\left[\bar{f}(s)\right] = \int_0^\infty e^{-st}(-t)\, f(t)\, dt.$$

$$(-1)\frac{d}{ds}\left[\bar{f}(s)\right] = \int_0^\infty e^{-st}\left[t\, f(t)\right] dt = \mathcal{L}\left[t f(t)\right]$$

This verifies the result for $n=1$ $\longrightarrow$ ①.

Let us assume the result to be true for $n=k$.

$$(-1)^k \frac{d^k}{ds^k}\left[\bar{f}(s)\right] = \mathcal{L}\left[t^k f(t)\right]. \longrightarrow ②$$

$$(-1)^k \frac{d^k}{ds^k}\left[\bar{f}(s)\right] = \int_0^\infty e^{-st}\left[t^k f(t)\right] dt$$

diff wrt s again we get.

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}\left[\bar{f}(s)\right] = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}\right)\left[t^k f(t)\right] dt$$

$$(-1)^k \frac{d^{k+1}}{ds^{k+1}}\left[\bar{f}(s)\right] = \int_0^\infty e^{-st}(-t)\left[t^k f(t)\right] dt$$

Multiplying (-1) we get.

$$(-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}\left[\bar{f}(s)\right] = \int_0^\infty e^{-st}\left[t^{k+1} f(t)\right] dt$$

$$(-1)^{k+1} \frac{d^{k+1}}{ds^{k+1}}\left[\bar{f}(s)\right] = \mathcal{L}\left[t^{k+1} f(t)\right] \longrightarrow ③$$

Comparing ② and ③.

we conclude that the result is true

for $n=k+1$

Hence by the principal of mathematical

induction

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n}\left[\bar{f}(s)\right] = (-1)^n \bar{f}^{(n)}(s)$$

This property is called the derivative of transform property.

Q1.10 If $\mathcal{L}\left[f(t)\right] = \bar{f}(s)$ then $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds$

Proof : we have $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$

$$\int_s^\infty \bar{f}(s)\, ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\, dt\right] ds.$$

$$= \int_0^\infty \int_s^\infty e^{-st} f(t)\, ds\, dt$$

On changing the order of integration

$$= \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)\, dt$$

$$= \int_0^\infty \left[0 - \frac{e^{-st}}{-t}\right] f(t)\, dt$$

$$\int_s^\infty \bar{f}(s)\, ds = \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right] dt$$

$$\int_s^\infty \bar{f}(s)\, ds = \mathcal{L}\left[\frac{f(t)}{t}\right]$$

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds$$

Properties at a glance

21.4 $\Rightarrow$ $\mathcal{L}[f(t)] = \bar{f}(s)$.

21.4 $\Rightarrow$ $\mathcal{L}[e^{at} f(t)] = \bar{f}(s-a)$. $\rightarrow$ Shifting Property

21.7 $\Rightarrow$ $\mathcal{L}\{f(at)\} = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$ $\Rightarrow$ Time scale

21.8 $\Rightarrow$ $\mathcal{L}\left[\int_0^t f(u)\,du\right] = \frac{1}{s} \bar{f}(s)$ $\Rightarrow$ Integration

21.9 $\Rightarrow$ $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [\bar{f}(s)]$ $\rightarrow$ Multiplication by t.

21.10 $\Rightarrow$ $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$. $\rightarrow$ Division by t

Problems

$\mathcal{L}[e^{at} f(t)] = \bar{f}(s-a)$

1) $e^{-2t}(2\cos 5t - \sin 5t)$

$f(t) = 2\cos 5t - \sin 5t$

$\mathcal{L}[f(t)] = 2 \cdot \frac{s}{s^2+5^2} - \frac{5}{s^2+5^2}$

$\mathcal{L}[f(t)] = \frac{2s}{s^2+25} - \frac{5}{s^2+25}$.

$\mathcal{L}[e^{-2t} f(t)] = \left[\frac{2s-5}{s^2+25}\right]_{s \to s+2}$

$\mathcal{L}[e^{-2t} f(t)] = \frac{2(s+2)-5}{(s+2)^2+25}$

$\mathcal{L}[e^{-2t}(2\cos 5t - \sin 5t)] = \frac{2s-1}{s^2+4s+29}$

2) $e^{-t} \cos^2 3t$ $\quad \cos 2x = 2\cos^2 x - 1$

$f(t) = \cos^2 3t$

$f(t) = \frac{1+\cos 6t}{2}$

$\mathcal{L}[f(t)] = \frac{1}{2}[1 + \cos 6t]$

$\mathcal{L}[f(t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$

$\mathcal{L}[e^{-t} f(t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]_{s \to s+1}$

$\mathcal{L}[e^{-t}\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s+1} + \frac{(s+1)}{(s+1)^2+36}\right]$

3) $f(t) = e^{-4t} t^{-5/2}$

$f(t) = t^{-5/2}$

$\mathcal{L}[f(t)] = \Gamma$

3) $(1 + 3t e^{2t})^2$

$f(t) = (1+3te^{2t})^2.$

$f(t) = 1^2 + 9t^2e^{4t} + 6te^{2t}$

$\mathcal{L}[f(t)] = \mathcal{L}(1) + 9\mathcal{L}[e^{4t}.t^2] + 6\mathcal{L}[t\, e^{2t}].$

$\mathcal{L}[f(t)] = \frac{1}{s} + 9\left[\frac{1}{s-4}\right]\mathcal{L}[t^2] + 6\left[\frac{1}{s-2}\right]\mathcal{L}[t].$

$\mathcal{L}[f(t)] = \frac{1}{s} + 9\left[\mathcal{L}[t^2)\right]_{s \to s-4} + 6\left[\mathcal{L}[t]\right]_{s \to s-2}$

$\mathcal{L}(t) = \frac{1}{s^2} \quad \left[t^n = \frac{n!}{s^{n+1}} = .\, t^1 = \frac{1}{s^{1+1}} = \frac{1}{s^2}\right]$

$\mathcal{L}(t^2) = \frac{2}{s^3} \quad \left[t^n = \frac{n!}{s^{n+1}} = t^2 = \frac{2}{s^{2+1}} = \frac{2}{s^3}\right]$

$\mathcal{L}[(1+3te^{2t})^2] = \frac{1}{s} + \frac{6.1}{s^2} + \frac{9.2}{s^3}$

$\mathcal{L}[(1+3te^{2t})^2] = \frac{1}{s} + \frac{6}{(s-2)^2} + \frac{18}{(s-4)^3}$

4) $\cosh t \sin^3 2t$

$\sin^3 t = \dfrac{3\sin t - \sin 3t}{4}$

$f(t) = \sin^3 2t$

$\sin 3x = 3\sin x - 4\sin^3 x$

$\mathcal{L}[\sin^3 2t] = \dfrac{3\sin 2t - \sin 6t}{4}$

$$\mathcal{L}[\sin^3 2t] = \frac{1}{4}[3 \cdot \sin 2t - \sin 6t]$$

$$\mathcal{L}[\sin^3 2t] = \frac{1}{4}\left[3 \cdot \frac{2}{s^2+4} - \frac{6}{s^2+36}\right]$$

$$\mathcal{L}[\sin^3 2t] = \frac{1}{4}\left[\frac{6}{s^2+4} - \frac{6}{s^2+36}\right]$$

$$\mathcal{L}[\sin^3 2t] = \frac{1}{4}\left[\frac{6(s^2+36) - 6(s^2+4)}{(s^2+4)(s^2+36)}\right]$$

$$\mathcal{L}[\sin^3 2t] = \frac{1}{4}\left[\frac{6s^2+216-6s^2-24}{(s^2+4)(s^2+36)}\right]$$

$$\mathcal{L}[\sin^3 2t] = \frac{48}{(s^2+4)(s^2+36)}$$

$$\mathcal{L}[\cosh t \sin^3 2t] = \mathcal{L}\left[\frac{e^t + e^{-t}}{2} \sin^3 2t\right]$$

$$= \frac{1}{2}\mathcal{L}\left[e^t \sin^3 2t + e^{-t}\sin^3 2t\right]$$

$$= \frac{1}{2}\left[\mathcal{L}(e^t\,(\sin^3 2t) + \mathcal{L}(e^{-t}\,(\sin^3 2t)\right]$$

