Module 4 : Numerical Methods - I

Importance of numerical methods for discrete data in the field of CS, EC and EE engineering:

Solutions of algebraic and transcedental equation

Equation involving algebraic quantites like $x, x^2, x^3, \sqrt{x}, 1/x$ etc are called as algebraic eq

Examples : ① $x^3 - 4x - 9 = 0$ ② $x^4 + x^3 = 80$

Equations involving non algebraic quantites like $e^x, \log x, \sin x, \tan x$ etc are transcedental eq

Eg : ① $xe^x - 2 = 0$ ② $x \log_e x - 12 = 0$ ③ $\tan x = 2x$

=> Regula Falsi method or Method of False Position :

This is the geometrical method for finding an approximate real root of the given eq

Formula $$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Problems :

1) Use the regula falsi method to find a real root of the equation $x^3 - 2x - 5 = 0$ correct to three decimal place.

Sol: Let $f(x) = x^3 - 2x - 5$

$f(0) = 0^3 - 2(0) - 5 = -5$

$f(1) = 1^3 - 2(1) - 5 = -6$

$f(2) = 2^3 - 2(2) - 5 = -1 < 0$

$f(3) = 3^3 - 2(3) - 5 = 16 > 0$

A real root lies in $(2,3) = (a,b) = b - a = 3 - 2 = 1$

$2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 3$

$f(x) = x^3 - 2x - 5$

$f(2.1) = (2.1)^3 - 2(2.1) - 5 = 9.261 - 4.2 - 5 = 0.061$

$f(2.1) = +0.061 > 0$

The roots lies in $(2, 2.1)$

$a = 2 \qquad f(a) = f(2) = -1$

$b = 2.1 \qquad f(b) = f(2.1) = 0.061$

We have.

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

I st approximation $x_1 = \dfrac{2(0.061) - 2.1(-1)}{0.061 + 1} = \dfrac{0.122 + 2.1}{1.061}$

$$x_1 = \frac{2.222}{1.061}$$

$$x_1 = 2.0942$$

II nd approximation

$f(x_1) = f(2.0942) = x^3 - 2x - 5$

$f(2.0942) = (2.0942)^3 - 2(2.0942) - 5$

$f(2.0942) = 9.18477 - 4.1884 - 5$

$= -3.923 \times 10^{-3} = -0.00392$

$f(2.0942) = -0.00392 < 0$

The root lies in $(2.0942, 2.1)$

$a = 2.0942 \qquad f(a) = -0.00392$

$b = 2.1 \qquad f(b) = 0.061$

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_2 = \frac{2.0942(0.061) - 2.1(-0.00392)}{0.061 - (-0.00392)}$$

$$x_2 = \frac{0.1277462 + 0.008232}{0.06492}$$

$x_2 = 2.09455$

x_1 and x_2 are close enough, x_2 is a better approximation than x_1

Thus the required approximate root correct to 3 decimal place is $\underline{\underline{2.095}}$.

2) Use regula falsi method to find the fourth root of 12 correct to three decimal places.

Sol) Let $x = \sqrt[4]{12} \quad \therefore x^4 = 12$ or $x^4 - 12 = 0$

$f(x) = x^4 - 12$ we have

$f(0) = 0 - 12 = -12 < 0$

$f(1) = 1 - 12 = -11 < 0$

$f(2) = 2^4 - 12 = 4 > 0$

A real root lies in $(1, 2) = (a, b) = b - a = 2 - 1 = 1$

1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9

$f(x) = x^4 - 12$

$f(1.1) = (1.1)^4 - 12 = 1.4641 - 12 = -10.5359 < 0$

$f(1.2) = (1.2)^4 - 12 = 2.0736 - 12 = -9.9264 < 0$

$\vdots$

$f(1.7) = (1.7)^4 - 12 = -3.6479 < 0$

$f(1.8) = (1.8)^4 - 12 = 10.4976 - 12 = -1.5024 < 0$

$f(1.9) = (1.9)^4 - 12 = 13.0321 - 12 = 1.0321 > 0$

$\therefore$ The root in the neighborhood of 2 lies in $(1.8, 1.9)$.

I[st] approximation : Let $a = 1.8$ $\quad f(a) = -1.5024$
$\qquad b = 1.9 \quad f(b) = 1.0321$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_1 = \frac{(1.8)(1.0321) - 1.9(-1.5024)}{1.0321 - (-1.5024)}$$

$$x_1 = \frac{1.85778 + 2.85456}{2.5345} = 1.8593$$

II step : $f(1.8593) = x^4 - 12 = (1.8593)^4 - 12$

$f(1.8593) = -0.0492 < 0$

$\therefore$ The root lies in $(1.8593, 1.9)$

Now $a = 1.8593 \quad f(a) = -0.0492$
$\quad b = 1.9 \quad f(b) = 1.0321$

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_2 = \frac{1.8593\,(1.0321) - 1.9\,(-0.0492)}{1.0321 - (-0.0492)}$$

$$x_2 = \frac{1.91898 + 0.09348}{1.0813} = 1.8612$$

III step : $f(x_2) = f(1.8612) = x^4 - 12$

$$= (1.8612)^4 - 12 = -0.00025 < 0$$

$\therefore$ The root lies in $(1.8612, 1.9)$

Now $a = 1.8612$ $\quad f(a) = -0.00025$

$b = 1.9$ $\quad f(b) = 1.0321$

Substituting in ① we obtain

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$x_3 = \frac{1.8612\,(1.0321) - 1.9\,(-0.00025)}{1.0321 - (-0.00025)}$$

$$x_3 = \frac{1.92094 + 0.000475}{1.03235} = 1.8612 \simeq 1.861$$

Thus the required fourth root of 12 correct to 3 decimal place is $\underline{\underline{1.861}}$.

3) Find the real root of the equation $\cos x = 3x - 1$ correct to three decimal by using regula falsi method.

Let $f(x) = \cos x + 1 - 3x$

$f(0) = \cos(0) + 1 - 3(0) = 1 + 1 = 2 > 0$

$f(1) = \cos(1) + 1 - 3(1) = \underline{0.540} \; -1.46 < 0$

A real root lies in (0,1) and we expect the root in the neighbourhood of 1

Consider 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1

$f(0.1) = \cos x + 1 - 3x$

$f(0.1) = \cos(0.1) + 1 - 3(0.1) = 1.695 > 0$

$f(0.1) = \cos(0.1) + 1 - 3(0.1) = 1.380 > 0$

$f(0.2) = \cos(0.2) + 1 - 3(0.2) =$

$f(0.6) = \cos(0.6) + 1 - 3(0.6) = 0.82 + 1 - 1.8 = 0.0253 > 0$

$f(0.7) = \cos(0.7) + 1 - 3(0.7) = 0.76 + 1 - 2.1 = -0.3352 < 0$

$\therefore$ The root lies in $(0.6, 0.7)$

Ist iteration : $a = 0.6 \quad f(a) = 0.0253$

$b = 0.7 \quad f(b) = -0.3352$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_1 = \frac{0.6(-0.3352) - 0.7(0.0253)}{-0.3352 - 0.0253} = \frac{-0.2188}{-0.3605} = 0.6069$$

-0.219

-0.361

$$x_1 = \frac{-0.20112 - 0.01771}{-0.3605} = \frac{-0.21883}{-0.3605} = 0.607$$

$x_1 = 0.607 = 0.6069$

IInd iteration

$f(x_1) = f(0.607) = \cos(0.607) + 1 - 3(0.607)$

$f(x_1) = 0.00036 > 0 \quad 0.0007$

$\therefore$ The root lies in $(0.607, 0.7)$

Now $a = 0.6079 \quad f(a) = 0.00036 = 0.0007$

$b = 0.7 \quad f(b) = -0.3352$

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$ $-0.2034 - 0.000$

$$x_2 = \frac{0.607(-0.3352) - 0.7(0.00036)}{-0.3352 - 0.00036} = \frac{-0.2039}{-0.3359}$$

$$x_2 = \frac{-0.2034664 - 0.000252}{-0.33556} = \frac{-0.2037184}{-0.33556}$$

$$x_2 = 0.607$$ 0.6070

Thus the real root correct to 3 decimal is $\underline{0.607}$

$\log e = \ln$

4) Find the smallest positive root of the equation $x^2 - \log_e x = 12$ by Regula-Falsi method

Sol: Let $f(x) = x^2 - \log_e x - 12$

$f(0) = 0 - \log_e(0) - 12$

$f(1) = 1^2 - \log_e(1) - 12 = -11 < 0$

$f(2) = 2^2 - \log_e(2) - 12 = 4 - 0.693 - 12 = -8.69 < 0$

$f(3) = 3^2 - \log_e(3) - 12 = 9 - \quad -12 = -4.0986 < 0$

$f(4) = 4^2 - \log_e(4) - 12 = 16 - \quad -12 = 2.6137 > 0$

$\therefore$ a real root lies in $(3, 4)$ and will be in the neighborhood of 4.

