TEXTBOOK FOR SECOND PUC

BASIC MATHEMATICS

DEPARTMENT OF PRE-UNIVERSITY EDUCATION

Sampige Road, 18th cross, Malleshwaram, Bengaluru-12

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Director's Message

Dear Students,

We at the Department of Pre-university Education, Karnataka strive to empower each student to dream big and equip them with the tools that enable them to reach new heights and successfully deal with the challenges of life. As Swami Vivekananda said, "**Real education is that which enables one to stand on one's own legs".**

The course contents in this book are designed with the objective of equipping you well for the next level of study.

We wish you well on your journey and look forward to you becoming a responsible citizen of the nation and give back to the betterment of the society.

With best wishes,

Sd/-
C. Shikha, IAS
Director
Department of Pre University Education
Bengaluru

ACKNOWLEDGEMENTS

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PREFACE

This book of **BASIC MATHEMATICS** is written in accordance with the New Syllabus prescribed by the pre-university Board of Karnataka for the **Academic year 2014-15** according to the CBSE pattern with the guidance of NCERT for the second year pre-university students of Commerce and Arts. Good Effort has been put forward to prepare this book in a very lucid and unambiguous manner.

A large number of worked examples and exercise problems are included so that the students understand the basic concept of the subject easily. Our endeavour is to provide a good student-friendly textbook. We hope that this book would serve the purpose of making the study of mathematics interesting as well as stimulating.

This Book is designed as a self contained, comprehensive class room textbook for the first year Pre-university students. Great Care has been taken while framing the syllabus keeping in mind that the students should be able to develop good skills by problem solving technique in various branch of mathematics namely **Algebra, Commercial Arithmetic, Trigonometry, Analytical Geometry and Calculus**. We, the committee chairman, co-ordinator and the members are extremely grateful to the Department of Pre-University Education for having given an opportunity along with guidance, encouragement and support in this endeavour.

Our sincere thanks to the textbook review committee for their valuable guidance and suggestion in sculpting/framing this textbook. .

Our thanks are due to the publishers, M/s Sidvin Global Solutions for their DTP and page layout design in bringing out this book in an elegant manner.

Inspite of our best efforts if some misprint and mistake have escaped our notice we thankfully acknowledge them and sincerely incorporate them in the subsequent edition. Any suggessions for improvement of this book are most welcome.

From

The Syllabus Committee

Chairman, Co-ordinator and the

Members of II PUC Basic Mathematics

NEW SYLLABUS

SUBJECT : II P.U.C BASIC MATHEMATICS (CODE : 75)

(2014–2015 onwards)

UNIT I : ALGEBRA **42 hrs**

CHAPTER 1. MATRICES AND DETERMINANTS (13 hrs)

CHAPTER 2. PERMUTATIONS AND COMBINATIONS (08 hrs)

CHAPTER 3. PROBABILITY (05 hrs)

CHAPTER 4. BINOMIAL THEOREM (06 hrs)

CHAPTER 5. PARTIAL FRACTIONS (04 hrs)

CHAPTER 6. MATHEMATICAL LOGIC (06 hrs)

UNIT II : COMMERCIAL ARITHMETIC **34 hrs**

CHAPTER 7. RATIOS AND PROPORTIONS (10 hrs)

CHAPTER 8. BILL DISCOUNTING (06 hrs)

CHAPTER 9. STOCKS AND SHARES (04 hrs)

CHAPTER 10. LEARNING CURVE (04 hrs)

CHAPTER 11. LINEAR PROGRAMMING PROBLEMS (06 hrs)

CHAPTER 12. SALES TAX AND VALUE ADDED TAX (04 hrs)

UNIT III :TRIGONOMETRY **12 hrs**

CHAPTER 13. HEIGHTS AND DISTANCES (04 hrs)

CHAPTER 14. COMPOUND ANGLES, MULTIPLE ANGLES, SUB MULTIPLES ANGLES AND TRANSFORMATION FORMULAE (08 hrs)

UNIT IV : ANALYTICAL GEOMETRY **10 hrs**

CHAPTER 15. CIRCLES (06 hrs)

CHAPTER 16. PARABOLA (04 hrs)

UNIT V : CALCULUS **42 hrs**

CHAPTER 17. LIMITS AND CONTINUITY OF A FUNCTION (8 hrs)

CHAPTER 18. DIFFERENTIAL CALCULUS (10 hrs)

CHAPTER 19. APPLICATION OF DERIVATIVES (08 hrs)

CHAPTER 20. INDEFINITE INTEGRALS (08 hrs)

CHAPTER 21. DEFINITE INTEGRALS AND ITS APPLICATION TO AREAS (08 hrs)

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COMPREHENSIVE SYLLABUS

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1.2. Matrices
1.3. Types of Matrices
1.4. Algebra of Matrices
1.5. Determinants
1.6. Expansion or evaluation of a determinant
1.7. Properties of Determinants
1.8. Solution of Linear Equations – (Cramer's Rule / det method)
1.9. Minors and Cofactor and Adjoint of a square matrix
1.10 Singular and non-singular Matrices
1.11. Inverse of a Matrix
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17.4. Types of Functions

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UNIT I - ALGEBRA

Chapter	Title	No. of Teaching hrs.
1.	MATRICES AND DETERMINANTS	13 hrs
2.	PERMUTATION & COMBINATIONS	08 hrs
3.	PROBABILITY	05 hrs
4.	BINOMIAL THEOREM	06 hrs
5.	PARTIAL FRACTIONS	04 hrs
6.	MATHEMATICAL LOGIC	06 hrs
	TOTAL TEACHING HOURS	**42 hrs**

Chapter

1 MATRICES AND DETERMINANTS

1.1 Introduction

The famous English mathematician James Joseph Sylvester (1814 - 97) introduced the term matrix into mathematics. The term was applied to denote a rectangular array of numbers. British mathematician Arthur Coyley (1821 - 95) actually originated and developed matrices.

W.R. Hamilton, G. Frobenius and others are to be reckoned as pioneers in the development of matrices. Matrices enables us to perform calculations like solving a system of linear equations is a very compact form. Matrices and determinants are very elegant and powerful mathematical tools used for solving problems that arise not only in mathematics but also different fields of engineering, physics, economics and statistics etc.

1.2 Matrices

A matrix is an arrangements of numbers (real or complex) in **rows** (horizontal lines) and **columns** (vertical lines) and the numbers are enclosed in square brackets or ordinary brackets or pair of double bars and are denoted by capital letters A,B,C…..

Example: $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & 5 & 6 \end{bmatrix} \begin{matrix} \rightarrow \text{ 1st row} \\ \rightarrow \text{ 2nd row} \end{matrix}$

$\downarrow \quad \downarrow \quad \downarrow$

1st col, 2nd col, 3rd col

Order of a matrix:

If a matrix A has 'm rows' and 'n column's then A is called the matrix of order $(m \times n)$ (read as m cross n or m by n)

Examples: (a) $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 6 & 5 \end{bmatrix}$ is a matrix of order 2×3

(b) $B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ is a matrix of order 2×1

(c) $C = \begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots\ldots\ldots & a_{2n} \\ a_{m1} & a_{m2} & a_{m3} \ldots\ldots\ldots & a_{mn} \end{bmatrix}$ is a matrix of order m × n

1.3 Types of Matrices

1. **Row matrix :** A matrix having only one row is called a row matrix

   Example: $\begin{bmatrix} 4 & 7 & 1 \end{bmatrix}$ is a row matrix of order 1×3.

2. **Column matrix:** A matrix having only one column is called a column matrix

   Example: $\begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix}$ is a column matrix of order 3×1.

3. **Null matrix or zero matrix:** A matrix in which all the elements are zeroes is called a null matrix or zero matrix and denoted by 0.

   Example: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 2×3.

4. **Square matrix:** A matrix in which the number of rows is equal to number of columns is called a square matrix.

   Example: $\begin{bmatrix} 2 & 4 \\ 7 & 8 \end{bmatrix}$ is a square matrix of order 2×2 or of order 2.

   $\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & 6 \\ 5 & 7 & 8 \end{bmatrix}$ is a square matrix of order 3×3 or of order 3.

   In a square matrix the diagonal from left top corner to right bottom corner is called the **principal diagonal or leading diagonal.**

5. **Rectangular matrix:** A matrix in which the number of rows is not equal to the number of columns is called a rectangular matrix

   Example: $\begin{bmatrix} 2 & 1 & 5 \\ 6 & 4 & 0 \end{bmatrix}$ is a rectangular matrix of order 2×3

6. **Diagonal matrix:** A diagonal matrix is a square matrix in which all elements except the elements in the principal diagonal are zeroes.