By shifting property $= \dfrac{1}{2}\left[\dfrac{1}{\cancel{s-1}}\mathcal{L}[\sin^3 2t] + \dfrac{1}{s+1}\mathcal{L}(\sin^3 2t)\right]$

$$= \frac{1}{2}\left[\mathcal{L}(\sin^3 2t)_{s\to s-1} + \mathcal{L}(\sin^3 2t)_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[\frac{48}{(s^2+4)(s^2+36)} + \frac{48}{(s^2+4)(s^2+36)}\right]$$

$$= \frac{1}{2}\left[\frac{48}{((s-1)^2+4)((s+1)^2+36)} + \frac{48}{[(s+1)^2+4][(s+1)^2+36]}\right]$$

$$\mathcal{L}\left[\cosh t \sin^3 2t\right) = \frac{1}{2}\left[\frac{48}{(s^2+1-2s+4)(s^2+1-2s+36} + \frac{48}{(s^2+1+2s+4)(s^2+1+2s+36)}\right.$$

$$\mathcal{L}(\cosh t \sin^3 2t) = \frac{48}{2}\left[\frac{1}{(s^2-2s+5)(s^2-2s+37)} + \frac{1}{(s^2+2s+5)(s^2+2s+37)}\right]$$

$$\mathcal{L}(\cosh t \sin^3 2t) = 24\left[\frac{1}{(s^2-2s+5)(s^2-2s+37)} + \frac{1}{(s^2+2s+5)(s^2+2s+37)}\right]$$

5) $\sinh at \sin at$

$$f(t) = \sinh at$$

$$f(t) = \left[\frac{e^{at}-e^{-at}}{2}\right]\sin at$$

$$f(t) = \frac{1}{2}\left[e^{at}\sin at - e^{-at}\sin at\right]$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\mathcal{L}\left[e^{at}\sin at - e^{-at}\sin at\right]$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\left[\frac{1}{s-a}\mathcal{L}(\sin at) - \frac{1}{s+a}\mathcal{L}(\sin at)\right]$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\left[\mathcal{L}(\sin at)_{s\to s-a} - \mathcal{L}(\sin at)_{s\to s+a}\right]$$

$$\mathcal{L}[\sin at] = \left[\frac{a}{s^2+a^2}\right]$$

$$\mathcal{L}[\sinh at \sin at] = \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2}\right]$$

$$\mathcal{L}[\sinh at \sin at] = \frac{1}{2}\left[\frac{a}{s^2+a^2-2sa+a^2} - \frac{a}{s^2+a^2+2as+a^2}\right]$$

$$\mathcal{L}[\sinh at \sin at] = \frac{a}{2}\left[\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right]$$

$$\mathcal{L}[\sinh at \sin at] = \frac{a}{2}\left[\frac{s^2+2a^2+2as-(s^2+2a^2-2as)}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$$

$$\mathcal{L}[\sinh at \sin at] = \frac{a}{2}\left[\frac{4as}{(s^2+2a^2-2as)\ (s^2+2a^2+2as)}\right]$$

$$= \frac{2a^2 s}{s^2(s^2+2a^2+2as)+2a^2(s^2+2a^2+2as)-2as\ (s^2+2a^2+2as)}$$

$$= \frac{2a^2 s}{s^4+2a^2s^2+2as^3+2a^2s^2+4a^4+4a^3s-2as^3-4a^3s-4a^2s^2}$$

$$= \frac{2a^2 s}{s^4+4a^4}$$

$$\mathcal{L}[\sinh at \sin at] = \frac{2a^2 s}{s^4+4a^4}$$

Multiplication by T

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$$

(1) $t \cos at$

Sol $f(t) = \cos at$

$$\mathcal{L}[f(t)] = \frac{s}{s^2+a^2}$$

$$\mathcal{L}[t\, f(t)] = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$\mathcal{L}[t \cos at] = -\left[\frac{(s^2+a^2)1 - s(2s)}{(s^2+a^2)^2}\right]$$

$$\mathcal{L}[t \cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

2) $f(t) = \sin t \implies t^3 \sin t$

$\mathcal{L}[f(t)] = \frac{1}{s^2+1}$

$\mathcal{L}[t^3 \sin t) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right) = -1 \frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)$

$\mathcal{L}[t^3 \sin t] = -\frac{d}{ds} \cdot \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$

$\mathcal{L}[t^3 \sin t] = -\frac{d}{ds}\frac{d}{ds}\left[\frac{s^2+1(0) - 1(2s)}{(s^2+1)^2}\right]$

$\mathcal{L}[t^3 \sin t] = -\frac{d}{ds}\frac{d}{ds}\left[\frac{-2s}{(s^2+1)^2}\right]$

$\mathcal{L}[t^3 \sin t] = +\frac{d}{ds}\left[\frac{(s^2+1)^2 2 - 2s\, 2(s^2+1)(2s)}{(s^2+1)^4}\right]$

$\mathcal{L}[t^3 \sin t] = \frac{d}{ds}\left[\frac{2(s^2+1)[(s^2+1) - 4s^2}{(s^2+1)^4}\right]$

$\mathcal{L}[t^3 \sin t] = \frac{d}{ds}\left[\frac{2(s^2+1)(1-3s^2)}{(s^2+1)^4}\right]$

$\mathcal{L}[t^3 \sin t) = \frac{d}{ds}\left[\frac{2(1-3s^2)}{(s^2+1)^3}\right]$

$= 2\frac{d}{ds}\left[\frac{1-3s^2}{(s^2+1)^3}\right]$

$= 2\left[\frac{(s^2+1)^3(-6s) - (1-3s^2)\, 3(s^2+1)^2(2s)}{(s^2+1)^6}\right]$

$= 2\left[\frac{(s^2+1)^3(-6s) - (1-3s^2)(s^2+1)^2(6s)}{(s^2+1)^6}\right]$

$= 2(-6s)\left[\frac{(s^2+1)^2[(s^2+1) - (1-3s^2)]}{(s^2+1)^6}\right]$

$$\mathcal{L}[t^3 \sin t] = -12s(s^2+1)^2\left[\frac{(s^2+1)+(1-3s^2)}{(s^2+1)^6}\right]$$

$$= \frac{-12s\,[s^2+1+1-3s^2]}{(s^2+1)^4}$$

$$= \frac{-12s\,(2-2s^2)}{(s^2+1)^4}$$

$$= \frac{24s\,(-1+s^2)}{(s^2+1)^4}$$

$$\mathcal{L}[t^3 \sin t] = \frac{24s\,(s^2-1)}{(s^2+1)^4}$$

3) $t^2 \sin at$

$f(t) = \sin at$

$$\mathcal{L}[f(t)] = \frac{a}{s^2+a^2}$$

$$\mathcal{L}[t^2 \sin at] = \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right)$$

$$= \frac{d}{ds}\left[\frac{s^2+a^2(0) - a(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2(-2a) + 2as\,2(s^2+a^2)(2s)}{(s^2+a^2)^4}$$

$$\mathcal{L}[t^2 \sin at] = \frac{2a(s^2+a^2)\left[-(s^2+a^2)+4s^2\right]}{(s^2+a^2)^4}$$

$$\mathcal{L}[t^2 \sin at] = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

4) $\cancel{\mathcal{L} f(t)}$ $t^5 \cdot e^{4t} \cosh 3t$

$$f(t) = \left[\frac{e^{3t}+e^{-3t}}{2}\right] t^5 e^{4t}$$

$$f(t) = \frac{1}{2}\left[e^{7t} t^5 + e^{t} t^5\right]$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\mathcal{L}\left[e^{7t} t^5\right] + \mathcal{L}\left[e^{t} t^5\right]$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\left[\frac{1}{s-7}\mathcal{L}[t^5]\right] + \frac{1}{s-1}\mathcal{L}[t^5]$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\left[\mathcal{L}[t^5]_{s\to s-7} + \mathcal{L}[t^5]_{s\to s-1}\right]$$

$$\mathcal{L}[t^5] = t^n = \frac{n!}{s^{n+1}} = \frac{5!}{s^6} = \frac{120}{s^6}$$

$$\mathcal{L}[t^5 \cdot e^{4t}\cosh 3t] = \frac{1}{2}\left[\frac{120}{(s-7)^6} + \frac{120}{(s-1)^6}\right]$$

$$\mathcal{L}[t^5 \cdot e^{4t}\cosh 3t] = 60\left[\frac{1}{(s-7)^6} + \frac{1}{(s-1)^6}\right]$$

5) $t^3 \cosh t$

$$f(t) = \cosh t$$

$$f(t) = t^3\left(\frac{e^{t}+e^{-t}}{2}\right)$$

$$f(t) = \frac{1}{3}\left(e^{t} t^3 + t^3 e^{-t}\right)$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\left[\mathcal{L}(t^3)_{s\to s-1} + \mathcal{L}(t^3)_{s\to s+1}\right].$$

$$\mathcal{L}(t^3) = \frac{n!}{s^{n+1}} = \frac{3!}{s^4} = \frac{6}{s^4}.$$

$$\mathcal{L}[t^3\cosh t] = \frac{1}{2}\left[\frac{6}{(s-1)^4} + \frac{6}{(s+1)^4}\right]$$

$$= 3\left[\frac{1}{(s-1)^4} + \frac{1}{(s+1)^4}\right]$$

6) Find the value of $\int_0^\infty t\, e^{-3t}\cos 2t\, dt$ using Laplace transforms ?