Now 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9

$f(x) = x^2 - \log_e x - 12$

$f(3.1) = (3.1)^2 - \log_e(3.1) - 12 =$

$f(3.6) = (3.6)^2 - \log_e(3.6) - 12 = 12.96 - \quad - 12 = -0.3209$

$f(3.7) = (3.7)^2 - \log_e(3.7) - 12 = 13.69 - \quad - 12 = 0.3817$

-0.017

-0.201

I st iteration Let $a = 3.6$ $f(a) = -0.3209$

$b = 3.7$ $f(b) = 0.3817$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_1 = \frac{3.6(0.3817) - 3.7(-0.3209)}{0.3817 - (-0.3209)}$$

$$x_1 = \frac{1.37412 + 1.18733}{0.3817 + 0.3209} = \frac{2.56145}{0.7026} = 3.6457$$

$x_1 = 3.6457$

II iteration : $f(x) = x^2 - \log_e x - 12$

$$f(3.6457) = (3.6457)^2 - \log_e(3.6457) - 12 = \cancel{0.0}$$
$$= -0.00242 < 0$$

The root lies in $(3.6457, 3.7)$.

Now $a = 3.6457$ $f(a) = -0.00242$

$b = 3.7$ $f(b) = 0.3817$

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_2 = \frac{3.6457(0.3817) - 3.7(-0.00242)}{0.3817 + 0.00242}$$

$$x_2 = \frac{1.39156369 + 0.008954}{0.38412}$$

$$x_2 = \frac{1.40051769}{0.38412}$$

$x_2 = 3.646042$

Thus the required root correct to 3 decimal place is 3.646

5) Find the root of the equation $xe^x - \cos x = 0$ by the method of false position ?

Sol: Let $f(x) = xe^x - \cos x$

$$f(0) = 0e^0 - \cos(0) = -1 < 0$$

$$f(1) = 1e^1 - \cos(1) = 2.178 > 0$$

Further we have 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9

$$f(0.5) = 0.5e^{0.5} - \cos(0.5) = -0.0532 < 0$$

$$f(0.6) = 0.6e^{0.6} - \cos(0.6) = 0.2679 > 0$$

The root lies in $(0.5, 0.6)$

I iteration : Let $a = 0.5$ $\quad f(a) = -0.0532$

$b = 0.6$ $\quad f(b) = 0.2679$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{0.5(0.2679) - 0.6(-0.0532)}{0.2679 - (-0.0532)}$$

$$x_1 = \frac{0.13395 + 0.03192}{0.2679 + 0.0532} = \frac{0.16587}{0.3211}$$

$$x_1 = \underline{\underline{0.5166}}$$

II iteration

$$f(x) = xe^x - \cos x$$

$$= 0.5166\, e^{0.5166} - \cos(0.15166)$$

$$= -0.0035 < 0$$

$\therefore$ The root lies in $(0.5166, 0.6)$

Now $a = 0.5166$ $f(a) = -0.0035$

$b = 0.6$ $f(b) = 0.2679$

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_2 = \frac{0.5166(0.2679) - 0.6(-0.0035)}{0.2679 - 0.5166}$$

$$x_2 = 0.5177$$

III iteration : $f(x) = xe^x - \cos x$

$$f(0.5177) = 0.5177e^{0.5177} - \cos(0.5177)$$

$$f(0.5177) = -0.00017 < 0$$

The root lies in $(0.5177, 0.6)$.

Now $a = 0.5177$ $f(a) = -0.00017$

$b = 0.6$ $f(b) = 0.2679$

$$x_3 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$x_3 = \frac{0.5177(0.2679) - 0.6(-0.00017)}{0.2679 - (-0.00017)}$$

$$x_3 = 0.5178$$

Thus the required root is 0.5178

Newtons - Raphson methods

In these method we locate an approximate real root of the given eq and improve its accuracy by an iterative process.

The first approximation of the root x_0 is given by.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The second approximation is obtained replacing x_0 by x_1

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \ldots \text{ and so on}$$

Problems :

1) use newtons - Raphson method to find a real root of the equation $x^3 - 2x - 5 = 0$ Correct to three decimal places ?

Sol : $f(x) = x^3 - 2x - 5$

$f(0) = 0 - 2(0) - 5 = -5 < 0$

$f(1) = 1 - 2(1) - 5 = -6 < 0$

$f(2) = 8 - 2(2) - 5 = -1 < 0$

$f(3) = 27 - 2(3) - 5 = 16 > 0$

A real root lies in $(2, 3)$ and let $x_0 = 2$

The first approximation is given by

$$\cancel{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)}}$$

$f(x) = x^3 - 2x - 5$

$f'(x) = 3x^2 - 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} \Rightarrow f(x) = x^3 - 2x - 5$$

$$f(2) = 8 - 2(2) - 5$$

$$f(2) = -1$$

$$f'(x) = 3x^2 - 2$$

$$f'(2) = 3(2)^2 - 2$$

$$f'(2) = 10$$

$$x_1 = 2 - \frac{(-1)}{10}$$

$$x_1 = 2 + \frac{1}{10}$$

$$x_1 = 2.1$$

$$f(2.1) = (2.1)^3 - 2(2.1) - 5$$

$$f(2.1) = 0.0610$$

$$f'(2.1) = 3x^2 - 2$$

$$f'(2.1) = 3(2.1)^2 - 2$$

$$f'(2.1) = 11.2300$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1 - \frac{f(2.1)}{f'(2.1)}$$

$$x_2 = 2.1 - \frac{0.0610}{11.2300}$$

$$x_2 = 2.1 - 0.0054$$

$$x_2 = 2.0946$$

$$f(2.0946) = 0.0005$$

$$f'(2.0946) = 11.1620$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = (2.0946) - \frac{f(2.0946)}{f'(2.0946)}$$

$$x_3 = 2.0946 - \frac{0.0005}{11.1620}$$

$$x_3 = 2.0946 - 0$$

$$x_3 = 2.0946$$

The required approximate to 3 decimal place is

2.0945

2) Show that a root of the equation $x^3 + 5x - 11 = 0$ lies b/w 1 and 2. Find the root by Newton-Raphson method (carry out 3 iteration)

Sol Let $f(x) = x^3 + 5x - 11$

$f(0) = -11 < 0$

$f(1) = 1 + 5 - 11 = -5 < 0$

$f(2) = 8 + 5(2) - 11 = 7 > 0$

A real root lies b/w $(1, 2)$ and let $x_0 = 1$

And we have $f(x) = x^3 + 5x - 11$

$f'(x) = 3x^2 + 5$

I Approximation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)}$$

$f(1) = 1 + 5(1) - 11 = -5$

$f'(1) = 3(1) + 5 = 8$

$$x_1 = 1 - \frac{(-5)}{8}$$

$x_1 = 1.625$

II Approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.625 - \frac{f(1.625)}{f'(1.625)}$$

$f(1.625) = (1.625)^3 + 5(1.625) - 11$

$f(1.625) = 1.4160$

$f'(1.625) = 3(1.625)^2 + 5$

$f'(1.625) = 12.9219$

$$x_2 = 1.625 - \frac{1.4160}{12.9219}$$

$x_2 = 1.625 - 0.1096$

$x_2 = 1.5154$

III Approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.5154 - \frac{f(1.5154)}{f'(1.5154)}$$

$f(1.5154) = (1.5154)^3 + 5(1.5154) - 11$

$f(1.5154) = 0.0570$

$f'(1.5154) = 3(1.5154)^2 + 5$

$f'(1.5154) = 11.8893$

$x_3 = \cancel{11.8893}$

$$x_3 = 1.5154 - \frac{0.0570}{11.8893}$$

$x_3 = 1.5154 - 0.0048$

$x_3 = 1.5106$

Thus the required root is 1.5106

3) Find a real root of the equation $x^3+x^2+3x+4=0$ applying Newton - Raphson method. Carry out 2 iteration

Sol: $f(x) = x^3+x^2+3x+4$

$f(0) = 0+0+3(0)+4 = 4 > 0$

$f(1) = 1+1+3+4 = 9 > 0$

$f(-1) = 1 > 0$

$f(-2) = -6 < 0$

A real root lies in $(-2,-1)$; let $x_0 = -1$

we also have.

$f(x) = x^3+x^2+3x+4$

$f'(x) = 3x^2+2x+3$

I Approximation

$\cancel{f(x)} =$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 - \frac{f(-1)}{f'(-1)}$$

$f(-1) = (-1)^3+(-1)^2+3(1)+4$

$f(-1) = 1$

$$x_1 = -1 - \frac{1}{4}$$

$x_1 = -1 - 0.250 = -1.2500$

II Approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1.2500 - \frac{f(-1.2500)}{f'(-1.2500)}$$

$$x_2 = -1.2500 - \frac{[(-1.25)^3+(-1.25)^2+3(1.25)+4]}{[3(-1.25)+2(+1.25)+3]}$$

$$x_2 = -1.2500 - \frac{(-0.1406)}{-5.1875}$$

$x_2 = -1.2500 - 0.0275$

$x_2 = -1.229$

Thus the required real root is $-1.2229 \simeq -1.223$

4) Use Newton-Raphson method to find $\sqrt[3]{37}$ correct to 3 decimal place

Sol: Let $x = \sqrt[3]{37}$

$x = (37)^{1/3}$

$x^3 = 37$

$x^3 - 37 = 0$

$f(x) = x^3 - 37$

$f(0) = -37 < 0$

$f(1) = -36 < 0$

$f(2) = -29 < 0$

$f(3) = -10 < 0$

$f(4) = 27 > 0$

A root lies b/w $(3, 4)$

at let $x_0 = 3$

$f(x) = x^3 - 37$

$f'(x) = 3x^2$

I first Approximation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 + \frac{f(3)}{f'(3)}$$

$$x_1 = 3 - \frac{(-10)}{27} = 3 + 0.3704$$

$x_1 = 3.370$

II nd Approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.370 - \frac{f(3.370)}{f'(3.370)}$$

$(3.370)^3 - 37 = 1.273$

$3(3.370)^2 = 34.071$

$$x_2 = 3.370 - \frac{1.273}{34.071}$$

$= 3.370 - 0.037$

$x_2 = 3.333$

III rd Approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.333 - \frac{f(3.333)}{f'(3.333)}$$

$(3.333)^3 - 37$

$37.026 - 37 = 0.026$

$3(3.333)^2 = 33.327$

$$x_3 = 3.333 - \frac{0.026}{33.327}$$

$x_3 = 3.333 - 0.001$

$x_3 = 3.332$

IV Approximation

$$f(3.332) = (3.332)^3 - 37 = 0.007$$

$$f'(3.332) = 3(3.332)^2 = 33.307$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 3.332 - \frac{f(3.332)}{f'(3.332)}$$

$$x_4 = 3.332 - \frac{0.007}{33.307} = 3.332 - (-0.000)$$

$$x_4 = 3.332$$

Thus the required $\sqrt[3]{37}$ correct to 3 decimal place is 3.332

5) Find a real root of the equation $xe^x - 2 = 0$ correct to three decimal places using Newton Raphson method.