   Example: $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ are diagonal matrices of order 2 and 3 respectively

7. **Scalar matrix:** A diagonal matrix in which all the diagonal elements are same is called a scalar matrix.

Example : $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

8. **Unit matrix or Identity matrix:** A diagonal matrix in which all the diagonal elements are equal to one is called unit matrix or identity matrix and denoted by I.

   Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Is an identity matrix of order 2.

   $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an identity matrix of order 3.

9. **Equal matrices:** Two matrices are said to be equal if and only if their orders are same and their corresponding elements are equal.

   Example: 1. $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$

A and B are equal matrices

2. $A = \begin{bmatrix} 1 & 3 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 6 \\ 3 & 7 \\ 5 & 8 \end{bmatrix}$

A and B are not equal as they are of different orders.

1.4 Algebra of matrices

1. **Scalar multiplication:** If A is any matrix and K is any scalar (constant) then KA is a matrix obtained by multiplying every element of A by K.

   If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $KA = \begin{bmatrix} Ka & Kb \\ Kc & Kd \end{bmatrix}$

2. **Addition of matrices:** Two matrices can be added or subtracted only if their orders are same. If A and B are matrices of order $m \times n$ then $A + B$ is of same order and it is defined as the matrix obtained by adding the corresponding elements of A and B.

   If $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 & 5 \\ 6 & 2 & 4 \end{bmatrix}$ then

   $A + B = \begin{bmatrix} 1-3 & 4+1 & 7+5 \\ 2+6 & 3+2 & 0+4 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 12 \\ 8 & 5 & 4 \end{bmatrix}$

   Subtraction of two matrices is similarly defined

$$A - B = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 1 & 5 \\ 6 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 \\ -4 & 1 & -4 \end{bmatrix}$$

The addition of matrices and scalar multiple of a matrix satisfy the following properties.

If A, B and C are matrices of same order then

(i) $A + B = B + A$ [Commutative law under matrix addition]

(ii) $A + (B + C) = (A + B) + C$ [Associative law under matrix addition]

(iii) $K.(A + B) = K.A + K.B$ [Scalar multiplication]

(iv) $A + 0 = 0 + A = A$ [0 is called the identity element under matrix addition]

WORKED EXAMPLES

Example 1 :

If $\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{-2} & \mathbf{0} \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{4} \\ \mathbf{1} & \mathbf{3} & \mathbf{2} \end{bmatrix}$

Find (i) $2A - 3B$ (ii) $4A + B$ (iii) $A + \frac{1}{3}B$

Solution :

(i) $$2A - 3B = 2\begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 0 \end{bmatrix} - 3\begin{bmatrix} 1 & -2 & 4 \\ 1 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 2 \\ 2 & -4 & 0 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 12 \\ 3 & 9 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 12 & -10 \\ -1 & -13 & -6 \end{bmatrix}$$

(ii) $$4A + B = 4\begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 4 \\ 1 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 12 & 4 \\ 4 & -8 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 4 \\ 1 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 10 & 8 \\ 5 & -5 & 2 \end{bmatrix}$$

(iii) $$A + \frac{1}{3}B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 & -2 & 4 \\ 1 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} & \frac{4}{3} \\ \frac{1}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{3} & \frac{7}{3} & \frac{7}{3} \\ \frac{4}{3} & -1 & \frac{2}{3} \end{bmatrix}$$

Example 2 :

If $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} & \mathbf{-2} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{4} \\ \mathbf{-2} & \mathbf{-1} & \mathbf{-3} \end{bmatrix}$ **and** $\mathbf{C} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{-4} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \end{bmatrix}$

Find $3A - 4B + 2C$

Solution :

$$3A - 4B + 2C = 3\begin{bmatrix} 1 & -3 & 1 \\ 3 & 2 & -2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 4 \\ -2 & -1 & -3 \end{bmatrix} + 2\begin{bmatrix} 1 & -2 & -4 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -9 & 3 \\ 9 & 6 & -6 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 16 \\ -8 & -4 & -12 \end{bmatrix} + \begin{bmatrix} 2 & -4 & -8 \\ 4 & 2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4+2 & -9-0-4 & 3-16-8 \\ 9+8+4 & 6+4+2 & -6+12+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -13 & -21 \\ 21 & 12 & 12 \end{bmatrix}$$

Example 3 :

If $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{-3} \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} \mathbf{-2} & \mathbf{4} \\ \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$**. Find** $\mathbf{2A + 3B}$

Solution :

$$2A + 3B = 2\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & -3 \end{bmatrix} + 3\begin{bmatrix} -2 & 4 \\ 3 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ 4 & 0 \\ 0 & -6 \end{bmatrix} + \begin{bmatrix} -6 & 12 \\ 9 & 6 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 10 \\ 13 & 6 \\ 3 & -6 \end{bmatrix}$$

Example 4 :

If $A=\begin{bmatrix}2 & 3 & 1\\ 1 & 0 & -2\end{bmatrix}$, $B=\begin{bmatrix}1 & -2 & -4\\ -1 & 1 & 3\end{bmatrix}$ **and** $C=\begin{bmatrix}5 & 2 & -1\\ 6 & 0 & 3\end{bmatrix}$

Find matrix X such that $3A-2B-X=C$

Solution :

$3A-2B-X=C$

$X=3A-2B-C$

$$X=3\begin{bmatrix}2 & 3 & 1\\ 1 & 0 & -2\end{bmatrix}-2\begin{bmatrix}1 & -2 & -4\\ -1 & 1 & 3\end{bmatrix}-\begin{bmatrix}5 & 2 & -1\\ 6 & 0 & 3\end{bmatrix}$$

$$X=\begin{bmatrix}6 & 9 & 3\\ 3 & 0 & -6\end{bmatrix}-\begin{bmatrix}2 & -4 & -8\\ -2 & 2 & 6\end{bmatrix}-\begin{bmatrix}5 & 2 & -1\\ 6 & 0 & 3\end{bmatrix}$$

$$X=\begin{bmatrix}6-2-5 & 9+4-2 & 3+8+1\\ 3+2-6 & 0-2-0 & -6-6-3\end{bmatrix}$$

$$X=\begin{bmatrix}-1 & 11 & 12\\ -1 & -2 & -15\end{bmatrix}$$

Example 5 :

Find x, y, z given that $4\begin{bmatrix}x & 2 & 3\\ 4 & 5 & -6\\ 7 & -8 & 9\end{bmatrix}-5\begin{bmatrix}3 & 2 & 1\\ 2 & y & -4\\ 9 & -8 & 7\end{bmatrix}=\begin{bmatrix}13 & -2 & 7\\ 6 & -5 & -4\\ -17 & 2z & 1\end{bmatrix}$

Solution :

$$\begin{bmatrix}4x & 8 & 12\\ 16 & 20 & -24\\ 28 & -32 & 36\end{bmatrix}-\begin{bmatrix}15 & 10 & 5\\ 10 & 5y & -20\\ 45 & -40 & 35\end{bmatrix}=\begin{bmatrix}13 & -2 & 7\\ 6 & -5 & -4\\ -17 & 2z & 1\end{bmatrix}$$

By equality of matrices

$4x-15=13$ $\quad$ $20-5y=-5$ $\quad$ $-32+40=2z$

$4x=13+15$ $\quad$ $20+5=5y$ $\quad$ $8=2z$

$4x=28$ $\quad$ $25=5y$ $\quad$ $\boxed{z=4}$

$\boxed{x=7}$ $\quad$ $\boxed{y=5}$

Example 6 :

Solve for x : $\begin{bmatrix}x^2 & 3\\ 1 & 2\end{bmatrix}+\begin{bmatrix}5x & 1\\ 4 & 3\end{bmatrix}=\begin{bmatrix}6 & 4\\ 5 & 5\end{bmatrix}$

Solution :

The given equation is $\begin{bmatrix} x^2+5x & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 5 & 5 \end{bmatrix}$

$x^2 + 5x = 6$

$x^2 + 5x - 6 = 0$

$x^2 + 6x - x - 6 = 0$

$x(x+6) - 1(x+6) = 0$

$(x+6) \quad (x-1) = 0$

$x + 6 = 0 \quad x - 1 = 0$

$\boxed{x = -6} \quad \boxed{x = 1}$

Example 7 :

If $\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & x-3 \\ y-4 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$**. Find** x **and** y**.**

Solution :

$\begin{bmatrix} 6 & x+2 \\ y-1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$

$x + 2 = 1, \quad y - 1 = 2$

$x = 1 - 2, \quad \boxed{y = 3}$

$\boxed{x = -1}$

Example 8 :

If $2A + B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$ **and** $3A + 2B = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix}$**. Find A and B.**

Solution :

$2A + B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$ ——— (1)

$3A + 2B = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix}$ ——— (2)

(1) $\times^{ly} 2 \rightarrow 4A + 2B = \begin{bmatrix} 4 & 6 & 2 \\ 2 & 8 & 0 \end{bmatrix}$

(2) $\rightarrow 3A + 2B = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

Subtracting, $A = \begin{bmatrix} 4 & 6 & 2 \\ 2 & 8 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & -5 \end{bmatrix}$

Substitute for A in equation (1)

$$B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & -5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -6 & 10 \end{bmatrix}$$

$\therefore A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & -5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -6 & 10 \end{bmatrix}$

Example 9 :

If $A + B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ and $2A - B = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}$. Find A and B.