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Sol

$$\int_0^\infty e^{-st}\, t\cos 2t\, dt = \mathcal{L}[t\cos 2t) \longrightarrow ①$$

$$\mathcal{L}(t\cos 2t) = \frac{s^2-4}{(s^2+4)^2}$$

Using these result in RHS of 1

$$\int_0^\infty e^{-st}\, t\cos 2t\, dt = \frac{s^2-4}{(s^2+4)^2}.$$

$s = 3$

$$\int_0^\infty e^{-3t}\, t\cos 2t\, dt = \frac{9-4}{(3^2+4)^2} = \frac{5}{(9+4)^2} = \frac{5}{13^2} = \frac{5}{169}$$

7) Show that $\int_0^\infty t\, e^{-2t}\sin 4t\, dt = \frac{1}{25}$

$$\int_0^\infty e^{-st}\, t\sin 4t\, dt = \mathcal{L}(t\sin 4t) \longrightarrow ①.$$

$$\mathcal{L}(\sin 4t) = \frac{4}{s^2+16}.$$

$$\mathcal{L}(t\sin 4t) = -\frac{d}{ds}\left(\frac{4}{s^2+16}\right) = \frac{8s}{(s^2+16)^2}$$

Hence eq ① become

$$\int_0^\infty e^{-st}\, t\sin 4t\, dt = \frac{8s}{(s^2+16)^2}$$

Put $s = 2$ we get :

$$\int_0^\infty e^{-2t}\, t \sin 4t \, dt = \frac{8 \cdot 2}{(4+16)^2} = \frac{16}{400} = \frac{1}{25}$$

Division by t

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds.$$

Imp 1) $\dfrac{\cos at - \cos bt}{t}$ wed

* $f(t) = \cos at - \cos bt$

$$\frac{d}{ds}\left(\log(s^2+a^2)\right) = \frac{2s}{s^2+a^2}$$

$$\mathcal{L}[f(t)] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\Big]_s^\infty$$

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\, ds.$$

$$\mathcal{L}\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$\mathcal{L}\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty \qquad \frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}$$

$$= \frac{1}{2}\left[\lim_{s\to\infty} \log \frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)} - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}.$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

2) $\dfrac{\sin ht}{t}$

$f(t) = \sin ht$ wed

$f(t) = \dfrac{e^{x} \mp e^{-x}}{2.}$

$f(t) = \dfrac{e^{t} - e^{-t}}{2.}$

$$L[f(t)] = \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1}\right] = \frac{1}{2}\left(\frac{s+1-s+1}{(s-1)(s+1)}\right)$$

$$= \frac{1}{2}\left(\frac{2}{(s(s+1)-1(s+1)}\right)$$

$$= \frac{1}{2}\left(\frac{2}{s^2-1}\right)$$

$$\bar{f}(s) = \frac{1}{s^2-1}$$

$$\frac{d}{ds}\left[\log(s^2-1)\right] = \frac{2s}{s^2-1}$$

$$\frac{1}{2s}\left[\log(s^2-1)\right] = \frac{1}{s^2-1}$$

$$L\left[\frac{\sin ht}{t}\right] = \int_s^\infty \bar{f}(s)\, ds.$$

$$= \int_s^\infty \frac{1}{s^2-1}\, ds.$$

$$\int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$$

$$= \frac{1}{2}\left[\log\left(\frac{s-1}{s+1}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{s(1-1/s)}{s(1+1/s)}\right] - \frac{1}{2}\log\left(\frac{s-1}{s+1}\right)$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s-1}{s+1}\right)\right] = \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

$$L\left[\frac{\sin ht}{t}\right] = \log\sqrt{\frac{s+1}{s-1}}$$

3) $\dfrac{\sin^2 t}{t}$ $\cos 2t = 1 - 2\sin^2 t$

No

$f(t) = \sin^2 t.$

$f(t) = \frac{1}{2}(1 - \cos 2t).$

$L[f(t)] = \frac{1}{2} L[1 - \cos 2t]$

$\bar{f}(s) = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right].$

$L\left[\frac{f(t)}{t}\right] = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds.$

$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty.$

$= \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]_s^\infty.$

$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{s}{s\sqrt{1+\left(\frac{4}{s^2}\right)}}\right] - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$

$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s}{\sqrt{s^2+4}}\right)\right].$

$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\left[\log\left(\frac{\sqrt{s^2+4}}{s}\right)\right]$

$\int \frac{s}{s^2+4} ds$

$s^2+4 = t$

$2s\,ds = dt$

$s\,ds = \frac{dt}{2}$

$\frac{1}{2}\int \frac{dt}{t}$

$\frac{1}{2}\log(t)$

$\frac{1}{2}\log(s^2+4)$

4) $\dfrac{\sin at}{t}$

No

$f(t) = \sin at$

$\bar{f}(s) = \frac{a}{s^2+a^2}.$

$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \frac{a}{s^2+a^2} ds$

$$\mathcal{L}\left[\frac{\sin at}{t}\right] = a \cdot \frac{1}{a}\left[\tan^{-1}(s/a)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/a).$$

$$= \frac{\pi}{2} - \tan^{-1}(s/a)$$

$$\mathcal{L}\left[\frac{\sin at}{t}\right] = \cot^{-1}(s/a)$$

No

5) Find $\mathcal{L}\left[\frac{e^{-at}-e^{-bt}}{t}\right] \Rightarrow \int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt$

$$\mathcal{L}\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty \left[\mathcal{L}(e^{-at}) - \mathcal{L}(e^{-bt})\right] dt$$

$$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds = \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \log\left[\frac{s(1+a/s)}{s(1+b/s)}\right] - \log\left(\frac{s+a}{s+b}\right).$$

$$= \log 1 - \log\left(\frac{s+a}{s+b}\right)$$

$$\mathcal{L}\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \log\left(\frac{s+b}{s+a}\right)$$

$$\int_0^\infty e^{-st}\left[\frac{e^{-at}-e^{-bt}}{t}\right] dt = \log\left(\frac{s+b}{s+a}\right)$$

Putting $s=0$.

$$\int_0^s e^{-0}\left[\frac{e^{-at}-e^{-bt}}{t}\right] dt = \log\left(b/a\right)$$

$$\int_0^s \frac{e^{-at}-e^{-bt}}{t}\, dt = \log\left(b/a\right)$$

1) Find the laplace transforms of $t^2 e^{-3t} \sin 2t$

~~$L(t^2 \sin 2t)$~~.

$f(t) = \sin 2t$

$L[f(t)] = L[\sin 2t]$

$L[f(t)] = \dfrac{2}{s^2+4}$

$L[\sin 2t] = \dfrac{2}{s^2+4}$

$L[t^2 \sin 2t) = (-1)^n \dfrac{d^2}{ds^2}(\bar{f}(s))$ wud.

$= (-1)^2 \dfrac{d}{ds}\left(\dfrac{d}{ds}\left(\dfrac{2}{s^2+4}\right)\right)$

$= \dfrac{d}{ds}\left(\dfrac{-4s}{(s^2+4)^2}\right]$

$= \dfrac{(s^2+4)^2(-4) + 4s \; 2(s^2+4)(2s)}{(s^2+4)^4}$

$L[t^2 \sin 2t] = \dfrac{4(3s^2-4)}{(s^2+4)^3}$

By Shifting property

$L\left[e^{-3t} t^2 \sin 2t\right] = \dfrac{4\left[3(s+3)^2-4\right]}{\left[(s+3)^2+4\right]^3}$

2) Using Laplace transforms evaluate $\int_0^\infty e^{-t}\, t \sin^2 3t \, dt$

$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$f(t) = \sin^2 3t$ $\qquad 1 - \sin^2 x = \cos^2 x$

$\qquad\qquad\qquad\qquad - \sin^2 x = -1 + \cos^2 x$

$\sin^2 3t = \dfrac{1 - \cos 6t}{2} = \dfrac{1 - \cos 2x}{2} = \dfrac{1 - \cos 2(3x)}{2}$ $\qquad \sin^2 x = 1 - \cos 2x.$

$\sin^2 3t = \frac{1}{2}(1 - \cos 6t).$

$\mathcal{L}[\sin^2 3t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$. $\quad \mathcal{L}(t \sin^3 3t] = (-1)^n \frac{d}{ds}[\bar{f}(s)]$ used.