Sol: Let $xe^x - 2 = 0$

$$f(x) = xe^x - 2.$$

$$f(0) = 0e^0 - 2 = -2 < 0$$

$$f(1) = 1e^1 - 2 = 0.7183 > 0$$

A real root lies in $(0, 1)$ and let $x_0 = 1$

$$f(x) = xe^x - 2$$

$$f'(x) = xe^x + e^x = e^x(x+1)$$

I st Approximation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(1) = 1e^1 - 2 = 0.718$

$f'(1) = 5.437$

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{0.718}{5.437} = 1 - 0.132 = 0.868.$$

II nd Approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$f(1) = 0.068$

$f'(1) = 1.143$

$$x_2 = 0.868 - \frac{f(0.868)}{f'(0.868)}$$

$$x_2 = 0.868 - \frac{0.068}{1.143} = 0.868 - 0.059 = \underline{\underline{0.809}}$$

4.450

$0.868 - 0.015 = 0.853$

III rd Approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.809 - \frac{f(0.809)}{f'(0.809)}$$

$$x_3 = 0.809 - \frac{(-0.183)}{4.062} = 0.809 + 0.045 = \underline{\underline{0.854}}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$0.853 - 0.00$

$$= 0.853 - \frac{f(0.853)}{f'(0.853)} = 0.853 - \frac{0.002}{4.348}$$

$$= \underline{\underline{0.853}}$$

IV Approximation

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 0.854 - \frac{f(0.854)}{f'(0.854)}$$

$$x_4 = 0.854 - \frac{0.006}{4.355} = 0.854 - 0.001 = \underline{0.853}$$

V Approximation

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$

$$x_5 = 0.853 - \frac{f(0.853)}{f'(0.853)} = 0.853 - \frac{0.002}{4.348} = 0.853 - 0.000$$

$x_5 = 0.853$

Thus the required real root correct to three decimal place is 0.853.

6) Using Newtons Raphson method find the root of the equation $\tan x = x$ that lies near $x = 4.5$ correct to 4 decimal places?

Soln $f(x) = \tan x - x$

$$f'(x) = \sec^2 x - 1 = \tan^2 x = (\tan x)^2$$

Also $x_0 = 4.5$

I Approximation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 4.5 - \frac{f(4.5)}{f'(4.5)} = 4.5 - \frac{0.1373}{21.5048} = 4.5 - 0.0064 = 4.4936$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 4.4936 - \frac{f(4.4936)}{f'(4.4936)} = 4.4936 - \frac{0.0039}{20.2271} = 4.4936 - 0.0002 = 4.4934$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 4.4934 - \frac{f(4.4934)}{f'(4.4934)} = 4.4934 + \frac{0.0002}{20.1889} = 4.4934 + 0 = 4.4934$$

Thus the required real root is 4.4934

Finite differences

7) Use Newtons - Raphson method to final a real root of $x \sin x + \cos x = 0$ near $x = \pi$ Carry out the iterations upto four decimal places?

Sol $f(x) = x \sin x + \cos x$

$f'(x) = x \cos x + \sin x - \sin x = x \cos x$

Also $x_0 = \pi$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi - \frac{f(\pi)}{f'(\pi)} = \cancel{3.147} - \cancel{\frac{f(3.147)}{f'(3.147)}}$$

$$= \pi - \frac{\pi \sin \pi + \cos \pi}{\pi \cos \pi}$$

$$= \pi - \frac{1}{\pi} = 2.8226 = 2.825$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.7986$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7984$$

Finite Differences

Let $y = f(x)$ be a function, in this funtion x is independent Variable and y is dependent Variable. By assigning a value for x, we get a corresponding Value $y = f(x)$ for the funtion.

The value assigned to x is Called the argument The corresponding value of the funtion is Called entry.

The difference between 2 consecutive value of x is called the interval of difference. It is denoted by h

Forward differences

Let $y = f(x)$ be a funtion and h is the interval of difference, then $f(x+h) - f(x)$ is Called the first forward difference of the funtion between the arguments x and $x+h$. It is denoted by $\Delta f(x)$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta f(x_0) = f(x_0+h) - f(x_0) \; ; \; \Delta y_0 = y_1 - y_0$$

$$\Delta f(x_1) = f(x_1+h) - f(x_1) \; ; \; \Delta y_1 = y_2 - y_1$$

$$\vdots$$

$$\Delta f(x_n) = f(x_n+h) - f(x_n) \; ; \; \Delta y_{n-1} = y_n - y_{n-1}$$

The difference of the first forward differences are called second forward differences

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 \quad , \quad \Delta^2 y_2 = \Delta y_3 - \Delta y_2$$

$$\Delta^2 y_{n-2} = \Delta y_{n-1} - \Delta y_{n-2}$$

Forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	...	$\Delta^n y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
				$\Delta^3 y_0$		
		Δy_1				$\Delta^n y_0$
x_2	y_2		$\Delta^2 y_1$			
		Δy_2		$\Delta^3 y_1$		
x_3	y_3					
			$\Delta^2 y_2$			
		Δy_3				
x_4	y_4					

The first entries in the table namely $\Delta y_0, \Delta^2 y_0, \dots \Delta^n y_0$ are called leading forward differences.

Backward differences:

Let $y = f(x)$ be a given function & h be the difference of interval, then the backward difference ∇ is defined as $\nabla f(x) = f(x) - f(x-h)$

$\nabla f(x_n) = f(x_n) - f(x_n - h)$ or $\nabla y_n = y_n - y_{n-1}$

$\nabla f(x_{n-1}) = f(x_{n-1}) - f(x_{n-1} - h)$ or $\nabla y_{n-1} = y_{n-1} - y_{n-2}$

$\vdots$

$\nabla y_2 = y_2 - y_1$

$\nabla y_1 = y_1 - y_0$

Similary second backward difference are defined

$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$

$\nabla^2 y_{n-1} = \nabla y_{n-1} - \nabla y_{n-2}$

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$

Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	....	$\nabla^n y$
x_0	y_0	∇y_1	$\nabla^2 y_2$			
x_1	y_1			$\nabla^3 y_3$		$\nabla^n y_n$
		∇y_2				
x_2	y_2		$\nabla^2 y_3$			
		∇y_3		$\nabla^3 y_n$		
x_3	y_3		$\nabla^2 y_n$			
$\vdots$						
x_{n-}	y_n	∇y_n				

The last entries in the table namely $\nabla y_n, \nabla^2 y_n, \nabla^3 y_n \dots \nabla^n y_n$ are called leading backward differences.

Example:

1) Construct the table of differences for the following data

x	0	10	20	30	40
$f(x)$	1	1.5	2.2	3.1	4.6

Evaluate $\Delta^2 f(20)$, $\Delta^3 f(10)$ and $\Delta^4 f(0)$

Sol: We shall construct the forward difference table

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 0$	$1 = y_0$	$\Delta y_0 = 0.5$	$\Delta^2 y_0 = 0.2$	$\Delta^3 y_0 = 0$	$\Delta^4 y_0 = 0.4$
$x_1 = 10$	$1.5 = y_1$	$\Delta y_1 = 0.7$	$\Delta^2 y_1 = 0.2$	$\Delta^3 y_1 = 0.4$	
$x_2 = 20$	$2.2 = y_2$	$\Delta y_2 = 0.9$	$\Delta^2 y_2 = 0.6$		
$x_3 = 30$	$3.1 = y_3$	$\Delta y_3 = 1.5$			
$x_4 = 40$	$4.6 = y_4$				

$$\Delta^2 f(20) = \Delta^2 f(x_2) = \Delta^2 y_2 = 0.6$$

$$\Delta^3 f(10) = \Delta^3 f(x_1) = \Delta^3 y_1 = 0.4$$

$$\Delta^4 f(0) = \Delta^4 f(x_0) = \Delta^4 y_0 = 0.4$$

Newtons forward interpolation formula.

The value of $y=f(x)$ at $x = x_0 + rh$ that is $y_r = f(x_0 + rh)$ is approximately given by

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \dots$$

$$+ \frac{r(r-1)(r-2)\dots(r-(n-1))}{n!}\Delta^n y_0 .$$

where r is any real number.

Newtons backward interpolation formula

The value of $y=f(x)$ at $x = x_n + rh$ that $y_r = f(x_n + rh)$ is approximately given by.

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \dots$$

$$+ \frac{r(r+1)(r+2)\dots(r+n-1)}{n!}\nabla^n y_n$$

where r is any real number.