Solution :

$$A + B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \quad \text{——— (1)}$$

$$2A - B = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} \quad \text{——— (2)}$$

Adding (1) and (2)

$$3A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3 & 0 \\ 6 & 3 \end{bmatrix} \div 3 \qquad \therefore \quad A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Substitute for A in equation (1)

$$A + B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - A$$

$$B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\therefore \quad B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

Example 10 :

If $B + A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ **and** $B - A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}$**. Find B.**

Solution :

$$B + A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 4 \end{bmatrix} \quad \text{——— (1)}$$

$$B - A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix} \quad \text{——— (2)}$$

Adding (1) and (2)

$$2B = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}$$

$$2B = \begin{bmatrix} 3 & 2 & 4 \\ 5 & 7 & 6 \end{bmatrix} \qquad \therefore \quad B = \begin{bmatrix} \frac{3}{2} & 1 & 2 \\ \frac{5}{2} & \frac{7}{2} & 3 \end{bmatrix}$$

3. **Transpose of a matrix:** If A is any matrix then the matrix obtained by interchanging the rows and columns of A is called the transpose of A and it is written as $\mathbf{A^1}$ **or** $\mathbf{A^T}$. In general, if A is a matrix of order $m \times n$, then its transpose A^1 is a matrix of order $n \times m$

4. **Symmetric and skew - symmetric matrices.**

A square matrix A is called a symmetric matrix if $\boxed{A^1 = A}$ and is called skew - symmetric matrix if $\boxed{A^1 = -A}$

Example (i) : $\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ are symmetric matrix

(ii) $\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ are skew symmetric matrices

Example 11 :

Write the transpose of each of the following matrices.

(i) $\mathbf{A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 4 & 5 \end{bmatrix}}$ **(ii)** $\mathbf{B = \begin{bmatrix} 1 & 6 & -2 \\ 4 & 2 & -3 \\ 3 & 5 & 7 \end{bmatrix}}$

(iii) $\mathbf{C} = \begin{bmatrix} \mathbf{5} & \mathbf{-1} \\ \mathbf{-3} & \mathbf{4} \end{bmatrix}$ **(iv)** $\mathbf{D} = \begin{bmatrix} \mathbf{7} & \mathbf{-5} & \mathbf{6} \end{bmatrix}$

Solution :

(i) $A^1 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 5 \end{bmatrix}$ (ii) $B^1 = \begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 5 \\ -2 & -3 & 7 \end{bmatrix}$

(iii) $C^1 = \begin{bmatrix} 5 & -3 \\ -1 & 4 \end{bmatrix}$ (iv) $D^1 = \begin{bmatrix} 7 \\ -5 \\ 6 \end{bmatrix}$

Example 12 :

If $\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{4} \\ \mathbf{3} & \mathbf{-1} \\ \mathbf{4} & \mathbf{0} \end{bmatrix}$ **show that** $\mathbf{(A^1)^1 = A}$

Solution :

$$A^1 = \begin{bmatrix} 2 & 3 & 4 \\ 4 & -1 & 0 \end{bmatrix}$$

$$(A^1)^1 = \begin{bmatrix} 2 & 4 \\ 3 & -1 \\ 4 & 0 \end{bmatrix}$$

$\therefore$ $\boxed{(A^1)^1 = A}$

Example 13 :

If $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{-2} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} & \mathbf{0} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{4} \\ \mathbf{6} & \mathbf{2} & \mathbf{5} \end{bmatrix}$

Verify that (i) $(A + B)^1 = A^1 + B^1$

(ii) $(A - B)^1 = A^1 - B^1$

Solution :

(i) $A + B = \begin{bmatrix} 3 & -2 & 4 \\ 2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 4 \\ 6 & 2 & 5 \end{bmatrix}$

$$A + B = \begin{bmatrix} 4 & -3 & 8 \\ 8 & 3 & 5 \end{bmatrix}$$

$$(A + B)^1 = \begin{bmatrix} 4 & 8 \\ -3 & 3 \\ 8 & 5 \end{bmatrix} \text{ ——— (1)}$$

$$A^1 + B^1 = \begin{bmatrix} 3 & 2 \\ -2 & 1 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ -1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$A^1 + B^1 = \begin{bmatrix} 4 & 8 \\ -3 & 3 \\ 8 & 5 \end{bmatrix} \text{ ——— (2)}$$

From (1) and (2) $\boxed{(A + B)^1 = A^1 + B^1}$

(ii) $A - B = \begin{bmatrix} 3 & -2 & 4 \\ 2 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 4 \\ 6 & 2 & 5 \end{bmatrix}$

$$A - B = \begin{bmatrix} 2 & -1 & 0 \\ -4 & -1 & -5 \end{bmatrix}$$

$$(A - B)^1 = \begin{bmatrix} 2 & -4 \\ -1 & -1 \\ 0 & -5 \end{bmatrix} \text{ ——— (1)}$$

$$A^1 - B^1 = \begin{bmatrix} 3 & 2 \\ -2 & 1 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ -1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$A^1 - B^1 = \begin{bmatrix} 2 & -4 \\ -1 & -1 \\ 0 & -5 \end{bmatrix} \text{ ——— (2)}$$

From (1) and (2) $\boxed{(A - B)^1 = A^1 - B^1}$

Example 14 :

If $\mathbf{A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}}$**. Find** $\mathbf{A + A^1}$ **and** $\mathbf{A - A^1}$

Solution : $A^1 = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

$$A+A^1=\begin{bmatrix}2&3\\4&5\end{bmatrix}+\begin{bmatrix}2&4\\3&5\end{bmatrix},\quad A-A^1=\begin{bmatrix}2&3\\4&5\end{bmatrix}-\begin{bmatrix}2&4\\3&5\end{bmatrix}$$

$$A+A^1=\begin{bmatrix}4&7\\7&10\end{bmatrix},\quad A-A^1=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

Example 15 :

If $A=\begin{bmatrix}1&-2&3\\4&5&-3\end{bmatrix}$ and K is a scalar, show that $(KA)^1=KA^1$.

Solution :

$$KA=K\begin{bmatrix}1&-2&3\\4&5&-3\end{bmatrix}=\begin{bmatrix}K&-2K&3K\\4K&5K&-3K\end{bmatrix}$$

$$(KA)^1=\begin{bmatrix}K&4K\\-2K&5K\\3K&-3K\end{bmatrix}\text{——— (1)}$$

$$KA^1=K\begin{bmatrix}1&4\\-2&5\\3&-3\end{bmatrix}=\begin{bmatrix}K&4K\\-2K&5K\\3K&-3K\end{bmatrix}\quad\text{..... (2)}$$

From (1) and (2) $\boxed{(KA)^1=KA^1}$

Example 16 :

If $A=\begin{bmatrix}1&4&2&-1\\3&-2&-4&0\end{bmatrix}$ and $B=\begin{bmatrix}3&0\\4&-2\\-1&-4\\1&2\end{bmatrix}$. Find (i) $3A+2B^1$ (ii) $2A^1-3B$

Solution :

$$A^1=\begin{bmatrix}1&3\\4&-2\\2&-4\\-1&0\end{bmatrix}\qquad B^1=\begin{bmatrix}3&4&-1&1\\0&-2&-4&2\end{bmatrix}$$

(i) $3A+2B^1=3\begin{bmatrix}1&4&2&-1\\3&-2&-4&0\end{bmatrix}+2\begin{bmatrix}3&4&-1&1\\0&-2&-4&2\end{bmatrix}$

$$= \begin{bmatrix} 3 & 12 & 6 & -3 \\ 9 & -6 & -12 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 8 & -2 & 2 \\ 0 & -4 & -8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 20 & 4 & -1 \\ 9 & -10 & -20 & 4 \end{bmatrix}$$

(ii) $2A^1 - 3B = 2\begin{bmatrix} 1 & 3 \\ 4 & -2 \\ 2 & -4 \\ -1 & 0 \end{bmatrix} - 3\begin{bmatrix} 3 & 0 \\ 4 & -2 \\ -1 & -4 \\ 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 6 \\ 8 & -4 \\ 4 & -8 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 12 & -6 \\ -3 & -12 \\ 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 6 \\ -4 & 2 \\ 7 & 4 \\ -5 & -6 \end{bmatrix}$$

Example 17 :

If $A + B = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 1 \\ 4 & 2 & 6 \end{bmatrix}$ and $A - B = \begin{bmatrix} 4 & 1 & -3 \\ 2 & -1 & 1 \\ -2 & 4 & 0 \end{bmatrix}$ Find A^1 and B^1.