$\mathcal{L}[t \sin^2 3t] = \frac{1}{2} \cdot (-1) \frac{d}{ds}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$

$\mathcal{L}[t \sin^2 3t] = \frac{1}{2} \cdot (-1)\left[\frac{-1}{s^2} - \frac{(s^2+36)(1) - s(2s)}{(s^2+36)^2}\right]$

$\mathcal{L}[t \sin^2 3t] = \frac{-1}{2}\left[\frac{-1}{s^2} - \frac{(s^2+36) - 2s^2}{(s^2+36)^2}\right]$

$\mathcal{L}[t \sin^2 3t] = \frac{-1}{2}\left[\frac{-1}{s^2} \mp \frac{(36 + s^2 - 2s^2)}{(s^2+36)^2}\right].$

$\mathcal{L}[t \sin^2 3t] = \frac{-1}{2}\left[\frac{-1}{s^2} - \frac{(36 - s^2)}{(s^2+36)^2}\right]$

$\mathcal{L}[t \sin^2 3t] = \frac{1}{2}\left[\frac{1}{s^2} + \frac{36 - s^2}{(s^2+36)^2}\right]$ Thus by $s = 1$

$\mathcal{L}[t \sin^2 3t] = \frac{1}{2}\left[\frac{1}{1} + \frac{36-1}{(1+36)^2}\right].$

$\mathcal{L}[t \sin^2 3t] = \frac{1}{2}\left[1 + \frac{35}{(37)^2}\right] = \frac{702}{1369}.$

$\int_0^\infty e^{-t} t \sin^2 3t \, dt = \mathcal{L}[t \sin^2 3t]$

$\int_0^\infty e^{-t} t \sin^2 3t \, dt = \frac{702}{1369}.$

3) Prove the following

① $\mathcal{L}\left[\frac{1}{2a}(\sin at + at\cos at)\right] = \frac{s^2}{(s^2+a^2)^2}$.

② $\mathcal{L}\left[\frac{1}{2a^3}(\sin at - at\cos at)\right] = \frac{1}{(s^2+a^2)^2}$.

Sol ① $\frac{1}{2a}\mathcal{L}[\sin at + at\cos at]$. Thus

$$\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$$

$$a\mathcal{L}[t\cos at] = \frac{a(s^2-a^2)}{(s^2+a^2)^2}$$

$$\mathcal{L}[\sin at + at\cos at] = \frac{a}{s^2+a^2} + \frac{a(s^2-a^2)}{(s^2+a^2)^2}$$

$$= a\left[\frac{1}{s^2+a^2} + \frac{(s^2-a^2)}{(s^2+a^2)^2}\right]$$

$$= a\left[\frac{s^2+a^2+s^2-a^2}{(s^2+a^2)^2}\right]$$

$$\mathcal{L}\left[\cancel{\tfrac{1}{2a}}(\sin at + at\cos at)\right] = \frac{2as^2}{(s^2+a^2)^2}$$

$$\mathcal{L}\left[\frac{1}{2a}(\sin at + at\cos at)\right] = \frac{\cancel{2a}s^2}{(s^2+a^2)^2}$$

② $\mathcal{L}\left[\frac{1}{2a^3}(\sin at - at\cos at)\right] = \frac{1}{(s^2+a^2)^2}$

$$\mathcal{L}[\sin at - at\cos at] = \frac{a}{s^2+a^2} - \frac{a(s^2-a^2)}{(s^2+a^2)^2}$$

$$= a\left[\frac{s^2+a^2 - (s^2-a^2)}{(s^2+a^2)^2}\right]$$

$$= a\left[\frac{s^2+a^2-s^2+a^2}{(s^2+a^2)^2}\right]$$

$$= \frac{2a^3}{(s^2+a^2)^2}.$$

$$L\left[\sin at - at\cos at\right] = \frac{2a^3}{(s^2+a^2)^2}.$$

$$L\left[\frac{1}{2a^3}\left(\sin at - at\cos at\right)\right] = \frac{1}{(s^2+a^2)^2}$$

4) $L[\sin\sqrt{t}.]$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

No

$$\sin(\sqrt{t}) = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots$$

$$L[\sin(\sqrt{t})] = L(t^{1/2}) - \frac{L(t^{3/2})}{6} + \frac{L(t^{5/2})}{120.} \longrightarrow (1)$$

$$L(t^{1/2}) = t^n = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}} = \frac{\Gamma(3/2)}{s^{3/2}.}$$

$$= \frac{\Gamma(3/2)}{s^{3/2}} = \frac{1/2\,\Gamma(1/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}.}$$

$$L(t^{3/2}) = \frac{\Gamma(5/2)}{s^{5/2}} = \frac{3/2 \cdot 1/2 \cdot \sqrt{\pi}}{s^{5/2}} = \frac{3\sqrt{\pi}}{4s^{5/2}.}$$

$$L(t^{5/2}) = \frac{\Gamma(7/2)}{s^{7/2}} = \frac{5/2 \cdot 3/2 \cdot 1/2 \cdot \sqrt{\pi}}{s^{7/2}} = \frac{15\sqrt{\pi}}{8s^{7/2}.}$$

Substitituting these values in ① we get

$$\mathcal{L}[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{8s^{5/2}} + \frac{\sqrt{\pi}}{64s^{7/2}} \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{32s^2} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1/4s}{1!} + \frac{(1/4s)^2}{2!} - \cdots\right].$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}.$$

$$\mathcal{L}[\sin\sqrt{t}] = \frac{1}{2s}\sqrt{\frac{\pi}{s}}\, e^{-1/4s}.$$

5) Find the laplace transform of $te^{2t} - \frac{2\sin 3t}{t}$

Sol

$$f(t) = te^{2t} - \frac{2\sin 3t}{t} \Rightarrow f(t) = f_1(t) - f_2(t)$$

$$\mathcal{L}[f(t)] = \mathcal{L}[f_1(t)] - \mathcal{L}[f_2(t)]$$

$$f_1(t) = te^{2t}$$

$$\mathcal{L}[f_1(t)] = \mathcal{L}[te^{2t}]$$

$$\mathcal{L}[f_1(t)] = \left[\frac{1}{s^2}\right]_{s\to s-2}. \qquad t^n = \frac{n!}{s^{n+1}} = \frac{1}{s^{1+1}} = \frac{1}{s^2}$$

$$\mathcal{L}[f_1(t)] = \frac{1}{(s-2)^2}.$$

$$\mathcal{L}[f_2(t)] = 2\mathcal{L}\left[\frac{\sin 3t}{t}\right].$$

$$= 2\int_s^\infty \mathcal{L}(\sin 3t)\, ds.$$

$$= 2\left(\frac{3}{s^2+9}\right]_s^\infty$$

$$\mathcal{L}[f^2(t)] = 2\left[\tan^{-1}(s/3)\right]_s^\infty$$

$$= 2\left[\frac{\pi}{2} - \tan^{-1}(s/3)\right]$$

$\tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2}$

$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

$$= 2\cot^{-1}(s/3)$$

$$\mathcal{L}[f_2(t)] = 2\cot^{-1}(s/3)$$

Thu

$$\mathcal{L}[f(t)] = \mathcal{L}[f_1(t)] - \mathcal{L}[f_2(t)]$$

$$\mathcal{L}[f(t)] = \frac{1}{(s-2)^2} - 2\cot^{-1}(s/3)$$

Laplace transform of Periodic function

Definition A function $f(t)$ is said to be a periodic function period $T > 0$ if $f(t+nT) = f(t)$.
where $n = 1, 2, 3, \ldots$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt.$$

Problems

1) If $f(t) = t^2$, $0 < t < 2$ and $f(t+2) = f(t)$ for $t > 2$, find $\mathcal{L}[f(t)]$

Sol $f(t)$ is a periodic function of period 2.