Problems:

1) Find $y(1.4)$ given the data

x	1	2	3	4	5
y	10	26	58	112	194.

Sol: Here we have to find y at $x = 1.4$

Since the value of $x = 1.4$ is in the first half table near $x = 1$

Newtons forward interpolation formula is appropriat

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 1$	$y_0 = 10$				
		16			
2	26		16		
		32		6	
3	58		22		0
		54		6	
4	112		28		
		82			
x_n 5	y_n 194				

We have Newtons forward interpolation formula

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \dots$$

where $r = \dfrac{x - x_0}{h}$

x = point at which is required $= 1.4$.

x_0 = initial point (first value of x) $= 1$

h = common interval length $= 1 \Rightarrow x_2 - x_1 = 2 - 1 = 1$

$$r = \frac{1.4 - 1}{1} = \frac{0.4}{1} = 0.4$$

$\Delta y_0 = 16$, $\Delta^2 y_0 = 16$ $\Delta^3 y_0 = 6$ $\Delta^4 y_0 = 0$

$$y(1.4) = 10 + (0.4)16 + \frac{(0.4)(0.4-1)(16)}{2} + \frac{(0.4)(0.4-1)(0.4-2)(6)}{6}$$

$y(1.4) = \underline{14.864}$

② Find $U_{0.5}$ from the data $u_0 = 225$, $U_1 = 238$
$U_2 = 320$, $U_3 = 340$

Sol The value of $x = 0.5$ is near to $x = 0$ and hence newtons forward interpolation formula

x	$u_x = y$	Δy	$\Delta^2 y$	$\Delta^3 y$
0	225			
		13		
			69	
1	238			-131
		82		
2	320		-62	
		20		
3	340			

we have newtons forward interpolation formula.

$$y_x = y_0 + x\,\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0 .$$

$$x = \frac{x - x_0}{h} = \frac{0.5 - 0}{1} = 0.5$$

$$\Delta y_0 = 13 \quad , \Delta^2 y_0 = 69 \quad , \Delta^3 y_0 = -131$$

$$y(0.5) = 225 + 0.5(13) + \frac{0.5(0.5-1)}{2!}(69) + \frac{(0.5)(0.5-1)(0.5-2)(-13}{6}$$

$$y(0.5) = \underline{\underline{214.6875}}$$

$$U_{0.5} = \underline{\underline{214.6875}}$$

3) The area of a circle (A) corresponding to diameter ~~105 using an ap~~ (D).

D	80	85	90	95	100
A	5026	5674	6362	7088	785

Find the area corresponding to diameter 105 using an appropriate interpolation form

Here we have to find A when D = 105

As this value 105 is near to end value 100

Newtons backward interpolation formula is

appropricab.

$x = D$	$y = A$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40		
		688		-2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
$x_n = 100$	y_n 7854				

Newtons backward interpolation formula.

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \dots$$

$$r = \frac{x - x_n}{h} = \frac{105 - 100}{5} = 1$$

$\nabla y_n = 766 \quad \nabla^2 y_n = 40 \quad \nabla^3 y_n = 2, \quad \nabla^4 y_n = 4$

$$y_r = 7854 + 1(766) + \frac{1(1+1)}{2}40 + \frac{1(1+1)(1+2)}{6}2 + \frac{(1)(2)(3)(4)}{24}(4)$$

$y_r = \underline{8666}$

Thus the area (A) corresponding to Diameter (D) = 105 is 8666

4) The following table give the values of $\tan x$ for $0.10 \le x \le 0.30$ Find $\tan(0.26)$

x	0.10	0.15	0.20	0.25	0.30
$\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Sol : Here the value $x = 0.26$ is near the end value 0.30.

Hence newtons backward interpolation formula is appropriate

The backward difference table is

x	$f(x) = y$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.10	0.1003				
		0.508			
			0.0008		
0.15	0.15111			0.0002	
		0.0516			0.0002
			0.0010		
0.20	0.2027			0.0004	
		0.0526			
			0.0014		
0.25	0.2553				
		0.0540			
0.30	0.3093				

Newton backward interpolation formula.

$$y_n = y_n + n\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n + \frac{n(n+1)(n+2)(n+3)}{4!}\nabla^4 y_n$$

$$n = \frac{x - x_n}{h} = \frac{0.26 - 0.30}{0.05} = \underline{-0.8}$$

(a)

$h = 0.15$

$\nabla y_n = 0.0540$, $\nabla^2 y_n = 0.0014$, $\nabla^3 y_n = 0.0004$

$\nabla^4 y_n = 0.0002$

$$y_x = y_n + x\nabla y_n + \frac{x(x+1)}{2!}\nabla^2 y_n + \frac{x(x+1)(x+2)}{3!}\nabla^3 y_n +$$

$$\frac{x(x+1)(x+2)(x+3)}{4!}\nabla^4 y_n + \dots$$

$$\cancel{x = \frac{x - x_n}{h} = 0.26 - 0.30}$$

$$f(0.26) = 0.3093 + (-0.8)(0.054) + \frac{(-0.8)(-0.8+1)}{2}(0.0014)$$

$$+ \frac{(-0.8)(-0.8+1)(-0.8+2)}{6}(0.0004) + \frac{(-0.8)(-0.8+1)(-0.8+2)(-0.8+3)}{24}(0.0002)$$

$f(0.26) = 0.26602$

$\tan(0.26) = \underline{\underline{0.2660}}$

5) Extrapolate for 25.4 given the data

x	19	20	21	22	23
y	91	100.25	110	120.25	131

$x = 25.4$ is near to the end value.

Hence Newton's backward interpolation formula is appropriate. The backward difference table is as follows.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
$x_0 = 19$	$y_0 = 91$			
		9.25		
20	100.25		0.5	
		9.75		0
21	110		0.5	
		10.25		0
22	120.25		0.5	
		10.75		
$x_n = 23$	$y_n = 131$			

$\nabla y_n = 10.75$

$\nabla^2 y_n = 0.5$

$\nabla^3 y_n = 0$

$$r = \frac{x - x_n}{h} = \frac{25.4 - 23}{20 - 19} = 2.4$$

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n$$

$$y_r = 131 + 2.4(10.75) + \frac{2.4(2.4+1)(0.5)}{2}$$

$y_r = 158.84$

$f(25.4) = \underline{158.84}$

6) $\sin 45° = 0.7071$, $\sin 50° = 0.7660$, $\sin 55° = 0.8192$

$\sin 60° = 0.8660$, find $\sin 57°$ using an interpolation formula.

Sol we have to find the value of $f(x) = \sin x$ at $x = 57°$ which is near the end value 60. Newtons Backward interpolation formula is applied

x	$f(x) = y$	∇y	$\nabla^2 y$	$\nabla^3 y$
$x_0 = 45$	$y_0 = 0.7071$			
		0.0589		
			-0.0057	
50	0.7660			-0.0007
		0.0532		
			-0.0064	
55	0.8192			
		0.0468		
$x_n = 60$	$y_n = 0.8660$			

$\nabla y_n = 0.0486$ $\quad \nabla^2 y_n = -0.0064$ $\quad \nabla^3 y_n = -0.0007$

$$r = \frac{x - x_n}{h} = \frac{57 - 60}{50 - 45} = -0.6$$

$$y_x = y_n + x\nabla y_n + \frac{x(x+1)}{2!}\nabla^2 y_n + x\frac{(x+1)(x+2)}{3!}\nabla^3 y_n + \dots$$

$$y_x = 0.8660 + (-0.6)(0.0468) + \frac{(-0.6)(-0.6+1)}{2}(-0.0064) + (-0.6)\frac{(-0.6+1)(-0.6+2)}{6}(-0.0007)$$

$y_x = 0.8387$

$f(57) = 0.8387$

Thus $\sin 57^\circ = 0.8387$ Imp

7) From the following table find the number of student who have obtained (a) less than 45 (b) between 40 and 45

Marks	30-40	40-50	50-60	60-70	70-80
No of Students	31	42	51	35	31

Sol: We shall reconstitute the given table with $f(x)$ representing the number of student less than x marks

less than 40 marks 31 students

less than 50 marks 31 + 42 = 73 students

less than 60 marks 73 + 51 = 124 students

less than 70 marks 124 + 35 = 159 students

less than 80 marks 159 + 31 = 190 students

we have the new table along with forward difference

a) we need to find $f(45)$ being the no of students Scoring less than 45

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31	42	9	-25	37
50	73	51	-16	12	
60	124	35	-4		
70	159	31			
80	190				

a) we shall find $f(45)$ by applying newtons forward interpolation formula

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}$$

$$r = \frac{x - x_0}{h} = \frac{45-40}{50-40} = 0.5$$

$$y_r = 31 + (0.5)(42) + \frac{0.5(0.5-1)}{2}9 + \frac{0.5(0.5-1)(0.5-2)}{6}(-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24}(37)$$

$y_r = 47.86 \simeq 48$

Thus the number of students obtaining less than 45 marks ~~48~~ is 48.

$\underline{\underline{f(45) = 48}}$

b) we need to find $f(45) - f(40)$.