Solution :

$$(A + B) + (A - B) = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 1 \\ 4 & 2 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 1 & -3 \\ 2 & -1 & 1 \\ -2 & 4 & 0 \end{bmatrix}$$

$$2A = \begin{bmatrix} 6 & 4 & -2 \\ 2 & 4 & 2 \\ 2 & 6 & 6 \end{bmatrix}$$

$$\div 2 \quad \therefore \quad A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 1 \\ 4 & 2 & 6 \end{bmatrix} - A$$

$$B = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 1 \\ 4 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 2 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 1 & 2 \\ -1 & 3 & 0 \\ 3 & -1 & 3 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 3 \\ -1 & 1 & 3 \end{bmatrix} \quad B^1 = \begin{bmatrix} -1 & -1 & 3 \\ 1 & 3 & -1 \\ 2 & 0 & 3 \end{bmatrix}$$

Example 18 :

Find x such that $\begin{bmatrix} 3 & x \\ 4 & 7 \end{bmatrix}$ is symmetric.

Solution :

$$A = \begin{bmatrix} 3 & x \\ 4 & 7 \end{bmatrix}$$

For symmetric $A = A^1$

$$\begin{bmatrix} 3 & x \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ x & 7 \end{bmatrix}$$

By equating $x = 4$

Example 19 :

If $\begin{bmatrix} 2a-1 & 5 \\ b+3 & 0 \end{bmatrix}$ is a skew symmetric matrix find a, b

Solution :

Let $A = \begin{bmatrix} 2a-1 & 5 \\ b+3 & 0 \end{bmatrix}$

for skew symmetric

$A = -A^1$

$$\begin{bmatrix} 2a-1 & 5 \\ b+3 & 0 \end{bmatrix} = -\begin{bmatrix} 2a-1 & b+3 \\ 5 & 0 \end{bmatrix}$$

equate the corresponding element

$2a - 1 = -2a + 1$ | $b + 3 = -5$

$4a = 2$ | $b = -8$

$a = \frac{1}{2}$

EXERCISE 1.1

I. Questions carrying one marks each.

1. If $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$. Find 2A and 3A.

2. If $A = \begin{bmatrix} 1 & -3 & 5 \\ 6 & 2 & 4 \end{bmatrix}$ Find $5A^1$.

3. If $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 & -1 \\ 1 & 5 & -2 \end{bmatrix}$. Find A + B and A – B.

4. If $\begin{bmatrix} x+y & 3 \\ 2 & -y+x \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find x and y.

5. If $\begin{bmatrix} x^2 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 2x & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$. Find x.

6. If $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Find $A - A^1$.

7. If $A = \begin{bmatrix} 2 & -x \\ x & -7 \end{bmatrix}$. Find $A + A^1$.

8. If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ -3 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$. Find 2A – 3B – C.

9. Find the matrix A. if $2A + B = \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$, where $B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$.

10. If $A = \begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}$, Find X such that $A - 2X = \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix}$.

11. If $A + B + C = 0$ where $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -4 & 2 \\ -3 & 3 \end{bmatrix}$. Find C.

12. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 1 \\ -3 & 4 \end{bmatrix}$ Verify $(A - B)^1 = A^1 - B^1$

II. Questions carrying two marks each.

1. If $A = \begin{bmatrix} 4 & -1 \\ 0 & 3 \\ 2 & -3 \end{bmatrix}$. Find (i) $\frac{5A}{2}$ (ii) $\frac{-2A}{3}$

2. If $A = \begin{bmatrix} 1 & 3 & -1 \\ -1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 3 & -2 \end{bmatrix}$. Find (i) $2A + 3B$ (ii) $A - 3B$ (iii) $A + \frac{1}{3}B$.

3. If $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$. Find $3A - 2B - 4C$

4. If $A = \begin{bmatrix} 1 & -3 \\ -4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ -5 & 1 \end{bmatrix}$ and 0 is a null matrix of order 2×2. Find the matrix C such that

   (i) $2C = A + B$ (ii) $A + C = 0$ (iii) $B + 5C = A$ (iv) $3A + 5B + 2C = 0$

5. Simplify (i) $2\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 4 & 2 \end{bmatrix} + 3\begin{bmatrix} 0 & 2 \\ 1 & 4 \\ 2 & 3 \end{bmatrix}$

   (ii) $4\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix} - 3\begin{bmatrix} 2 & 0 & -1 \\ 4 & 2 & 5 \end{bmatrix}$

   (iii) $\begin{bmatrix} -1 & 2 & 4 \\ 3 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4 & 1 & -2 \\ 2 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix}$

6. Find x and y given that $\begin{bmatrix} -9 \\ 2 \end{bmatrix} - \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

7. If $A=\begin{bmatrix}3 & 1 & 4\\ 5 & 6 & 3x+1\end{bmatrix}$ and $B=\begin{bmatrix}3 & 5\\ 1 & 6\\ 4 & 3\end{bmatrix}$. Find x given that $A=B^1$.

8. Solve for x and y, $x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}3\\5\end{bmatrix}+\begin{bmatrix}4\\6\end{bmatrix}=\begin{bmatrix}12\\17\end{bmatrix}$.

9. Find x and y given that $\begin{bmatrix}x+y & 3\\ -1 & x-y\end{bmatrix}=\begin{bmatrix}4 & 3\\ -1 & 8\end{bmatrix}$.

10. If $\begin{bmatrix}2 & 3\\ 7 & 5\end{bmatrix}+\begin{bmatrix}2 & x-2\\ y-1 & 5\end{bmatrix}=\begin{bmatrix}4 & 1\\ 7 & 10\end{bmatrix}$ Find x and y.

III. Questions carrying three marks each.

1. Find A and B if

(a) $2A+B=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$ and $A-3B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.

(b) $2A+B=\begin{bmatrix}2 & 3 & 1\\ 1 & 4 & 0\end{bmatrix}$, $3A+2B=\begin{bmatrix}4 & 6 & 1\\ 2 & 3 & 5\end{bmatrix}$

(c) $2A-3B=\begin{bmatrix}2 & -4\\ -12 & 1\end{bmatrix}$, $A+5B=\begin{bmatrix}1 & 24\\ 33 & 7\end{bmatrix}$

(d) $2A+B=\begin{bmatrix}3 & -1\\ -2 & 5\end{bmatrix}$ and $A-2B=\begin{bmatrix}4 & 2\\ -1 & 5\end{bmatrix}$.

2. If $A=\begin{bmatrix}3 & 8 & 1\\ 2 & -6 & 3\\ 7 & 4 & -5\end{bmatrix}$, $B=\begin{bmatrix}4 & 0 & 2\\ 6 & 2 & 3\\ 1 & 3 & 2\end{bmatrix}$. Verify $3(A+B)=3A+3B$.

3. Find x, y, a, b if $\begin{bmatrix}3x+4y & 2 & x-2y\\ a+b & 2a-b & -1\end{bmatrix}=\begin{bmatrix}2 & 2 & 4\\ 5 & -5 & -1\end{bmatrix}$.

4. Find x if $\begin{bmatrix}x^3 & 1\\ 3 & 3\end{bmatrix}+\begin{bmatrix}-2x^2 & 3\\ 1 & 4\end{bmatrix}=\begin{bmatrix}-x & 4\\ 4 & 7\end{bmatrix}$.

5. If $A=\begin{bmatrix}2 & -1 & 3\\ -2 & 1 & 0\end{bmatrix}$, $B=\begin{bmatrix}-1 & 0 & 4\\ 5 & 1 & -3\end{bmatrix}$ and $C=\begin{bmatrix}4 & -1 & 1\\ 6 & 2 & -1\end{bmatrix}$. Find the matrix X such that $2A-3B-X=C$.