$T = 2$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}\, t^2\,dt$$

$$L[f(t)] = \frac{1}{1-e^{-2s}} \int_0^2 t^2 e^{-st}\, dt \qquad UV = U\int V - U'\iint V + U''\iiint V$$

Apply Bernoullis rule of integration by parts

$$L[f(t)] = \frac{1}{1-e^{-2s}} \left\{ \left[t^2 \frac{e^{-st}}{-s} \right]_0^2 - \left[2t \frac{e^{-st}}{s^2} \right]_0^2 + 2\left[\frac{e^{-st}}{-s^3} \right]_0^2 \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \frac{-1}{s}[4e^{-2s} - 0] - \frac{2}{s^2}[2e^{-2s} - 0] - \frac{2}{s^3}(e^{-2s} - 1) \right\}$$

$$L[f(t)] = \frac{1}{1-e^{-2s}} \left\{ -\frac{4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3} + \frac{2}{s^3} \right\}$$

$$= \frac{2}{1-e^{-2s}} \left\{ -\frac{2e^{-2s}}{s} - \frac{2e^{-2s}}{s^2} - \frac{e^{-2s}}{s^3} + \frac{1}{s^3} \right\}$$

$$= \frac{2}{1-e^{-2s}} \left\{ \frac{-2s^2e^{-2s} - 2se^{-2s} - e^{-2s} + 1}{s^3} \right\}$$

$$= \frac{2}{s^3(1-e^{-2s})} \left\{ 1 - 2s^2e^{-2s} - 2se^{-2s} - e^{-2s} \right\}$$

$$= \frac{2}{s^3(1-e^{-2s})} \left\{ 1 - e^{-2s}(2s^2 + 2s + 1) \right\}$$

Imp

2) Find the laplace transform of the function $f(t) = E\sin(\pi t/w)$; $0 < t < w$ given $f(t+w) = f(t)$

Sol
$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt \qquad \text{Here } T = w.$$

$$L[f(t)] = \frac{1}{1-e^{-sw}} \int_0^w e^{-st} E \sin(\pi t/w)\, dt$$

$$= \frac{E}{1-e^{-sw}} \left[\frac{e^{-st}}{s^2 + (\pi/w)^2} \left\{ -s\sin(\pi t/w) - \frac{\pi}{w}\cos(\pi t/w) \right\} \right]_0^w$$

$\frac{s^2w^2+\pi^2}{w^2}$

$$= \frac{E}{1-e^{-ws}} \cdot \frac{1}{s^2+(\pi/w)^2}\left[e^{-sw}\left(\frac{\pi}{w}\right)-1\left(\frac{-\pi}{w}\right)\right]$$

$$= \frac{E\pi}{w(1-e^{-ws})} \cdot \frac{1}{\frac{s^2}{1}+\frac{\pi^2}{w^2}}\left[e^{-sw}+1\right]$$

$$= \frac{E\pi}{w(1-e^{-ws})} \cdot \frac{1}{\frac{s^2w^2+\pi^2}{w^2}}\left[e^{-sw}+1\right).$$

$$= \frac{E\pi}{w(1-e^{-ws})} \cdot \frac{w^2}{w^2s^2+\pi^2}\left(1+e^{-ws}\right).$$

$$= \frac{E\pi w}{w^2s^2+\pi^2} \cdot \frac{1+e^{-ws}}{1-e^{-ws}}$$

$$= \frac{E\pi w}{w^2s^2+\pi^2} \cdot \frac{e^{ws/2}+e^{-ws/2}}{e^{ws/2}-e^{-ws/2}}.$$

$$\mathcal{L}[f(t)] = \frac{E\pi w}{w^2s^2+\pi^2}\coth\left(\frac{ws}{2}\right)$$

Imp

3) If $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a-t, & a \le t \le 2a \end{cases}$, $f(t+2a) = f(t)$.

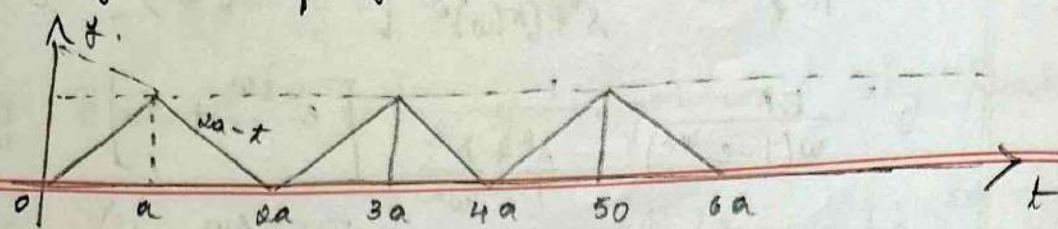
① Sketch the graph of $f(t)$ as a periodic function

② Show that $\mathcal{L}[f(t)] = \frac{1}{s^2}\tanh(as/2)$.

Sol ① Let $f(t) = y$ and $y = t$ is a straight line passing through the origin making an angle 45° with the t axis.

$y = 2a - t$ or $y + t = 2a$ or $t/2a + y/2a = 1$ is a straight line passing through the points $(2a, 0)$ and $(0, 2a)$

The graph of $y = f(t)$ is as follows.



The periodic function $f(t)$ is called the triangular wave function.

(2) $T = 2a$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2as}} \left[\int_0^a t e^{-st}\,dt + \int_a^{2a} (2a-t) e^{-st}\,dt \right]$$

Apply Bernoulli's rule to each of the integrals.

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2as}} \left\{ \left[t \frac{e^{-st}}{-s} - (1)\frac{e^{-st}}{s^2} \right]_0^a + \left[(2a-t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2} \right]_0^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{-1}{s} a e^{-as} \cancel{- \frac{1}{s^2}(e^{-as} - e^{-0})} \right] \cancel{+ \left[(2a-t) e \right.}$$

$$\cancel{- \frac{1}{s} 2a e^{-st}}$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2as}} \left[\frac{-1}{s}(a e^{-as} - 0) - \frac{1}{s^2}(e^{-as} - e^0) \right] + \left[\frac{-1}{s}(2a e^{-st}) - t e^{-st} + \frac{e^{-st}}{s^2} \right]_a^{2a}$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{-1}{s}(a e^{-as}) - \frac{1}{s^2}(e^{-as} - 1) + \left[\frac{-1}{s}(2a e^{-2as} - 2a e^{-as}) - (2a e^{-2as} - a e^{-as}) + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2} \right] \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{-1}{s}(a e^{-as}) - \frac{1}{s^2}(e^{-as} - 1) - \frac{1}{s}(0 - a e^{-as}) + \frac{1}{s^2}(e^{-2as} - e^{-as}) \right]$$

$$= -\frac{1}{s}\left[\underset{0}{2a e^{-2as}} - 2a e^{as} - \underset{0}{2a e^{-2as}} + a e^{-as} \right.$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2as}}\left[-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{ae^{-as}}{s} + \frac{e^{-2as}}{s^2} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-\cancel{sae^{-as}} - e^{-as} + 1 + \cancel{sae^{-as}} + e^{-2as} - e^{-as}}{s^2}\right]$$

$$= \frac{1}{s^2(1-e^{-2as})}\left[1 - 2e^{-as} + e^{-2as}\right] \quad \therefore (1-e^{-as})^2 = 1^2 + e^{-2as} - 2e^{-as}$$

$$= \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

Fund.

$$= \frac{(1-e^{-as})}{s^2(1+e^{-as})}$$

$$= \frac{e^{as/2} - e^{-as/2}}{s^2(e^{as/2}+e^{-as/2})}$$ when we have multiplied both the numerator and denominator by $e^{as/2}$

$$\mathcal{L}[f(t)] = \frac{2\sinh(as/2)}{s^2 \cdot 2\cosh(as/2)} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

$$\mathcal{L}[f(t)] = \frac{1}{s^2}\tanh(as/2)$$

Unit Step funtion (Heaviside funtion)

The unit step funtion $u(t-a)$ or Heaviside funtion $H(t-a)$ is defined as.

$$u(t-a) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$$ where a is a positive constant

$-ae^{-as})$

Properties associated with the unit step function.

① $\mathcal{L}[u(t-a)] = \dfrac{e^{-as}}{s}$

② $\mathcal{L}[f(t-a)\, u(t-a)] = e^{-as} \bar{f}(s)$ where

$\mathcal{L}[f(t)] = \bar{f}(s)$.

Working procedure for problems:

Type 1 — To find $\mathcal{L}[F(t)\, u(t-a)]$ where $F(t)$ is a polynomial in t.

Step 1 — $F(t) = f(t-a)$ which implies that $F(t+a) = f(t)$

This is to replace t by $t+a$ to obtain

$f(t)$ and find $\mathcal{L}[f(t)] = \bar{f}(s)$

Step 2 — $\mathcal{L}[F(t)\, u(t-a)] = \mathcal{L}[f(t-a)\, u(t-a)] = e^{-as} \bar{f}(s)$

by property 2.

Type 2 — Given $f(t)$ as as discontinuous function to find $\mathcal{L}[f(t)]$ by expressing $f(t)$ in terms of unit step function.