But $f(40) = 31$ by data

$f(45) - f(40) = 48 - 31 = \underline{\underline{17}}$

Thus the number of students scoring marks between 40 and 45 is $\underline{\underline{17}}$

8) Estimate the probable number of persons in the age group 20 to 25 years from the following data

Income per day (Rs)	Under 10	10-20	20-30	30-40	40-50
Number of persons	20	45	115	210	115

Sol: The given data is reconstituted with $f(x)$ representing the number of persons less than age of Rs. x years. That is, less than 10 years = 20

less than 20 year = 20 + 45 = 65

less than 30 years = 65 + 115 = 180

less than 40 years = 180 + 210 = 390

less than 50 years = 390 + 115 = 505

we have to find f(25) and f(20) which respectively estimates the number of persons having age less than 25 years and less than 20 year.

The difference is $f(25) - f(20)$

The new table along with the forward differences is as follows

x	$f(x) = y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	20	45	70	25	-215
20	65	115	95	-190	
30	180	210	-95		
40	390	115			
50	505				

we have newtons forward interpolation formula

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0$$

$$r = \frac{x - x_0}{h} = \frac{25 - 10}{20 - 10} = 1.5$$

$$y_r = 20 + 1.5(45) + \frac{1.5(1.5-1)}{2!}70 + \frac{1.5(1.5-1)(1.5-2)}{6}25 + \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{24}(-215)$$

$y_r = 107.15 \simeq 107$

$f(25) = 107$

$f(20) = 65$

Hence $f(25) - f(20) = 107 - 65 = 42$

Thus the number of person in the age group of 20 to 25 years is 42.

9) Compute $U_{14.2}$ from the following table to applying Newtons backward interpolation formula

x	10	12	14	16	18
U_x	0.240	0.281	0.318	0.352	0.384

we shall apply Newtons backward interpolation formula

x	$U_x = y$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	0.240				
		0.041			
12	0.281		-0.004		
		0.037		0.001	
14	0.318		-0.003		0
		0.034		0.001	
16	0.352		-0.002		
		0.032			
18	0.384				

we have newtons backward interpolation formula

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n$$

$$r = \frac{x - x_n}{h} = \frac{14.2 - 18}{12 - 10} = -1.9$$

$$y_r = 0.384 + (-1.9)(0.032) + \frac{(-1.9)(-1.9+1)}{2}(-0.002) + \frac{(-1.9)(-1.9+1)(-1.9+2)}{6}(0.001)$$

$y_r = 0.3215$

$U_x = 0.3215 \Rightarrow U_{14.2} = 0.3215$

10) The population of a town is given by table

Year	1951	1961	1971	1981	1991
Population in thousands	19.96	39.65	58.81	77.21	94.61

Using newtons forward and Backward interpolation formula, calculate the increase in the population from the year 1955 to 1985 ? **Imp**

Sol : we Shall first form the finite difference

x	y	I difference	II diff	III diff	IV diff
1951	19.96				
		19.69			
1961	39.65		−0.53		
		19.16		−0.23	
1971	58.81		−0.76		−0.01
		18.4		−0.24	
1981	77.21		−1		
		17.4			
1991	94.61				

Case 1 : To find $y(1955)$ $\quad r = \frac{x - x_0}{h} = \frac{1955-1951}{10} = 0.4$

we have from the table

$x_0 = 1951$, $y_0 = 19.96$, $\Delta y_0 = 19.69$ $\quad \Delta^2 y_0 = -0.53$

$\Delta^3 y_0 = -0.23 \quad \Delta^4 y_0 = -0.01$

we have Newtons forward interpolation formula

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0$$

$$y_r = 19.96 + 0.4(19.69) + \frac{0.4(0.4-1)}{2}(-0.53) + \frac{0.4(0.4-1)(0.4-2)}{6}(-0.23)$$

$$+ \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}(-0.01)$$

$y(1955) = 27.89$

Case 2: To find $y(1985)$

we have from the table $\nabla^4 y_n = -0.01$

$x_n = 1991$, $y_n = 94.61$ $\nabla y_n = 17.4$, $\nabla^2 y_n = -1$, $\nabla^3 y_n = -0.24$

we have Newtons Backward interpolation formula

$$y_x = y_n + x\nabla y_n + \frac{x(x+1)}{2!}\nabla^2 y_n + \frac{x(x+1)(x+2)}{3!}\nabla^3 y_n + \frac{x(x+1)(x+2)(x+3)}{4!}$$

$$x = \frac{x - x_n}{h} = \frac{1985 - 1991}{10} = -0.6$$

$$y_x = 94.61 + (-0.6)(17.4) + \frac{(-0.6)(-0.6+1)}{2!}(-1) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(-0.24)$$

$$+ \frac{(-0.6)(0.4)(1.4)(2.4)}{24}(-0.01)$$

$y(1985) = 84.3$

Thus the increase in population from the year 1955 to 1985 is $84.3 - 27.89 = 56.41$ thousands.

Newtons divided difference formula or Newtons general interpolation formula

Statement: If $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ be a set of values of an unknown funtion $f(x)$ corresponding to the values of x; $x_0, x_1, x_2, \ldots, x_n$ at unequal intervals.

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \cdots$$

11) In a table given below, the values of y are consecutive terms of a series of which 23.6 is the 6^{th} term. Find the first and tenth term of the series.

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

<u>Sol</u>: we need to compute $y(1)$ and $y(10)$. Newtons forward and backward interpolation formula

x	y	I difference	II difference	III difference	IV diff
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

case ① : To find $y(1)$

$$q = \frac{x - x_0}{h}$$

we have from the table

$$q = \frac{1-3}{1} = -2$$

$x_0 = 3$, $y_0 = 4.8$, $\Delta y_0 = 3.6$, $\Delta^2 y_0 = 2.5$, $\Delta^3 y_0 = 0.5$

we have newtons forward interpolation formula

$$y_x = y_0 + x \Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 y_0$$

$$y_x = 73.9 + (-2)\, 3.6 + \frac{(-2)(-2-1)}{2!}\, 2.5 + \frac{(-2)(-2-1)(-2-2)}{3!}(0.5)$$

$$y(1) = 3.1$$

Case (ii) To find $y(10)$

we have from the table

$x_n = 9$, $y_n = 73.9$ $\nabla y_n = 21.1$ $\nabla^2 y_n = 4.5$ $\nabla^3 y_n = 0.5$

we have Newtons Backward interpolation

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \ldots$$

$$r = \frac{x - x_n}{h} = r = \frac{10-9}{1} = 1$$

$$y(10) = 73.9 + 1(21.1) + \frac{(1)(2)}{2!}(4.5) + \frac{(1)(2)(3)}{3!}(0.5)$$

$$y(10) = 100$$

Problems

1) Use Newtons divided difference formula to $f(4)$ given the data.

x	0	2	3	6
$f(x)$	-4	2	14	158

we shall first form the table of divided difference.

x	$f(x)$	1 DD	2^{nd} DD	3^{rd} DD
$x_0 = 0$	$f(x_0)$ -4	$f(x_0, x_1)$ $\frac{2-(-4)}{2-0} = 3$	$f(x_0, x_1, 2)$ $\frac{12-3}{3-0} = 3$	$f(x_0, x_1, x_2, x_3)$ $\frac{9-3}{6-0} = 1$
x_1 2	$f(x_1)$ 2	$f(x_1, x_2)$ $\frac{14-2}{3-2} = 12$	$f(x_1, x_2, x_3)$ $\frac{48-12}{6-2} = 9$	
x_2 3	$f(x_2)$ 14	$f(x_2, x_3)$ $\frac{158-14}{6-3} = 48$		
x_3 6	$f(x_3)$ 158			

we have newtons divided difference formula

We have newtons divided Difference formula

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3)$$

$$f(4) = 40$$

2) Fit an interpolating polynomial for the data $u_{10} = 355$, $u_0 = -5$, $u_8 = -21$, $u_1 = -14$, $u_4 = -125$ by using newtons general interpolation formula and hence find u_2

x	$u_x = f(x)$	1DD	2nd DD	3rd DD
$x_0 = 0$	$f(x_0) = -5$			
		$f(x_0, x_1)$ $\frac{-14-(-5)}{1-0} = -9$		
$x_1 = 1$	$f(x_1) = -14$		$f(x_0, x_1, x_2)$ $\frac{-37-9}{4-0} = -7$	
		$f(x_1, x_2)$ $\frac{-125-(-14)}{4-1} = -37$		$f(x_0, x_1, x_2, x_3)$ $\frac{9-(-7)}{8-0} = 2$
$x_2 = 4$	$f(x_2) = -125$		$f(x_1, x_2, x_3)$ $\frac{26-(-37)}{8-1} = 9$	
		$f(x_2, x_3)$ $\frac{-21+125}{8-4} = 26$		$f(x_1 x_2 x_3 x_4)$ $\frac{27-9}{10-1} = 2$
$x_3 = 8$	$f(x_3) = -21$		$f(x_2, x_3 x_4)$ $\frac{188-26}{10-4} = 27$	
		$f(x_3 x_4)$ $\frac{355-(-21)}{10-8} = 188$		
$x_4 = 10$	$f(4)\ 355$			

We have newtons general interpolation formula

$$f(x) = u_x = 2x^3 - 17x^2 + 6x - 5$$
$$f(2) = u_2 = 2(2)^3 - 17(2)^2 + 6(2) - 5$$
$$f(2) = -45$$

3) Find a cubic polynomial which passes through the points $(2,4)$ $(4,56)$ $(9,711)$ $(10,980)$ and hence estimate the dependent variable corresponding to the value of independent variable 3, 5, 7, 11 also express the interploating polynomial in powers of $(x-1)$ and hence estimate the value of $f(x)$ at $x=1.1$ and 1.5

Sol: We have to find $f(x)$ where we have by data $f(2)=4$, $f(4)=56$, $f(9)=711$ $f(10)=980$.