6. Find a, b, c given that $4\begin{bmatrix}a & 2 & 3\\ 4 & 5 & -6\\ 7 & -8 & 9\end{bmatrix}-5\begin{bmatrix}3 & 2 & 1\\ b & 6 & -4\\ 9 & -8 & 7\end{bmatrix}=\begin{bmatrix}13 & -2 & 7\\ 6 & -5 & -4\\ -17 & 2c & 1\end{bmatrix}$

Answers 1.1

I. 1. $\begin{bmatrix}2 & -4\\ 6 & 8\end{bmatrix}.\begin{bmatrix}3 & -6\\ 9 & 12\end{bmatrix}$ 2. $\begin{bmatrix}5 & 30\\ -15 & 10\\ 25 & 20\end{bmatrix}$ 3. $\begin{bmatrix}4 & -2 & 3\\ 0 & 8 & -4\end{bmatrix},\begin{bmatrix}-2 & 6 & 5\\ -2 & -2 & 0\end{bmatrix}$

4. $x=\frac{-3}{2}, y=-\frac{1}{2}$ 5. $x=-1$ 6. 0 7. $\begin{bmatrix}4 & 0\\ 0 & -14\end{bmatrix}$

8. $\begin{bmatrix}-5 & -10\\ 11 & 4\end{bmatrix}$ 9. $\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ -1 & \frac{-3}{2}\end{bmatrix}$ 10. $\begin{bmatrix}1 & \frac{-5}{2}\\ 1 & 4\end{bmatrix}$ 11. $\begin{bmatrix}-5 & 2\\ 2 & -2\\ 2 & -7\end{bmatrix}$

II. 1. (i) $\begin{bmatrix}10 & \frac{-5}{2}\\ 0 & \frac{15}{2}\\ 5 & \frac{-15}{2}\end{bmatrix}$ (ii) $\begin{bmatrix}\frac{-8}{3} & \frac{2}{3}\\ 0 & -2\\ \frac{-4}{3} & 2\end{bmatrix}$ 2. (i) $\begin{bmatrix}14 & 3 & 4\\ 1 & 9 & -2\end{bmatrix}$ (ii) $\begin{bmatrix}-11 & 6 & -7\\ -4 & -9 & 8\end{bmatrix}$ (iii) $\begin{bmatrix}\frac{7}{3} & \frac{8}{3} & \frac{-1}{3}\\ \frac{-2}{3} & 1 & \frac{4}{3}\end{bmatrix}$

3. $\begin{bmatrix}19 & -8\\ -1 & 10\end{bmatrix}$ 4. (i) $\begin{bmatrix}2 & \frac{1}{2}\\ \frac{-9}{2} & 0\end{bmatrix}$ (ii) $\begin{bmatrix}-1 & 3\\ 4 & 1\end{bmatrix}$ (iii) $\begin{bmatrix}\frac{-2}{5} & \frac{-7}{5}\\ \frac{1}{5} & \frac{-2}{5}\end{bmatrix}$ (iv) $\begin{bmatrix}-9 & \frac{-11}{2}\\ \frac{37}{2} & -1\end{bmatrix}$

5. (i) $\begin{bmatrix} 4 & 8 \\ 9 & 12 \\ 14 & 13 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & -4 & 11 \\ 0 & -2 & -15 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & \frac{5}{2} & 3 \\ 4 & 0 & \frac{7}{2} \\ \frac{5}{2} & 1 & 0 \end{bmatrix}$ 6. $x = -14,\ y = 3$

7. $x = \frac{-2}{3}$ 8. $x = 1,\ y = 2.$ 9. $x = 6,\ y = -2$ 10. $x = 0,\ y = 1.$

III. 1. (a) $A = \begin{bmatrix} \frac{3}{7}, & \frac{-2}{7} \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{7} & \frac{-3}{7} \\ \frac{-2}{7} & \frac{1}{7} \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 5 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -6 & 10 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 4 \\ 6 & 1 \end{bmatrix}$ (d) $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$

3. $x = 2, y = -1, a = 0, b = 5$ 4. $x = 0, x = 1$ 5. $\begin{bmatrix} 3 & -1 & -7 \\ -25 & -3 & 10 \end{bmatrix}$

6. $a = 7, b = 2, c = 4.$

Matrix Multiplication

If A is a matrix of order m × p and B is a matrix of order p × n, then the product AB is defined and its order is m × n. The product AB exists only **when the number of columns of A is equal to the number of rows of B.**

A $\rightarrow$ m × p (m rows, p columns)
B $\rightarrow$ p × n (p rows, n columns)
AB $\rightarrow$ m × n (m rows, n columns)

Consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

$$AB = \begin{bmatrix} a_{11} & \longrightarrow & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \downarrow \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$= \begin{bmatrix} \text{1st row} \times \text{1st column} & \text{1st row} \times \text{2nd column} \\ \text{2nd row} \times \text{1st column} & \text{2nd row} \times \text{2nd column} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

$$BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} & b_{11}a_{13} + b_{12}a_{23} \\ b_{21}a_{11} + b_{22}b_{21} & b_{21}a_{12} + b_{22}a_{22} & b_{21}a_{13} + b_{22}a_{23} \\ b_{31}a_{11} + b_{32}b_{21} & b_{31}a_{12} + b_{32}a_{22} & b_{31}a_{13} + b_{32}a_{23} \end{bmatrix}$$

WORKED EXAMPLES

Example 1 : If $A = \begin{bmatrix} 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 4 & 3 \\ 1 & 0 & -3 & -5 \end{bmatrix}$. Find A B.

Solution :

$$AB = \overrightarrow{\begin{bmatrix} 1 & 4 \end{bmatrix}} \downarrow \begin{bmatrix} 2 & -1 & 4 & 3 \\ 1 & 0 & -3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times2+4\times1 & 1\times-1+4\times0 & 1\times4+4\times-3 & 1\times3+4\times-5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4 & -1+0 & 4-12 & 3-20 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -1 & -8 & -17 \end{bmatrix}$$

Example 2 :

If $A = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & 5 \end{bmatrix}$. Find AB

Solution:

$$AB = \overrightarrow{\begin{bmatrix} 4 \\ -3 \end{bmatrix}} \downarrow \begin{bmatrix} 3 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 4 & 20 \\ -9 & -3 & -15 \end{bmatrix}$$

Example 3 :

If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ **and** $B = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ **Find AB.**

Solution:

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times -2 & + & 3\times 5 \\ 2\times -2 & + & 4\times 5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & + & 15 \\ -4 & + & 20 \end{bmatrix} = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$$

Example 4 :

If $A = \begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 4 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} 3 & 1 & -5 \\ 5 & 4 & -2 \end{bmatrix}$**. Find AB and BA.**

Solution :

$$AB = \begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & -5 \\ 5 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 3+-1\times 5 & 2\times 1+-1\times 4 & 2\times -5+-1\times -2 \\ -3\times 3+1\times 5 & -3\times 1+1\times 4 & -3\times -5+1\times -2 \\ 4\times 3+0\times 5 & 4\times 1+0\times 4 & 4\times -5+0\times -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6-5 & 2-4 & -10+2 \\ -9+5 & -3+4 & 15-2 \\ 12+0 & 4+0 & -20-0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -8 \\ -4 & 1 & 13 \\ 12 & 4 & -20 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 1 & -5 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 2+1\times -3+-5\times 4 & 3\times -1+1\times 1+-5\times 0 \\ 5\times 2+4\times -3+-2\times 4 & 5\times -1+4\times 1+-2\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6-3-20 & -3+1-0 \\ 10-12-8 & -5+4-0 \end{bmatrix} = \begin{bmatrix} -17 & -2 \\ -10 & -1 \end{bmatrix}$$

Example 5 :

If $A = \begin{bmatrix} 2 & -1 & 3 \\ -3 & 1 & 4 \\ -2 & -1 & 5 \end{bmatrix}$**. Find** A^2**.**

Solution :

$$A^2 = A \times A$$

$$= \begin{bmatrix} 2 & -1 & 3 \\ -3 & 1 & 4 \\ -2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ -3 & 1 & 4 \\ -2 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times2+-1\times-3+3\times-2 & 2\times-1+-1\times1+3\times-1 & 2\times3+-1\times4+3\times5 \\ -3\times2+1\times-3+4\times-2 & -3\times-1+1\times1+4\times-1 & -3\times3+1\times4+4\times5 \\ -2\times2+-1\times-3+5\times-2 & -2\times-1+-1\times1+5\times-1 & -2\times3+-1\times4+5\times5 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3-6 & -2-1-3 & 6-4+15 \\ -6-3-8 & 3+1-4 & -9+4+20 \\ -4+3-10 & 2-1-5 & -6-4+25 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -6 & 17 \\ -17 & 0 & 15 \\ -11 & -4 & 15 \end{bmatrix}$$