Step 1 — we express $f(t)$ in term of unit step function

Step 2 — we find $\mathcal{L}[f(t)]$

Unit step function (Heaviside function)

Express the following function in terms of heaviside unit step function and hence find their laplace transform

1) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$

1) ~~$f(t) = \{ \cos t, 0 <$~~ $\quad \mathcal{L}[F(t)\, u(t-a)]$ by property

~~$f(t) = [\cos t\,(t-0) - \cos t\,(t-\pi)] + [\sin t\,(u$~~

$f(t) = \cos t\, u(t-0) - \cos t\, u(t-\pi) + \sin u(t-\pi).$

$f(t) = \cos t\, u(t) + \sin t\, u(t-\pi) - \cos t\, u(t-\pi).$

$f(t) = \cos t + (\sin t - \cos t)\, u(t-\pi)$

$\mathcal{L}[f(t)] = \mathcal{L}[\cos t] + \mathcal{L}[(\sin t - \cos t)\, u(t-\pi)] \longrightarrow (1)$

$F(t-\pi) = \sin t - \cos t$

$F(t) = \sin(t+\pi) - \cos(t+\pi).$

$F(t) = -\sin t + \cos t.$

$\mathcal{L}[F(t)] = -\dfrac{1}{s^2+1} + \dfrac{s}{s^2+1}$

$\bar{F}(s) = \dfrac{-1+s}{s^2+1}$

$\bar{F}(s) = \dfrac{s-1}{s^2+1}.$

$$\mathcal{L}[F(t)u(t-a)] = \mathcal{L}[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$$

$$\mathcal{L}[f(t-\pi)u(t-\pi)] = \frac{e^{-\pi s}(s-1)}{s^2+1}$$

$$\mathcal{L}[f(t)] = \mathcal{L}(\cos t) + \mathcal{L}[(\sin t - \cos t)u(t-\pi)]$$

$$\mathcal{L}[f(t)] = \frac{s}{s^2+1} + \frac{e^{-\pi s}(s-1)}{s^2+1}$$

$$\mathcal{L}[f(t)] = \frac{s + e^{-\pi s}(s-1)}{s^2+1}$$

2) $$f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ t^2, & t > 2. \end{cases}$$

$\mathcal{L}[f(t)]$

By Property

$$\mathcal{L}[F(t)u(t-a)] = \mathcal{L}[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$$

$$f(t) = 1 - 1u(t-1) + t\,u(t-1) - t\,u(t-2) + t^2u(t-2).$$

$$f(t) = 1 + t\,u(t-1) - 1u(t-1) + t^2u(t-2) - t\,u(t-2)$$

Apply $\mathcal{L}$ $$f(t) = 1 + (t-1)u(t-1) + (t^2-t)u(t-2).$$

$$\mathcal{L}[f(t)] = \mathcal{L}[1] + \mathcal{L}[(t-1)u(t-1)] + \mathcal{L}[(t^2-t)u(t-2)] \rightarrow (1)$$

$F(t-1) = t-1 \rightarrow (2)$

$F(t) = t$

$G(t-2) = t^2 - t \rightarrow (3)$

$G(t) = (t+2)^2 - (t+2)$

$G(t) = t^2 + 3t + 2. \rightarrow$ Doubt

$$\mathcal{L}[F(t)] = \frac{1}{s^2}.$$

$$\bar{F}(s) = \frac{1}{s^2}$$

$$\mathcal{L}[G(t)] = \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}.$$

$$\bar{G}(s) = \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}$$

$\mathcal{L}[F(t-1)u(t-1)] = e^{-s}\bar{F}(s)$ $\quad$ $\mathcal{L}[G(t-2)u(t-2)] = e^{-2s}\bar{G}(s)$

$\mathcal{L}[(t-1)u(t-1)] = \frac{e^{-s}}{s^2}$ $\quad$ $\mathcal{L}[(t^2-t)u(t-2)] = e^{-2s}\left(\frac{2}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right)$

Apply in eq (1)

$$\mathcal{L}[f(t)] = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left(\frac{2}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right)$$

MQP 3) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > 2\pi \end{cases}$

By Property

$$\mathcal{L}[f(t)u(t-a)] = \mathcal{L}[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$$

$\mathcal{L}[f(t)]$

~~$f(t) = \cos t\, u(t-\pi)$~~

$f(t) = \cos t - \cancel{\cos t}\, u(t-\pi) + \cos 2t\, u(t-\pi) - \cos 2t\, u(t-2\pi)$
$+ \cos 3t\, u(t-2\pi)$

$f(t) = \cos t + \cos 2t\, u(t-\pi) - \cos t\, u(t-\pi) + \cos 3t\, u(t-2\pi)$
$- \cos 2t\, u(t-2\pi)$.

$f(t) = \cos t + (\cos 2t - \cos t)\, u(t-\pi) + (\cos 3t - \cos 2t)\, u(t-2\pi)$

$\mathcal{L}[f(t)] = \mathcal{L}[\cos t] + \mathcal{L}[(\cos 2t - \cos t)\, u(t-\pi)] + \mathcal{L}[(\cos 3t - \cos 2t)\, u(t-2\pi)]$ $\rightarrow$ (1)

$F(t-\pi) = \cos 2t - \cos t \rightarrow$ (2) $\quad$ $G(t-2\pi) = \cos 3t - \cos 2t \rightarrow$ (3)

Doubt $\quad F(t) = \cos 2(t+\pi) - \cos(t+\pi)$ $\quad$ $G(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi)$

$\cos(360+\theta) = \cos\theta$ $\quad \mathcal{L}[F(t)] = \mathcal{L}[\cos 2t] + \mathcal{L}[\cos t]$ $\quad$ $\mathcal{L}[G(t)] = \mathcal{L}[\cos 3t] - \mathcal{L}[\cos 2t]$

$\cos(180+\theta) = -\cos\theta$ $\quad \mathcal{L}[F(t)] = \frac{s}{s^2+4} + \frac{s}{s^2+1}$ $\quad$ $\mathcal{L}[G(t)] = \frac{s}{s^2+9} - \frac{s}{s^2+4}$

$\bar{F}(s) = \frac{s}{s^2+4} + \frac{s}{s^2+1}$ $\quad$ $\bar{G}(s) = \frac{s}{s^2+9} - \frac{s}{s^2+4}$

$$\mathcal{L}[F(t-a)u(t-a)] = e^{-as}\bar{F}(s) \quad \text{and}$$

$$\mathcal{L}[G(t-a)u(t-a)] = e^{-as}\bar{G}(s)$$

$$\mathcal{L}[F(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{F}(s)$$

$$\mathcal{L}[(\cos 2t - \cos t)u(t-\pi)] = e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right)$$

$$\mathcal{L}[G(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\bar{G}(s)$$

$$\mathcal{L}[(\cos 3t - \cos 2t)u(t-2\pi)] = e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

From eq ①

$$\mathcal{L}[f(t)] = \mathcal{L}[\cos t) + \mathcal{L}[(\cos 2t - \cos t)u(t-\pi)] + \mathcal{L}[(\cos 3t - \cos 2t)u(t-\pi)]$$

$$\mathcal{L}[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

$$\mathcal{L}[f(t)] = \frac{s}{s^2+1} + se^{-\pi s}\left(\frac{1}{s^2+4} + \frac{1}{s^2+1}\right) + se^{-2s}\left(\frac{s(s^2+4) - s(s^2+9)}{(s^2+4)(s^2+9)}\right)$$

$$\mathcal{L}[f(t)] = \frac{s}{s^2+1} + se^{-\pi s}\left(\frac{1}{s^2+4} + \frac{1}{s^2+1}\right) + se^{-2s}\left(\frac{-5}{(s^2+4)(s^2+9)}\right)$$

$$\mathcal{L}[f(t)] = \frac{s}{s^2+1} + se^{-\pi s}\left(\frac{1}{s^2+4} + \frac{1}{s^2+1}\right) - \frac{5se^{-2s}}{(s^2+4)(s^2+9)}$$

MQP 4) Express the following functions in terms of Heaviside unit step function and hence find the Laplace transform

$$f(t) = \begin{cases} t^2, & 0 < t \leq 2 \\ 4t, & t > 2 \end{cases}$$

Sol By property

$$L[f(t)\,u(t-a)] = L[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$$

$$f(t) = t^2 - t^2 u(t-2) + 4t\,u(t-2).$$

$$f(t) = t^2 + 4t\,u(t-2) - t^2 u(t-2)$$

$$f(t) = t^2 + (4t - t^2)\,u(t-2).$$

$$L[f(t)] = L[t^2] + L[(4t-t^2)\,u(t-2)] \longrightarrow (1).$$

$f(t) = 1$

$L[f(t)] = \frac{1}{s}$

$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s}\right)$

$= \frac{d}{ds}\left(-\frac{1}{s^2}\right) = \frac{2}{s^3}$

$$F(t-2) = 4t - t^2.$$

$$F(t) = 4(t+2) - (t+2)^2.$$

$$F(t) = 4 - t^2$$

$$\bar{F}(s) = L[F(t)] = \frac{4}{s} - \frac{2}{s^3}$$

$$L[F(t-2)\,u(t-2)] = e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right).$$

$$L[F(t)] = \frac{2}{s^3} + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right)$$

5) Define Heaviside unit step function. Using unit step function find the laplace transform of

$$f(t) = \begin{cases} \sin t & 0 \leq t < \pi \\ \sin 2t & \pi \leq t < 2\pi \\ \sin 3t & t \geq 2\pi \end{cases}$$

By property

$$L[f(t)\,u(t-a)] = L[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$$

~~$f(t) = \sin 2t \mp \sin 2t\, u(t-\pi)$~~

$f(t) = \sin t - \sin t\, u(t-\pi) + \sin 2t\, u(t-\pi) - \sin 2t\, u(t-2\pi)$
$+ \sin 3t\, u(t-2\pi)$

$f(t) = \sin t + \sin 2t\, u(t-\pi) - \sin t\, u(t-\pi) + \sin 3t\, u(t-2\pi)$
$- \sin 2t\, u(t-2\pi)$.