The divided difference table is as follows

x	$f(x)$	1DD	2DD	3DD
$x_0=2$	$f(x_0)$ 4	$f(x_0,x_1)$ $\frac{56-4}{4-2}=26$	$f(x_0,x_1,x_2)$ $\frac{131-26}{9-2}=15$	$f(x_0,x_1,x_2,x_3)$ $\frac{23-15}{10-2}=1$
$x_1=4$	$f(x_1)$ 56	$f(x_1,x_2)$ $\frac{711-56}{9-4}=131$	$f(x_1,x_2,x_3)$ $\frac{269-131}{10-4}=23$	
$x_2=9$	$f(x_2)$ 711	$f(x_2,x_3)$ $\frac{980-711}{10-9}=269$		
$x_3=10$	$f(x_3)$ 980			

We have newtons divided difference formula

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+\ldots$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+\ldots$$

$$f(x)=4+x-2(26)+(x-2)(x-4)(15)+(x-2)(x-4)(x-9)(1)$$
$$=4+(x-2)[26+(x-4)(15)+(x-4)(x-9)]$$
$$=4+(x-2)[26+15x-60+x^2-13x+36]$$
$$f(x)=4+(x-2][x^2+2x+2]$$

$f(x) = x^3 - 2x$ is the required polynomial.

$f(3) = 3^3 - 2(3) = 21$

$f(5) = 5^3 - 2(5) = 115$

$f(7) = 7^3 - 2(7) = 329$

$f(11) = 11^3 - 2(11) = 130$.

we shall now express the polymomil $x^3 - 2x$ in powers of $(x-1)$

$f(x) = x^3 - 2x$

$= \{(x-1)^3 + 3x^2 - 3x + 1\} - 2x$

$= (x-1)^3 + 3x^2 - 5x + 1$

$= (x-1)^3 + 3\{(x-1)^2 + 2x - 1\} - 5x + 1$

$= (x-1)^3 + 3(x-1)^2 + x - 2$

$= (x-1)^3 + 3(x-1)^2 + \{(x-1) + 1\} - 2$

Thus $f(x) = (x-1)^3 + 3(x-1)^2 + (x-1) - 1$

Putting $x = 1.1$

$f(1.1) = (1.1-1)^3 + 3(1.1-1)^2 + (1.1-1) - 1 = -0.869$

Putting $x = 1.5$

$f(1.5) = (1.5-1)^3 + 3(1.5-1)^2 + (1.5-1) - 1 = 0.375$

4) Using Newtons divided difference formula find $f(8)$, $f(15)$ from the following data

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

we have [newtons divided difference formula

The divided difference table is as shown

x	$f(x_i)$	1 DD	2^{nd} DD
x_0: 4	$f(x_0)$ 48	$f(x_0, x_1)$ $\frac{100-48}{5-4} = 52$	$f(x_0, x_1, x_2)$ $\frac{97-52}{7-4} = 15$
x_1 5	$f(x_1)$ 100	$f(x_1, x_2)$ $\frac{294-100}{7-5} = 97$	$f(x_1, x_2, x_3)$ $\frac{202-97}{10-5} = 21$
x_2 7	$f(x_2)$ 294	$f(x_2, x_3)$ $\frac{900-294}{10-7} = 202$	$f(x_2, x_3, x_4)$ $\frac{310-202}{11-7} = 27$
x_3 10	$f(x_3)$ 900	$f(x_3, x_4)$ $\frac{1210-900}{11-10} = 310$	$f(x_3, x_4, x_5)$ $\frac{409-310}{13-10} = 33$
x_4 11	$f(x_4)$ 1210	$f(x_4, x_5)$ $\frac{2028-1210}{13-11} = 409$	
x_5 13	$f(x_5)$ 2028		

3^{rd} DD	4^{th} DD
$f(x_0, x_1, x_2, x_3)$ $\frac{21-15}{10-4} = 1$	0
$f(x_1, x_2, x_3, x_4)$ $\frac{27-21}{11-5} = 1$	0
$f(x_2, x_3, x_4, x_5)$ $\frac{33-27}{13-7} = 1$	

$f(x) = x^3 - x^2$

$f(8) = 448$

$f(15) = 3150$

$f(x) = 48 + (x-4)(52) + (x-4)(x-5)15 + (x-4)(x-5)(x-7)(1)$

$f(x) = (x-4)[48 + 52 + (x-5)(15) + (x-5)(x-7)]$

$f(x) = (x-4)[48 + 52 + 15x - 75 + (x(x-7) - 5(x-7))]$

$f(x) = (x-4)[48 + 52 + 15x - 75 + (x^2 - 7x - 5x + 35]$

$f(x) = (x-4)[48 + 52 + 15x - 75 + (x^2 - 7x$

$f(x) = (x-4)[x^2 + 3x + 60]$

5) Determine $f(x)$ as a polynomial in x for the following data using Newtons divided difference formula

x	-4	-1	0	2	5
y	1245	33	5	9	1335

The divided difference table is

x	$f(x)$	1st DD	2nd DD	3rd DD	4th DD
$x_0=-4$	$f(x_0)\ 1245$	$f(x_0,x_1)$ $\frac{33-1245}{-1+4}=404$	$f(x_0,x_1,x_2)$ $\frac{-28+404}{0+4}=94$	$f(x_0,x_1,x_2,x_3)$ $\frac{10-94}{2+4}=-14$	
$x_1=-1$	$f(x_1)\ 33$	$f(x_1,x_2)$ $\frac{5-33}{0+1}=-28$	$f(x_1,x_2,x_3)$ $\frac{2-28}{2+1}=10$	$f(x_1,x_2,x_3,x_4)$ $\frac{88-10}{5+1}=13$	
$x_2=0$	$f(x_2)\ 5$	$f(x_2,x_3)$ $\frac{9-5}{2-0}=2$	$f(x_2,x_3,x_4)$ $\frac{442-2}{5-0}=88$		
$x_3=2$	$f(x_3)\ 9$	$f(x_3,x_4)$ $\frac{1335-9}{5-2}=442$			
$x_4=5$	$f(x_4)\ 1335$				

4th DD

$f(x_0,x_1,x_2,x_3,x_4)$

$\frac{13+14}{5+4}=3$

On substituting Newton divided difference formula

$$y=f(x)=1245+(x-4)(-404)+(x+4)(x+1)\,94+(x+4)(x+1)(x)(-14)+(x+4)(x+1)(x)(x-2)\,3.$$

$$f(x)=1245+(x-4)\left[(x+1)94+(x+1)x(-14)+(x+1)(x-2)(x)3\right]$$

$$f(x)=1245+(x-4)\left[-404+94x+94-14x^2-14x+3x^3-3x^2-6x\right]$$

$$=1245+(x-4)\left[3x^3-17x^2+74x-310\right].$$

5) Determine $f(x)$ as a polynomial in x for the following data using Newtons divided difference formula

x	-4	-1	0	2	5
y	1245	33	5	9	1335

The divided difference table is

x	$f(x)$	1st DD	2nd DD	3rd DD	4th DD
$x_0=-4$	$f(x_0)\,1245$	$f(x_0,x_1)$ $\frac{33-1245}{-1+4}=404$	$f(x_0,x_1,x_2)$ $\frac{-28+404}{0+4}=94$	$f(x_0,x_1,x_2,x_3)$ $\frac{10-94}{2+4}=-14$	
$x_1=-1$	$f(x_1)\,33$	$f(x_1,x_2)$ $\frac{5-33}{0+1}=-28$	$f(x_1,x_2,x_3)$ $\frac{2-28}{2+1}=10$	$f(x_1,x_2,x_3,x_4)$ $\frac{88-10}{5+1}=13$	
$x_2=0$	$f(x_2)\,5$	$f(x_2\,x_3)$ $\frac{9-5}{2-0}=2$	$f(x_2,x_3,x_4)$ $\frac{442-2}{5-0}=88$		
$x_3=2$	$f(x_3)\,9$	$f(x_3,x_4)$ $\frac{1335-9}{5-2}=442$			
$x_4=5$	$f(x_4)\,1335$				

4th DD

$f(x_0,x_1,x_2,x_3,x_4)$

$\frac{13+14}{5+4}=3$

On substituting Newton divided difference formula

$$y=f(x)=1245+(x-4)(-404)+(x+4)(x+1)\,94+(x+4)(x+1)(x)(-14)+(x+4)(x+1)(x)(x-2)\,3.$$

$$f(x)=1245+(x-4)\left[(x+1)\,94+(x+1)\,x(-14)+(x+1)(x-2)(x)\,3\right]$$

$$f(x)=1245+(x-4)\left[-404+94x+94-14x^2-14x+3x^3-3x^2-6x\right]$$

$$=1245+(x-4)\left[3x^3-17x^2+74x-310\right].$$

$f(x) = 1245 + 3x^4 - 17x^3 + 74x^2 - 310x + 12x^2 - 68x^2 + 296x - 1240$

$f(x) = 3x^4 - 5x^3 + 6x^2 - \underline{\underline{14x + 5}}$.

Lagranges formula for interpolation and inverse interpolation.

Statement : If $y_0 = f(x_0)$ $y_1 = f(x_1)$, $y_2 = f(x_2) \ldots, y_n = f(x_n)$ be a set of values of an unknown function $y = f(x)$ corresponding to the values of x; $x_0, x_1, x_2, \ldots x_n$ not ne

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)y_0}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} + \frac{(x-x_0)(x-x_2)(x-3)\cdots(x-x_n)y_1}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} + \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)y_2}{(x_2-x_0)(x_2-x_1)(x_3-x_3)\cdots(x_2-x_n)} + \cdots +$$

$$\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})y_n}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

Problem :

1) The following the table gives the normal weights of babies during first eight months of life.