Example 6 :

If $A = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} -1 & 5 \\ 6 & 2 \end{bmatrix}$**. Show that** $(AB)^1 = B^1A^1$**.**

Solution :

$$AB = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 6 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2\times-1+3\times6 & 2\times5+3\times2 \\ -4\times-1+1\times6 & -4\times5+1\times2 \end{bmatrix} = \begin{bmatrix} -2+18 & 10+6 \\ 4+6 & -20+2 \end{bmatrix} = \begin{bmatrix} 16 & 16 \\ 10 & -18 \end{bmatrix}$$

$$(AB)^1 = \begin{bmatrix} 16 & 10 \\ 16 & -18 \end{bmatrix} \text{ ——— (1)}$$

$$A^1 = \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix}, \quad B^1 = \begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix}$$

$$B^1A^1 = \begin{bmatrix} -1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix}$$

$$B^1A^1 = \begin{bmatrix} -1\times2+6\times3 & -1\times-4+6\times1 \\ 5\times2+2\times3 & 5\times-4+2\times1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+18 & 4+6 \\ 10+6 & -20+2 \end{bmatrix} = \begin{bmatrix} 16 & 10 \\ 16 & -18 \end{bmatrix} \text{ ——— (2)}$$

From (1) and (2)

$$\boxed{(AB)^1 = B^1A^1}$$

Example 7 :

If $A = \begin{bmatrix} 3 & -2 & 5 \end{bmatrix}$. Find AA^1 and A^1A.

Solution :

$$AA^1 = \overrightarrow{\begin{bmatrix} 3 & -2 & 5 \end{bmatrix}} \downarrow \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

$$= [3\times3 + -2\times-2 + 5\times5]$$

$$= [9 + 4 + 25] = [38]$$

$$A^1A = \overrightarrow{\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}} \downarrow \begin{bmatrix} 3 & -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -6 & 15 \\ -6 & 4 & -10 \\ 15 & -10 & 25 \end{bmatrix}$$

Example 8 :

If $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & -4 \\ 4 & -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 4 & 3 \\ 5 & -1 & 1 \end{bmatrix}$ then find AB and BA. Verify $AB \neq BA$

Solution :

$$AB = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & -4 \\ 4 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ -1 & 4 & 3 \\ 5 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times1+-1\times-1+3\times5 & 2\times2+-1\times4+3\times-1 & 2\times-3+-1\times3+3\times1 \\ 1\times1+3\times-1+-4\times5 & 1\times2+3\times4+-4\times-1 & 1\times-3+3\times3+-4\times1 \\ 4\times1+-2\times-1+0\times5 & 4\times2+-2\times4+0\times-1 & 4\times-3+-2\times3+0\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+1+15 & 4-4-3 & -6-3+3 \\ 1-3-20 & 2+12+4 & -3+9-4 \\ 4+2+0 & 8-8-0 & -12-6+0 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & -3 & -6 \\ -22 & 18 & 2 \\ 6 & 0 & -18 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 4 & 3 \\ 5 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & -4 \\ 4 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2-12 & -1+6+6 & 3-8-0 \\ -2+4+12 & 1+12-6 & -3-16+0 \\ 10-1+4 & -5-3-2 & 12+4+0 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 11 & -5 \\ 14 & 7 & -19 \\ 13 & -10 & 16 \end{bmatrix} \text{ and } \boxed{AB \neq BA}$$

Example 9 :

If $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ Prove that $A^2 - 4A - 5I = 0$, where I is unit matrix and 0 is the null matrix of order 2×2.

Solution :

$$A^2 = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 9+8 & 6+2 \\ 12+4 & 8+1 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 16 & 9 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 17 & 8 \\ 16 & 9 \end{bmatrix} - 4\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 17-12-5 & 8-8-0 \\ 16-16-0 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \quad \text{(null matrix)}$$

Example 10 :

If $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ **and** $C = \begin{bmatrix} 1 & -4 & +1 \\ -2 & 5 & -3 \\ 3 & 6 & 5 \end{bmatrix}$ **verify (AB)C = A (BC)**

Solution :

$$AB = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+0 & 15+2 & 18+4 \\ 4+0 & 5+0 & 6+0 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 17 & 22 \\ 4 & 5 & 6 \end{bmatrix}$$

$$(AB)\,C = \begin{bmatrix} 12 & 17 & 22 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -4 & +1 \\ -2 & 5 & -3 \\ 3 & 6 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 12-34+66 & -48+85+132 & 12-51+110 \\ 4-10+18 & -16+25+36 & 4-15+30 \end{bmatrix}$$

$$(AB)\,C = \begin{bmatrix} 44 & 169 & 71 \\ 12 & 45 & 19 \end{bmatrix} \text{ ———— (1)}$$

$$BC = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -4 & 1 \\ -2 & 5 & -3 \\ 3 & 6 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4-10+18 & -16+25+36 & 4-15+30 \\ 0-2+6 & 0+5+12 & 0-3+10 \end{bmatrix} = \begin{bmatrix} 12 & 45 & 19 \\ 4 & 17 & 7 \end{bmatrix}$$

$$A\,(BC) = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 12 & 45 & 19 \\ 4 & 17 & 7 \end{bmatrix}$$

$$A\,(BC) = \begin{bmatrix} 36+8 & 135+34 & 57+14 \\ 12+0 & 45+0 & 19+0 \end{bmatrix} = \begin{bmatrix} 44 & 169 & 71 \\ 12 & 45 & 19 \end{bmatrix} \text{ ———— (2)}$$

From (1) and (2)

$\boxed{(AB)C = A\,(BC)}$

Example 11 :

If $A=\begin{bmatrix}2 & 3\\ 1 & -1\end{bmatrix}$ **and** $B=\begin{bmatrix}-2\\ -1\end{bmatrix}$**. Find the matrix X such that AX = B**

Solution:

A is of 2 × 2 order and B is of 2 × 1 order. The product AX is of 2 × 1 order implies X is of order 2 × 1.

Let the required matrix be $X=\begin{bmatrix}x\\ y\end{bmatrix}$

Now $AX = B$

$$\Rightarrow \begin{bmatrix}2 & 3\\ 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-2\\ -1\end{bmatrix}$$

$$=\begin{bmatrix}2x+3y\\ x-y\end{bmatrix}=\begin{bmatrix}-2\\ -1\end{bmatrix}$$

$2x + 3y = -2$ ———— (1)

$x - y = -1$ ———— (2)

Solving (1) and (2), $x = -1, y = 0$.

$$\therefore\ X=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-1\\ 0\end{bmatrix}$$

Example 12 :

Find x **and** y **if** $\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}1 & 5\\ -2 & 3\end{bmatrix}=\begin{bmatrix}2 & 0\end{bmatrix}$

Solution:

$$\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}1 & 5\\ -2 & 3\end{bmatrix}=\begin{bmatrix}2 & 0\end{bmatrix}$$

$$[x-2y \quad 5x+3y] = [2\ \ 0]$$

$x - 2y = 2$ ———— (1)

$5x + 3y = 0$ ———— (2)

(1) X5 gives, $5x - 10y = 10$

(–) $5x + 3y = 0$

$-13y = 10$

$\therefore\ \boxed{y=\frac{-10}{13}}$ substitute in (1)

$$x - 2y = 2$$

$$x - 2\left(\frac{-10}{13}\right) = 2$$

$$x = 2 - \frac{20}{13}$$

$$x = \frac{6}{13}$$

$$\therefore \boxed{x = \frac{6}{13}, \quad y = \frac{-10}{13}}$$

Example 13 :

Find x, y, z given that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -4 \\ 4 & -1 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Solution :

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -4 \\ 4 & -1 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & +2 \\ -6 & +1 & -4 \\ -8 & -1 & -3 \end{bmatrix} = \begin{bmatrix} -1 \\ -9 \\ -12 \end{bmatrix}$$

$\therefore x = -1, \; y = -9, \; z = -12.$

Example 14 :

Find the product of $[x \;\; y \;\; z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Solution :

$$\begin{bmatrix} xa + yh + zg & xh + yb + zf & xg + yf + zc \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \left[(xa + yh + zg)x + (xh + yb + zf)y + (xg + yf + zc)z \right]$$

$$= [ax^2 + xyh + xzg + xyh + y^2b + yzf + xzg + yzf + z^2c]$$

$$= [ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx]$$

Example 15 :

Solve for x, y, z: $\begin{bmatrix} x & 2 & -3 \\ 5 & y & 2 \\ 1 & -1 & z \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 3 \\ 19 & -5 & 16 \\ 1 & -3 & 0 \end{bmatrix}$

Solution :

$$\begin{bmatrix} 3x+8-6 & -x+4-0 & 2x+10-9 \\ 15+4y+4 & -5+2y+0 & 10+5y+6 \\ 3-4+2z & -1-2+0 & 2-5+3z \end{bmatrix} = \begin{bmatrix} 5 & 3 & 3 \\ 19 & -5 & 16 \\ 1 & -3 & 0 \end{bmatrix}$$

$3x + 2 = 5$ $\quad$ $4y + 19 = 19$ $\quad$ $2z - 1 = 1$

$3x = 3$ $\quad$ $4y = 0$ $\quad$ $2z = 2$

$\therefore\ x = 1$ $\quad$ $y = 0$ $\quad$ $z = 1$

Note:

(i) $(A + B^1) = A^1 + B^1,\ (A - B)^1 = A^1 - B^1$

(ii) $(AB)^1 = B^1A^1$

(iii) $A(BC) = (AB)C$

(iv) $(A^1)^1 = A$

(v) $A(B + C) = AB + AC$

(vi) $AI = IA = A$ [I = Identity Matrix]

(vii) $AB \neq BA$, If $AB = BA$ then either A or B is an Identity Matrix of same order.