$f(t) = \sin t + (\sin 2t - \sin t)\, u(t-\pi) + (\sin 3t - \sin 2t)\, u(t-2\pi)$

$\mathcal{L}[f(t)] = \mathcal{L}[\sin t] + \mathcal{L}[(\sin 2t - \sin t)\, u(t-\pi)] + \mathcal{L}[(\sin 3t - \sin 2t)\, u(t-2\pi)] \longrightarrow ①$

$\mathcal{L}[\sin t] = \frac{1}{s^2+1} \longrightarrow ②$

$\mathcal{L}[(\sin 2t - \sin t)\, u(t-\pi)]$

$F(t-\pi) = \sin 2t - \sin t$

$F(t-\pi+\pi) = \sin 2(t+\pi) - \sin(t+\pi)$.

$\sin(360+\theta) = \sin\theta$

$F(t) = \sin(2\pi + 2t) - \sin(\pi + t)$.

$\sin(180+\theta) = -\sin\theta$

$F(t) = \sin 2t + \sin t$

$\mathcal{L}[F(t)] = \frac{2}{s^2+4} + \frac{1}{s^2+1}$

$\Rightarrow \mathcal{L}[F(t-\pi)\, u(t-\pi)] = e^{-\pi s}\bar{F}(s)$

$\mathcal{L}[F(\sin 2t - \sin t)\, u(t-\pi)] = e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right] \longrightarrow ③$

$\bar{F}(s) = \frac{2}{s^2+4} + \frac{1}{s^2+1} \longrightarrow ③$

$\mathcal{L}[(\sin 3t - \sin 2t)\, u(t-2\pi)]$.

$G(t-2\pi) = \sin 3t - \sin 2t$

$G(t) = \sin 3(t+2\pi) - \sin 2(t+2\pi)$.

$\sin(360+\theta) = \sin\theta$.

$\sin(180+\theta) = -\sin\theta$

$G(t) = \sin 3t - \sin 2t$

$\mathcal{L}[G(t)] = \frac{3}{s^2+9} - \frac{2}{s^2+4}$.

$\bar{G}(s) = \frac{3}{s^2+9} - \frac{2}{s^2+4} \longrightarrow ④$

$$\mathcal{L}[G(t-2\pi)\, u(t-2\pi)] = e^{-2\pi s}\, \bar{G}(s)$$

$$\mathcal{L}[(\sin 3t - \sin 2t)\, u(t-2\pi)] = e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right] \rightarrow (4)$$

From eq (1)

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin t] + \mathcal{L}[(\sin 2t - \sin t)\, u(t-\pi)] + \mathcal{L}[(\sin 3t - \sin 2t)\, u(t-2\pi)]$$

$$\mathcal{L}[f(t)] = \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{2}{s^2+4} + \frac{1}{s^2+1}\right] + e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]$$

Inverse Laplace Transforms:

$$\mathcal{L}[f(t)] = \bar{f}(s) = \mathcal{L}^{-1}[\bar{f}(s)] = f(t).$$

Eg: $\mathcal{L}[1] = \frac{1}{s} \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$

$\mathcal{L}[\cos at] = \frac{s}{s^2+a^2} \Rightarrow \mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

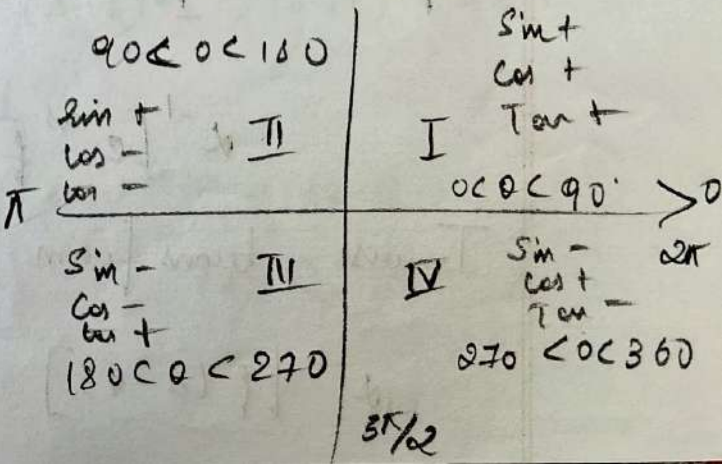
Problem Find the inverse laplace of the transform.

1) $\frac{1}{s+2} + \frac{3}{2s+5} - \frac{4}{3s-2}$

$$\mathcal{L}^{-1}\left[\cancel{\frac{1}{s^2+6^2}}\right] \quad \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \frac{3}{2}\mathcal{L}^{-1}\left[\frac{1}{s+5/2}\right] - \frac{4}{3}\mathcal{L}^{-1}\left(\frac{1}{s-2/3}\right)$$

Thus the required inverse laplace transform is

$$e^{-2t} + 3/2\, e^{-5t/2} - 4/3\, e^{2t/3}$$



2) $\frac{s+2}{s^2+36} + \frac{4s-1}{s^2+25}$

$\mathcal{L}^{-1}\left[\frac{s}{s^2+6^2}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s^2+6^2}\right] + 4\mathcal{L}^{-1}\left[\frac{s}{s^2+5^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2+5^2}\right]$

Thus the required inverse laplace transform is

$\cos 6t + \frac{1}{3}\sin 6t + 4\cos 5t - \frac{1}{5}\sin 5t$

3) $\frac{(s+2)^3}{s^6}$

$\mathcal{L}^{-1}\left[\frac{s^3+6s^2+12s+8}{s^6}\right]$

$\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) + 6\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) + 12\mathcal{L}^{-1}\left[\frac{1}{s^5}\right) + 8\mathcal{L}^{-1}\left(\frac{1}{s^6}\right)$

$\frac{t^2}{2!} + 6\cdot\frac{t^3}{3!} + 12\frac{t^4}{4!} + 8\frac{t^5}{5!}$

Thus the required inverse laplace transform is

$\frac{t^2}{2} + t^3 + \frac{t^4}{2} + \frac{t^5}{15}$

Computation of the inverse theorems

$e^{-as}\bar{f}(s)$

Laplace transform $\mathcal{L}\left[f(t-a)\,u(t-a)\right] = e^{-as}\bar{f}(s)$

$$\mathcal{L}^{-1}\left[e^{-as}\bar{f}(s)\right] = f(t-a)\,u(t-a)$$

Inverse transform by completing the square

$$\mathcal{L}^{-1}\left[\bar{f}(s-a)\right] = e^{at}\mathcal{L}^{-1}\left[\bar{f}(s)\right]$$

(1) Find the inverse laplace transform of

$$\frac{s+5}{s^2-6s+13}$$

$$L^{-1}\left[\frac{s+5}{s^2-6s+13}\right] = L^{-1}\left[\frac{s+5}{(s-3)^2+4}\right].$$

$$L^{-1}\left[\frac{s+5}{(s-3)^2+2^2}\right] = L^{-1}\left[\frac{s-3+3+5}{(s-3)^2+2^2}\right].$$

$$L^{-1}\left[\frac{(s-3)+8}{(s-3)^2+2^2}\right]$$

Here $a = 3$ and $(s-3)$ Changes to s.

$$= e^{3t} L^{-1}\left[\frac{s+8}{s^2+2^2}\right].$$

$$= e^{3t}\left[L^{-1}\left(\frac{s}{s^2+2^2}\right) + 8L^{-1}\left(\frac{1}{s^2+2^2}\right)\right]$$

$$= L^{-1}\left(\frac{s+5}{s^2-6s+13}\right) = e^{3t}(\cos 2t + 4\sin 2t)$$

2) $$\frac{s^2}{(s+1)^3}.$$

$L^{-1}\left[\frac{s^2}{(s+1)^3}\right]$ we need to express s^2 in terms of $(s+1)$

$$s^2 = (s+1)^2 - 2s - 1 = (s+1)^2 - 2(s+1) + 2 - 1$$

$$s^2 = (s+1)^2 - 2(s+1) + 1$$

$$L^{-1}\left[\frac{s^2}{(s+1)^3}\right] = L^{-1}\left[\frac{(s+1)^2 - 2(s+1) + 1}{(s+1)^3}\right]$$

Here $a = -1$ and $(s+1)$ Changes to s.