Age (in months)	0	2	5	8
Weights (in pounds)	6	10	12	16

Estimate the weight of the baby at the age of seven month using Lagranges interpolation formula

Sol : Let $x_0 = 0$ $x_1 = 2$ $x_2 = 5$ $x_3 = 8$ $x = 7$

$y_0 = 6$ $y_1 = 10$ $y_2 = 12$ $y_3 = 16$ $y = ?$

we have Lagranges interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$f(7) = \frac{(7-2)(7-5)(7-8)}{(0-2)(-5)(-8)} 6 + \frac{(7)(7-5)(7-8)}{(2)(2-5)(2-8)} 10 + \frac{(7)(5)(-1)(12)}{5(3)(-3)}$$

$$+ \frac{7(5)(2)(16)}{8(6)(3)}$$

$$f(7) = \frac{(5)(2)(-1)(6)}{(-2)(-5)(-8)} + \frac{(7)(2)(-1)(10)}{2(-3)(-6)} + \frac{7(5)(-1)(12)}{5(3)(-3)} + \frac{7(5)(2)(16)}{(8)(6)(3)}$$

$$f(7) = \frac{3}{4} - \frac{35}{9} + \frac{28}{3} + \frac{70}{9} = 13.97 \simeq 14$$

Thus the approximate weight at the age of 7 months is 14 pounds.

2) If $y(1) = 3$, $y(3) = 9$, $y(4) = 30$, $y(6) = 132$ find Lagranges interpolation polynomial that takes on these values.

Sol: $y(x_0) = y_0$

$x_0 = 1$ $x_1 = 3$ $x_2 = 4$ $x_3 = 6$ $y = f(x) = ?$

$y_0 = 3$ $y_1 = 9$ $y_2 = 30$ $y_3 = 132$

we have Lagranges interpolation formula.

$$y = f(x) = \frac{(x-3)(x-4)(x-6)(3)}{(-2)(-3)(-5)} + \frac{(x-1)(x-4)(x-6)\,9}{(2)(-1)(-3)}$$

$$+ \frac{(x-1)(x-3)(x-6)\,30}{(3)(1)(-2)} + \frac{(x-1)(x-3)(x-4)\,132}{(5)(3)(2)}$$

$$f(x) = \frac{-1}{10}(x-3)(x-4)(x-6) + \frac{3}{2}(x-1)(x-4)(x-6) -$$

$$5(x-1)(x-3)(x-6) + \frac{22}{5}(x-1)(x-3)(x-4)$$

$$f(x) = \frac{1}{10}\big[-(x-3)(x-4)(x-6) + 15(x-1)(x-4)(x-6)$$

$$-50(x-1)(x-3)(x-6) + 44(x-1)(x-3)(x-4)\big]$$

We can use the basic expansion formula

$$(x-a)(x-b)(x-c) = x^3 - x^2(a+b+c) + x(ab+bc+ca) - abc$$

$$f(x) = \frac{1}{10}\big[x^3 - x^2(3+4+6) + x(3\cdot4+4\cdot6+6\cdot3) - 3\cdot4\cdot6\big]$$

$$f(x) = \frac{1}{10}\big[x^3 - x^2(13) + x(54) - 72) + 15(x^3 - 11x^2 + 34x - 24)$$

$$-50(x^3 - 10x^2 + 27x - 18) + 44(x^3 - 8x^2 + 19x - 12)\big]$$

$$= \frac{1}{10}\big[8x^3 - 4x^2 - 58x + 84\big]$$

Thus the required polynomial is $f(x)$.

$$f(x) = \frac{1}{5}\big[4x^3 - 2x^2 - 29x + 42\big]$$

3) The observed values of a function are respectively 108, 120, 72 and 63 at four positions 3, 7, 9, 10 of the independent variable. What is the best estimate you can give for the value of the function at the position 6 of the independent variable?

Sol: For the function $y = f(x)$,

x is the independent variable and y is the dependent variable.

$x_0 = 3 \quad x_1 = 7 \quad x_2 = 9 \quad x_3 = 10 \quad x = 6$

$y_0 = 168 \quad y_1 = 120 \quad y_2 = 72 \quad y_3 = 63 \quad y = ?$

We have lagranges interpolation formula.

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3+x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(6-7)(6-9)(6-10)}{(3-7)(3-9)(3-10)} 168 + \frac{(6-3)(6-9)(6-10)}{(7-3)(7-9)(7-10)} 120 +$$

$$\frac{(6-3)(6-7)(6-10)}{(9-3)(9-7)(9-10)} 72 + \frac{(6-3)(6-7)(6-9)}{(10-3)(10-7)(10-9)} 63$$

$$= \frac{(-1)(-3)(-4)}{(-4)(-6)(-7)} 168 + \frac{(3)(-3)(-4)}{4(-2)(-3)} 120 + \frac{3(-1)(-4)}{66)(2)(-1)} 72 +$$

$$\frac{3(-1)(-3)}{7(3)(1)} 63$$

$$y = 12 + 180 - 72 + 27 = 147$$

Thus the estimate at position 6 of independent variable is $\underline{147}$

4) Using Lagranges interpolation formula find $f(9)$ from the following

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Sol. $x_0 = 5$, $x_1 =$ [illegible], $x_2 = 11$, $x_3 = 13$, $x_4 = 17$

$y_0 = 150$ $y_1 = 392$ $y_2 = 1452$ $y_3 = 2366$ $y_4 = 5202$.

$x = 9$, $y = ?$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} y_4$$

$$y = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} 2366 +$$

$$\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} 5202$$

$$y = f(x) = \frac{(2)(-2)(-4)(-8)}{(-2)(-6)(-8)(-12)} 150 + \frac{(4)(-2)(-4)(-8)}{(2)(-4)(-6)(-10)} 392 + \frac{(4)(2)(-4)(-8)}{(6)(4)(-2)(-6)} 1452$$

$$= \frac{(4)(2)(-2)(-8)}{(8)(6)(2)(-4)} 2366 + \frac{(4)(2)(-2)(-4)}{(12)(10)(6)(4)} 5202.$$

Thus $f(9) = \underline{\underline{810}}$

Numerical Integration :

Simpson's One third rule ;

$$I = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots+y_{n-2})\right]$$

Simpson's three eighth rule :

$$I = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots+y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})\right]$$

a) to apply simpsons 1/3 rd rule, n must be a multiple of 2.

b) to apply simpson's 3/8 th rule n must be a multiple of 3.

Problems:

1) Evaluate $\int_0^6 3x^2\,dx$ dividing the interval [0, 6] into six equal parts by applying

(a) Simpsons 1/3 rd rule

(b) Simpsons 3/8 th rule

1/3

Sol: Let us divide [0, 6] into 6 equal parts (n=6)

length of each strip $\frac{6-0}{6} = 1 = h$

The point of division are $x = 0, 1, 2, 3, 4, 5, 6$

$y = 3x^2$

The set of value of x and y are represented

x	0	1	2	3	4	5	6
$y = 3x^2$	0	3	12	27	48	75	108
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

a) Simpsons $1/3$ rd rule

$$\int_a^b y\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2})\right]$$

$n=6$, $h=1$, $a=0$, $b=6$, $y=3x^2$

$$\int_0^6 3x^2\,dx = \frac{1}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$\int_0^6 3x^2\,dx = \frac{1}{3}\left[(0+108) + 4(3+27+75) + 2(12+48)\right]$$

$$= \frac{1}{3}\left[108 + 420 + 120\right] = \frac{1}{3}(648)$$

$$\int_0^6 3x^2\,dx = 216$$

b) Simpsons $3/8$th rule.

$$\int_a^b y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots+y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})\right]$$

$n=6$, $h=1$

$$\int_0^6 3x^2\,dx = \frac{3}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2y_3\right]$$

$$= \frac{3}{8}\left[(0+108) + 3(3+12+48+75) + 2(27)\right]$$

$$= \frac{3}{8}\left[108 + 414 + 54\right] = \frac{3}{8}[576]$$

$$\int_0^6 3x^2\,dx = 216$$

2) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using simpsons $1/3$ rd rule taking four equal strips and hence deduce an approximate value of π

Sol Let us divide $[0,1]$ into 4 equal strips

$(n = 4)$

length of each strip $h = \frac{1-0}{4} = \frac{1}{4}$

The points of division are $x = 0, \frac{1}{4}, \frac{2}{4} = \frac{1}{2}, \frac{3}{4}, \frac{4}{4} = 1$

$x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

By data $y = \frac{1}{1+x^2}$

At $x = 0$ $\quad y = \frac{1}{1+0} = 1$

At $x = \frac{1}{4}$ $\quad y = \frac{1}{1+(1/4)^2} = \frac{16}{17}$

1/3

At $x = \frac{1}{2}$ $\quad y = \frac{1}{1+(\frac{1}{2})^2} = \frac{4}{5}$

At $x = \frac{3}{4}$ $\quad y = \frac{1}{1+(3/4)^2} = \frac{16}{25}$

At $x = 1$ $\quad y = \frac{1}{1+1^2} = \frac{1}{2}$

Now we have the following table

x	0	1/4	1/2	3/4	1
$y = \frac{1}{1+x^2}$	1	16/17	4/5	16/25	1/2
	y_0	y_1	y_2	y_3	y_4