EXERCISE 1.2

I. Questions carrying one marks each.

1. If $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Find AB.

2. If $A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$. Find A^2.

3. If $A = \begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}$. Find A^2.

4. If $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 2 \\ -4 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$. Find AB.

5. If $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find x and y.

6. If $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find x,y,z.

7. If $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} 4 & 8 & 4 \\ 1 & 2 & 1 \\ 3 & 6 & 2 \end{bmatrix}$ Find x,y,z.

8. If $A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$ and $B = [1\ \ 3\ \ 4]$. Find AB.

9. Find A.B, if $A = [1\ \ 3\ \ 2]$ and $B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

10 Find x, if $[2\ \ x\ \ 2] \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = [3]$.

II. Questions carrying two and three marks each.

1. If $A = \begin{bmatrix} 2 & -2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ -2 & 0 \end{bmatrix}$. Find AB and BA.

2. If $A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $B = [1\ \ 4\ \ 2]$. Find AB and BA.

3. If $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ Find AA^1 and A^1A.

4. If $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & 2 \\ 2 & 0 & 5 \end{bmatrix}$, Find A^2.

5. If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Find AB^1 and A^1B.

6. If $A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ -1 & 4 \end{bmatrix}$. Show that $(AB)^1 = B^1A^1$.

7. If $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$

   Verify that (i) $A(BC) = (AB)C$

   (ii) $A(B+C) = AB + AC$

8. If $A = \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix}$. Prove that $A^2 - A - 3I = 0$.

9. If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ Show that $A^2 - 4A + 3I = 0$.

10. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Show that $A^2 - 5A = 2I$.

11. Slove for x, y:

   (i) $\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$

   (ii) $\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

   (iii) $\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \end{bmatrix}$

12. If $x = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $z = [2 \quad 1]$. Verify that $x(yz) = (xy)z$.

III. 3 and 5 marks questions.

1. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix}$. and $C = \begin{bmatrix} 1 & -3 & -1 \\ -2 & 4 & 5 \\ 1 & 3 & -2 \end{bmatrix}$. Verify $(AB)C = A(BC)$.

2. If $A=\begin{bmatrix}1 & 2 & -3\\ 1 & -4 & 1\\ 0 & 5 & 3\end{bmatrix}$, $B=\begin{bmatrix}4 & -2 & -3\\ 2 & -4 & -1\\ 0 & 1 & 3\end{bmatrix}$, $C=\begin{bmatrix}4 & 1 & 2\\ 0 & 3 & 1\\ -1 & -3 & 4\end{bmatrix}$. Verify $(A+B)\,C = AC + BC$

3. If $A=\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$, $B=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ and $C=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}$ where $i^2 = -1$. show that

   (i) $A^2 = B^2 = C^2 = -I$

   (ii) $AB = -BA = -C$.

4. If $A=\begin{bmatrix}-1 & 1 & 0\\ 3 & -3 & 3\\ 5 & -5 & 5\end{bmatrix}$, $B=\begin{bmatrix}0 & 4 & 3\\ 1 & -3 & -3\\ -1 & 4 & 4\end{bmatrix}$

   Show that $A^2 . B^2 = A^2$.

5. If $f(x) = x^2 - 5x + 7$, find $f(A)$ where $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$.

6. If $A=\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & 4\\ 0 & 1 & 2\end{bmatrix}$, $B=\begin{bmatrix}2 & 2 & -4\\ -4 & 2 & 4\\ 2 & -1 & 5\end{bmatrix}$ show that $BA = 6I$

7. If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $A + 2B = A^2$, find B.

8. If $\begin{bmatrix}1 & 2\\ 4 & 7\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Find a,b,c,d.

9. If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

   Show that $A^2 - (a+d)A - (bc - ad)\,I$ is a null matrix.

10. If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$. Prove that $A^2 - 4A + 5I$ is a null matrix of order 3×3.

ANSWERS 1.2

I. 1. $\begin{bmatrix} 0 \\ 10 \end{bmatrix}$ 2. $\begin{bmatrix} 1 & -4 \\ 0 & 9 \end{bmatrix}$ 3. $\begin{bmatrix} 7 & 1 \\ -2 & 14 \end{bmatrix}$ 4. $\begin{bmatrix} 11 \\ 7 \\ -18 \end{bmatrix}$ 5. $x = 2, y = 1$

6. $x = 0, y = 1, z = -1$ 7. $x = 1, y = 2, z = 1$ 8. $\begin{bmatrix} 1 & 3 & 4 \\ 4 & 12 & 16 \\ 2 & 6 & 8 \end{bmatrix}$

9. [12] 10. $x = \frac{-3}{4}$

II. 1. $\begin{bmatrix} 10 & -2 \\ 11 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -5 \\ -4 & 4 \end{bmatrix}$ 2. $\begin{bmatrix} 2 & 8 & 4 \\ -1 & -4 & -2 \\ 3 & 12 & 6 \end{bmatrix}$, [4] 3. $\begin{bmatrix} 14 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 5 & -2 \\ 5 & 10 & -3 \\ -2 & -3 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 12 & -5 & 14 \\ 9 & 3 & 16 \\ 16 & -2 & 29 \end{bmatrix}$ 5. $\begin{bmatrix} -1 & 16 \\ -5 & 14 \end{bmatrix}, \begin{bmatrix} 0 & -6 \\ 11 & 13 \end{bmatrix}$

11. (i) $x = \frac{12}{5}, y = \frac{-26}{5}$ (ii) $x = \frac{-2}{5}, y = \frac{-1}{5}$ (iii) $x = \frac{-13}{5}, y = \frac{7}{5}$

III. 5. $A^2 - 5A + 7I = 0$ 7. $\begin{bmatrix} 3 & 4 \\ 6 & 9 \end{bmatrix}$ 8. $a = -7, b = 2, c = 4, d = -1$.

1.5 Determinant

The idea of a determinant was believed to be originated from a Japanese mathematician seki kowa (1683) while systematizing the old Chinese method of solving simultaneous equations whose coefficients were represented by calculating bamboos or sticks. Later the German mathematician Gottfried Wilhelm Von Leibnitz formally developed determinants. The present vertical notation was given in 1841 by Arthur clayey. Determinants was invented independently by **Cramer** whose well known rule for solving simultaneous equations was published is 1750.

1.6 Determinant of a square matrix

If the elements of a matrix are real numbers and order of the matrix is n × n, then we denote the set of matrices by the symbol M (n, R). Thus, M (2, R), M (3, R) are the set of all (2 × 2) and (3 × 3) matrices whose elements are real numbers.

Every square matrix is associated with a real number called its determinant.

For Example, the matrix $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ is associated with the determinant denoted by $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$ whose value is $(2 \times 4) - (1 \times 3) = 5$.

A determinant is defined as a (mapping) function from the set of square matrices to the set of real numbers. Denoting a determinant by Δ, $\Delta : M(2, R) \rightarrow R$. This mapping represents second order determinants as a function (mapping) from the set M (2, R) of all (2×2) matrices to the set of real numbers (R).

Example: $\Delta = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\, a_{22} - a_{21}\, a_{12}$ which is written as $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

The meaning of this statement is that the determinant of the square matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $(a_{11}\, a_{22} - a_{21}\, a_{12})$ which is written as the determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

Similarly, Δ M (3, R) $\rightarrow$ R. This mapping represents third order determinants is a function from the set M (3,R) of all 3×3 determinants to the set of real numbers.