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s+1)^3}\right] = \mathcal{L}^{-1}\left[\frac{(s+1)^2 - 2(s+1) + 1}{(s+1)^3}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s+1)^3}\right] = \mathcal{L}^{-1}\left[\frac{(s+1)^2}{(s+1)^3} - \frac{2(s+1)}{(s+1)^3} + \frac{1}{(s+1)^3}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s+1)^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{1}{(s+1)^3}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s+1)^3}\right] = e^{-t}\left[\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\frac{2}{s^2} + \mathcal{L}^{-1}\frac{1}{s^3}\right]$$

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s+1)^3}\right] = e^{-t}\left(1 - 2t + \frac{t^2}{2}\right)$$

3) $\dfrac{s+1}{s^2+s+1}$

$$s^2+s+1 = \left(s+\frac{1}{2}\right)^2 - \frac{1}{4} + 1 = \left(s+1/2\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\mathcal{L}^{-1}\left[\frac{s+1}{s^2+s+1}\right] = \mathcal{L}^{-1}\left[\frac{(s+1/2)+1/2}{(s+1/2)^2 + (\sqrt{3}/2)^2}\right]$$

$$= e^{-t/2}\,\mathcal{L}^{-1}\left[\frac{s+1/2}{s^2+(\sqrt{3}/2)^2}\right]$$

$$= e^{-t/2}\left\{\mathcal{L}^{-1}\left(\frac{s}{s^2+(\sqrt{3}/2)^2}\right) + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+(\sqrt{3}/2)^2}\right]\right.$$

$$\mathcal{L}^{-1}\left(\frac{s+1}{s^2+s+1}\right) = e^{-t/2}\left\{\cos\frac{\sqrt{3}t}{2} + \frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t/2\right)\right\}$$

Imp

Inverse transform by the method of partial fractions:

1) $\dfrac{1}{s(s+1)(s+2)(s+3)}$

$$\frac{1}{s(s+1)(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + \frac{D}{s+3}.$$

$$1 = A(s+1)(s+2)(s+3) + B\,s(s+2)(s+3) + C(s)(s+1)(s+3) + D\,s(s+1)(s+2).$$

Put $s = 0 \implies A = 1/6$

$s = -1 \implies B = -1/2$

$s = -2 \implies C = 1/2$

$s = -3 \implies D = -1/6$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)(s+3)}\right] = \frac{1}{6}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{2}\frac{1}{s+2} - \frac{1}{6}\frac{1}{s+3}$$

$$= \frac{1}{6}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{6}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{6}(1) + \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-3t}$$

2) $\dfrac{s^2}{(s^2+1)(s^2+4)}$ take $s^2 = t$

$$\frac{t}{(t+1)(t+4)} = \frac{A}{(t+1)} + \frac{B}{(t+4)}$$

$$t = A(t+4) + B(t+1)$$

$t = -1 \implies A = -1/3$

$t = -4 \implies B = 4/3$

$$\frac{t}{(t+1)(t+4)} = \frac{-1}{3}\cdot\frac{1}{t+1} + \frac{4}{3}\cdot\frac{1}{t+4}$$

$t = s^2$

$$\frac{s^2}{(s^2+1)(s^2+4)} = \frac{-1}{3}\frac{1}{s^2+1} + \frac{4}{3}\frac{1}{s^2+4}$$

$$L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right] = \frac{-1}{3}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{4}{3}L^{-1}\left[\frac{1}{s^2+4}\right].$$

$$L^{-1}\left[\frac{s^2}{(s^2+1)(s^2+4)}\right] = \frac{+1}{3}(2\sin 2t - \sin t)$$

3) $$\frac{(3s+1)e^{-3s}}{(s-1)(s^2+1)}$$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{(s^2+1)}$$

$$3s+1 = A(s^2+1) + \cancel{B(s+)}(Bs+C)(s-1)$$

$s=1 \implies A=2$

$s=0 \implies C=1$

$0 = A+B \implies B=-2$

$$L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = 2L^{-1}\left[\frac{1}{s-1}\right] - 2L^{-1}\left[\frac{1}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2e^{+t} - 2\cos t + \sin t$$

$$L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right]. = \left[2e^{t-3} - 2\cos(t-3) + \sin(t-3)\right]u(t-3)$$

4) $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

Sol $\dfrac{A}{s-1} + \dfrac{Bs+C}{(s^2+2s+5)}$.

$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$

Put $s=1 \Rightarrow A=1$

Put $s=0 \Rightarrow C=2$.

$0 = A+B \Rightarrow B=-1$

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{(s-1)} + \frac{-s+2}{(s^2+2s+5)} = \frac{1}{s-1} + \frac{2-s}{(s+1)^2+4}.$$

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{3-(s+1)}{(s+1)^2+4}.$$

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{3-(s+1)}{(s+1)^2+4)}\right]$$

$$= e^t + e^{-t} L^{-1}\left[\frac{3-s}{s^2+4}\right].$$

$$= e^t + e^{-t}\left[3L^{-1}\left(\frac{1}{s^2+4}\right) - L^{-1}\left(\frac{s}{s^2+4}\right)\right]$$

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = e^t + e^{-t}\left[\frac{3}{2}\cdot\sin 2t - \cos 2t\right]$$

Unit impulse function

$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$, $f(t) = e^{-st}$, $L\{\delta(t-a)\} = e^{-as}$

Problem:

1) Evaluate $\int_0^\infty \sin 2t\ \delta(t - \pi/4)\,dt$

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

$$\int_0^\infty \sin 2t\,\delta\left(t - \frac{\pi}{4}\right) dt = \sin\left(2 \cdot \frac{\pi}{4}\right) = \underline{\underline{1}}$$

2) Evaluate $L\ \frac{1}{t}\delta(t-a)$.

$$L\{\delta(t-a)\} = e^{-as}$$

$$L\left[\frac{1}{t}\delta(t-a)\right] = \int_s^\infty L[\delta(t-a)]\,ds.$$

$$= \int_s^\infty e^{-as}\,ds = \left[\frac{e^{-as}}{-a}\right]_s^\infty = \underline{\underline{\frac{1}{a}e^{-as}}}$$

3) An impulsive Voltage $E\delta(t)$ is applied to a circuit consisting of LRC in series with zero initial conditions. If i be the current at any subsequent time t, Find the limit of i as $t \to 0$?

Sol: The equation of the Circuit

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = E\delta(t)$$

where $i = 0$ when $t = 0$

Taking Laplace transform of both sides

$$L[s\bar{i} - i(0)] + R\bar{i} + \frac{1}{C}\frac{1}{s}\bar{i} = E$$

$$\left(s^2 + \frac{R}{L}s + \frac{1}{CL}\right)\bar{i} = \frac{E}{L}s \quad \text{or}$$

$$(s^2 + 2as + a^2 + b^2)\bar{i} = (E/L)s$$

$R/L = 2a$ and $1/CL = a^2 + b^2$

$$\bar{i} = \frac{E}{L}\frac{(s+a) - a}{(s+a)^2 + b^2} = \frac{E}{L}\left[\frac{s+a}{(s+a)^2+b^2} - a \cdot \frac{1}{(s+a)^2+b^2}\right]$$

On inversion we get

$$i = \frac{E}{L}\left\{e^{-at}\cos bt - \frac{a}{b}e^{-at}\sin bt\right\}$$

where $i = 0$ when $t = 0$ the limit of the current which is E/L.

4) A beam is simply supported at its end $x = 0$ and is clamped at other end $x = l$. It carries a load w at $x = l/4$. Find the resulting deflection at any point.

Sol The differential equation for deflection is

$$\frac{d^4y}{dx^4} = \frac{w}{EI}\delta(x - l/4)$$

Taking Laplace transform.

$$s^4\bar{y} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = \frac{w}{EI} e^{-ls/4}$$

using ; $y(0) = 0$, $y''(0) = 0$, $y'(0) = c_1$, $y'''(0) = c_2$

$$\bar{y} = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{w}{EI}\frac{e^{-ls/4}}{s^4}$$

on inversion it gives

$$y = c_1 x + c_2\frac{x^3}{3!} + \frac{w}{EI}\frac{(x-l/4)^3}{3!}\, u(x - l/4).$$

$$y = c_1 x + \frac{1}{6}c_2 x^3 \qquad 0 < x < l/4$$

$$y = c_1 x + \frac{1}{6}c_2 x^3 + \frac{w}{6EI}(x - l/4)^3 , \quad l/4 < x < l \qquad \rightarrow (1)$$

using $y(l) = 0$ and $y'(l) = 0$

$$0 = c_1 l + \frac{1}{6}c_2 l^3 + 9wl^3/128EI \quad \text{and}$$

$$0 = c_1 + \frac{1}{2}c_2 l^2 + 9wl^2/32EI$$

$$c_1 = 9wl^2/256\,EI$$

$$c_2 = -81w/128EI$$

substituting the value of c_1 and c_2 in (1) we get deflection at any part