Simpson's 1/3 rd for $n=4$ is given by

$[(y_0 + y_n) + 4(y_1 + y_3) + 2y_2]$

$$\int_a^b y\,dx = \frac{h}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1/4}{3}\left[\left(1 + \frac{1}{2}\right) + 4\left(\frac{16}{17} + \frac{16}{25}\right) + 2 \cdot \frac{4}{5}\right]$$

$= 0.7854$

Sol Let us divide $[0,1]$ into 4 equal strips $(n=4)$

length of each strip $h = \frac{1-0}{4} = \frac{1}{4}$

The points of division are $x = 0, \frac{1}{4}, \frac{2}{4} = \frac{1}{2}, \frac{3}{4}, \frac{4}{4} = 1$

$x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

By data $y = \frac{1}{1+x^2}$

At $x = 0$ $\quad y = \frac{1}{1+0} = 1$

At $x = \frac{1}{4}$ $\quad y = \frac{1}{1+(1/4)^2} = \frac{16}{17}$

1/3

At $x = \frac{1}{2}$ $\quad y = \frac{1}{1+(\frac{1}{2})^2} = \frac{4}{5}$

At $x = \frac{3}{4}$ $\quad y = \frac{1}{1+(3/4)^2} = \frac{16}{25}$

At $x = 1$ $\quad y = \frac{1}{1+1^2} = \frac{1}{2}$

Now we have the following table

x	0	1/4	1/2	3/4	1
$y = \frac{1}{1+x^2}$	1	16/17	4/5	16/25	1/2
	y_0	y_1	y_2	y_3	y_4

Simpsons 1/3 rd for $n=4$ is given by

$[(y_0+y_n) + 4(y_1+y_3) + 2y_2]$

$$\int_a^b y\,dx = \frac{h}{3}\left[(y_0+y_4) + 4(y_1+y_3) + 2y_2\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1/4}{3}\left[\left(1+\frac{1}{2}\right) + 4\left(\frac{16}{17}+\frac{16}{25}\right) + 2\cdot\frac{4}{5}\right]$$

$= 0.7854$

To deduce the value of π

$\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1}x]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

we must have $\frac{\pi}{4} = 0.7854$

$\pi = 4(0.7854) = 3.1476$

3) Evaluate $\int_0^1 \frac{dx}{1+x}$ taking seven ordinates by applying Simpsons 3/8th rule; Hence deduce the value of $\log_e 2$?

Sol 7 ordinates means that given interval [0, 1] must be divided into 6 equal parts

that is $n = 6$

The length of each part is $\frac{1-0}{6} = \frac{1}{6} = h$

By data $y = \frac{1}{1+x}$

At $x = 0$ $y = \frac{1}{1+0} = 1$ At $x = \frac{4}{6}$ $y = \frac{1}{1+\frac{4}{6}} = \frac{3}{5}$

At $x = \frac{1}{6}$ $y = \frac{1}{1+1/6} = \frac{6}{7}$ At $x = \frac{5}{6}$ $y = \frac{1}{1+\frac{5}{6}} = \frac{6}{11}$

At $x = \frac{2}{6}$ $y = \frac{1}{1+2/6} = \frac{3}{4}$ At $x = \frac{6}{6}$ $y = \frac{1}{1+\frac{6}{6}} = \frac{1}{2}$

At $x = \frac{3}{6}$ $y = \frac{1}{1+3/6} = \frac{2}{3}$

x	0	1/6	2/6	3/6	4/6	5/6	6/6
$y = 1/1+x$	1	6/7	3/4	2/3	3/5	6/11	1/2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson 3/8th rule for $n = 6$ is

$$\int_a^b y\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{1}{16}\left[\left(1+\frac{1}{2}\right)+3\left(\frac{6}{7}+\frac{3}{4}+\frac{3}{5}+\frac{6}{11}\right)+2\cdot\frac{2}{3}\right]$$

$$\int_0^1 \frac{1}{1+x}\,dx = 0.6932$$

To deduce the value of $\log_e 2$.

$$\int_0^1 \frac{1}{1+x}\,dx = \left[\log_e(1+x)\right]_0^1$$

$$= \log_e(1+1) - \log(1+0)$$

$$= \log_e 2 - \log_e 1$$

$$0.6932 = \log_e 2$$

$\log(4x+5)\Big]_0^1$

$\log(4+5) - \log(4(0)+5)$

$\log(9) - \log(4)$

The actual value of $\log_e 2 = \underline{0.693147}$

4) Find the approximate value of $\int_0^{\pi/2} \sqrt{\cos\theta}\,d\theta$ by Simpsons 1/3 rd rule by dividing $[0, \pi/2]$ into 6 equal parts?

Sol: Let us divide $[0, \pi/2]$ into 6 equal parts

$n = 6$

$$h = \frac{\pi/2 - 0}{6} = \frac{\pi}{2}\times\frac{1}{6} = \frac{\pi}{12} = \frac{180}{12} = 15^\circ$$

The value of θ and corresponding of $f(\theta) = \sqrt{\cos\theta}$ correct to four decimal places are

θ	0°	15°	30°	45°	60°	75°	90°
$\sqrt{\cos\theta}$	1	0.9828	0.9306	0.8409	0.7071	0.5087	0.0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpsons 1/3 rd rule for $n = 6$ is given by

$$\int_a^b y \, d\theta = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

where $h = \dfrac{\frac{\pi}{2}-0}{6} = \dfrac{\pi}{12}$

1/3

$$\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta = \frac{\pi/12}{3}\left[(1) + 4(0.9828 + 0.8409 + 0.5087) + 2(0.9306 + 0.7071)\right]$$

$$\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta = 1.1873$$

5) Evaluate $\int_0^3 \frac{dx}{(1+x)^2}$ by Simpson's 3/8 th rule

HW

Let us take $n = 3$

Length of Strips $(h) = \dfrac{3-0}{3} = 1$

x	0	1	2	3
$y = \frac{1}{(1+x)^2}$	1	$1/4$	$1/9$	$1/16$
	y_0	y_1	y_2	y_3

Simpsons 3/8 th rule for $n = 3$ is given by

$$\int_a^b y\, dx = \frac{3h}{8}\left[(y_0+y_3) + 3(y_1+y_2)\right]$$

$$\int_0^3 \frac{1}{(1+x)^2}\, dx = \frac{3}{8}\left[\left(1 + \frac{1}{16}\right) + 3\left(\frac{1}{4} + \frac{1}{9}\right)\right]$$

$$= 0.8047$$

6) Use Simpsons $\frac{1}{3}$rd rule with seven ordinates to evaluate $\int_2^8 \frac{dx}{\log_{10} x}$

1/3

Sol Seven ordinates means that given interval $[2,8]$ must be divided into 6 equal parts

$n = 6$

Length of each part $h = \frac{8-2}{6} = 1$

The value of $x = 2, 3, 4, 5, 6, 7, 8$

x =	2	3	4	5	6	7	8
$y = \frac{1}{\log_{10} x}$ =	3.3219	2.0959	1.6610	1.4307	1.2851	1.1833	1.1073
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpsons $\frac{1}{3}$rd rule for $n=6$ is given by

$$\int_a^b y\,dx = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$\int_2^8 \frac{dx}{\log_{10} x} = \frac{1}{3}\left[(3.3219 + 1.1073) + 4(2.0959 + 1.4307 + 1.1833) + 2(1.661 + 1.2851)\right]$$

$$\int_2^8 \frac{dx}{\log_{10} x} = 9.7203$$

7) Use Simpsons $3/8$th rule to obtain the approximate value of $\int_0^{0.3} (1-8x^3)^{1/2}\,dx$ by considering 3 equal intervals.

Sol : The length of each interval

$h = \frac{0.3-0}{3} = 0.1$

$n = 3$

The value of x = 0, 0.1, 0.2, 0.3

x	0	0.1	0.2	0.3
$y = (1-8x^3)^{1/2}$	1	0.996	0.9675	0.8854
	y_0	y_1	y_2	y_3

Simpson's 3/8th rule for $n = 3$ is given by

$$\int_a^b y\,dx = \frac{3h}{8}\left[(y_0 + y_3) + 3(y_1 + y_2)\right]$$

$$\int_0^{0.3} (1-8x^3)^{1/2}\,dx = \frac{3(0.1)}{8}\left[(1 + 0.8854) + 3(0.996 + 0.9675)\right]$$

$$\int_0^{0.3} (1-8x^3)^{1/2}\,dx = 0.2926$$

Trapezoidal rule:

$$\int f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

Problems:

1) Use the trapezoidal rule to estimate the integral $\int_0^2 e^{x^2}\,dx$ taking 10 interval.

Sol: $y = e^{x^2}$, $h = \frac{[2-0]}{10} = 0.2$, $n = 10$

x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y	~~1.0408~~ 1	~~1.10735~~ 1.0408	~~1.4333~~ 1.1735	~~1.8964~~ 1.4333	1.8964	2.7182	4.2206	7.0993	12.9352	25.5337	54.5981

By trapezoidal rule we have.

$$\int_0^1 e^{x^2}\,dx = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)\right]$$

$$= \frac{0.2}{2}\left[\right. \quad 17.0621$$

2) Divide the interval $[0, 6]$ into six parts of each of width $h = 1$; The value of $f(x) = \frac{1}{1+x^2}$ are given below?

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.05884	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Sol $\int_a^b \frac{1}{1+x^2} dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$

$= \underline{\underline{1.4108}}$