Example : $\Delta \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\ (a_{22}\ a_{33} - a_{32}\ a_{23}) - a_{12}\ (a_{21}\ a_{33} - a_{31}\ a_{23}) +$

$a_{13}\,(a_{21}\, a_{32} - a_{31}\, a_{22})$ which is written as $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Example :

(i) $\begin{vmatrix} 1 & 3 \\ 5 & -8 \end{vmatrix} = (1)\,(-8) - (5)\,(3) = -8 - 15 = -23$

(ii) $\begin{vmatrix} 0 & 3 \\ -2 & -5 \end{vmatrix} = (0)\,(-5) - (-2)\,(3) = 0 + 6 = 6$

(iii) $\begin{vmatrix} 1 & 7 \\ 2 & 8 \end{vmatrix} = 8 - 14 = -6$

(iv) $\begin{matrix} + & - & + \end{matrix}$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\,(a_{22}\,a_{33} - a_{32}\,a_{23}) - a_{12}\,(a_{21}\,a_{33} - a_{31}\,a_{23}) + a_{13}\,(a_{21}\,a_{32} - a_{31}\,a_{22})$$

To write this expansion, multiply the element a_{11} by the second order determinant which is got by deleting the first row and first column. Next, multiply - a_{12} by the determinant obtained by deleting the first row and second column. Then, multiply a_{13} by the determinant obtained by deleting first row and third column. Sum of these products is the value of the determinant.

Example:

$$\Delta = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 4 & 5 \\ 0 & -3 & 6 \end{vmatrix} = 2\begin{vmatrix} 4 & 5 \\ -3 & 6 \end{vmatrix} - 3\begin{vmatrix} 1 & 5 \\ 0 & 6 \end{vmatrix} + (-2)\begin{vmatrix} 1 & 4 \\ 0 & -3 \end{vmatrix}$$

$$= 2\,[24 + 15] - 3\,[6 - 0] - 2\,[-3 - 0]$$

$$= 2\,[39] - 3\,[6] - 2\,[-3]$$

$$= 78 - 18 + 6$$

$$\Delta = 66$$

WORKED EXAMPLES

Example 1 :

Evaluate: $\begin{vmatrix} \mathbf{3} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{-3} & \mathbf{-2} & \mathbf{5} \end{vmatrix}$

Solution :

$\begin{matrix} + & - & + \end{matrix}$

$$\begin{vmatrix} 3 & 0 & -1 \\ 1 & 2 & 4 \\ -3 & -2 & 5 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 4 \\ -2 & 5 \end{vmatrix} - 0\begin{vmatrix} 1 & 4 \\ -3 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ -3 & -2 \end{vmatrix}$$

$= 3\,[10+8] - 0\,[5+12] - 1\,[-2+6]$

$= 3\,[18] - 0 - 1\,(4)$

$= 54 - 4 = 50$

Example 2 : Solve for x: $\begin{vmatrix} x & 4 \\ 4 & x \end{vmatrix} = 0$

Solution :

$$\begin{vmatrix} x & 4 \\ 4 & x \end{vmatrix} = 0$$

$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm 4$

Example 3 :

Solve for x: $\begin{vmatrix} 2 & x & 3 \\ 4 & 1 & 6 \\ 1 & 2 & 7 \end{vmatrix} = 0$

Solution :

$$2\begin{vmatrix} 1 & 6 \\ 2 & 7 \end{vmatrix} - x\begin{vmatrix} 4 & 6 \\ 1 & 7 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 0$$

$2\,[7-12] - x\,[28-6] + 3\,[8-1] = 0$

$2\,[-5] - x\,(22) + 3\,(7) = 0$

$-10 - 22x + 21 = 0$

$-22x + 11 = 0$

$-22x = -11$

$\therefore \quad \boxed{x = \frac{1}{2}}$

Example 4 :

Solve for x: $\begin{vmatrix} x & 2 & -1 \\ 2 & 5 & x \\ -1 & 2 & x \end{vmatrix} = 0$

Solution :

$$x\begin{vmatrix} 5 & x \\ 2 & x \end{vmatrix} - 2\begin{vmatrix} 2 & x \\ -1 & x \end{vmatrix} - 1\begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix} = 0$$

$x\,[5x - 2x] - 2\,[2x + x] - 1\,[4 + 5] = 0$

$x\,[3x] - 2\,(3x) - 1\,(9) = 0$

$3x^2 - 6x - 9 = 0$

$3[x^2 - 2x - 3] = 0$

$x^2 - 2x - 3 = 0$

$x^2 - 3x + x - 3 = 0$

$(x - 3) + 1\,(x - 3) = 0$

$(x - 3)\,(x + 1) = 0$

$x - 3 = 0$ or $x + 1 = 0$

$\boxed{x = 3 \quad x = -1}$

Example 5 :

If $\begin{vmatrix} 2x & -4 \\ -2 & x \end{vmatrix} = 0$**, then find** x**.**

Solution :

$2x^2 - 8 = 0$

$2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

Example 6 :

Find x **if** $\begin{vmatrix} x & 3 \\ 12 & x \end{vmatrix} = 0$

Solution :

$x^2 - 36 = 0$

$x^2 = 36$

$x = \pm\sqrt{36}$

$x = \pm 6$

Example 7 :

Find $|AB|$ **if** $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$

Solution :

$$AB = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 0+2 \\ 6+1 & 0-2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 & 2 \\ 7 & -2 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 8 & 2 \\ 7 & -2 \end{vmatrix}$$

$$= -16 - 14 = -30$$

Example 8 :

Show that $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

Solution :

$$LHS = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$$

$$= a\,[bc - f^2] - h\,[hc - gf] + g\,[hf - bg]$$

$$= abc - af^2 - h^2c + fgh + fgh - bg^2$$

$$= abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= \text{R.H.S}$$

Example 9 : Evaluate: $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Solution :

$$a\begin{vmatrix} c & a \\ a & b \end{vmatrix} - b\begin{vmatrix} b & a \\ c & b \end{vmatrix} + c\begin{vmatrix} b & c \\ c & a \end{vmatrix}$$

$$= a\,[bc - a^2] - b\,[b^2 - ca] + c\,[ab - c^2]$$

$$= abc - a^3 - b^3 + abc + abc - c^3$$

$$= 3abc - a^3 - b^3 - c^3$$

Example 10 :

For what value of x, $\begin{vmatrix} 2 & 3 & 1 \\ x & 2 & 5 \\ 1 & 3 & 4 \end{vmatrix} = 0$?

Solution : $2\begin{vmatrix}2 & 5\\3 & 4\end{vmatrix} - 3\begin{vmatrix}x & 5\\1 & 4\end{vmatrix} + 1\begin{vmatrix}x & 2\\1 & 3\end{vmatrix} = 0$

$2\,[8 - 15] - 3\,[4x - 5] + 1\,[3x - 2] = 0$

$2\,(-7) - 3\,(4x - 5) + 1\,(3x - 2) = 0$

$-14 - 12x + 15 + 3x - 2 = 0$

$-9x + 1 = 0$

$-9x = -1 \therefore \boxed{x = \frac{1}{9}}$

EXERCISE 1.3

I. One and Two marks questions:

1. Evaluate:

(i) $\begin{vmatrix}2 & -3\\1 & 4\end{vmatrix}$ (ii) $\begin{vmatrix}2 & 0\\0 & 5\end{vmatrix}$ (iii) $\begin{vmatrix}6 & 7\\5 & 4\end{vmatrix}$ (iv) $\begin{vmatrix}x & y\\y & x\end{vmatrix}$ (v) $\begin{vmatrix}a+b & a-b\\a & b\end{vmatrix}$

2. Solve for x:

(i) $\begin{vmatrix}x & 7\\7 & x\end{vmatrix} = 0$ (ii) $\begin{vmatrix}2 & -x\\x & -2\end{vmatrix} = 0$ (iii) $\begin{vmatrix}2x+1 & 3\\2 & 4\end{vmatrix} = 0$ (iv) $\begin{vmatrix}3 & x\\4 & 5\end{vmatrix} = -2$ (v) $\begin{vmatrix}1 & x\\x & 1\end{vmatrix} = 0$

3. Evaluate:

(i) $\begin{vmatrix}2 & 3 & -2\\3 & 4 & 0\\1 & -1 & 5\end{vmatrix}$ (ii) $\begin{vmatrix}3 & -1 & 2\\1 & 3 & -2\\1 & 4 & 3\end{vmatrix}$ (iii) $\begin{vmatrix}2 & 0 & 0\\0 & 4 & 0\\0 & 0 & 5\end{vmatrix}$

(iv) $\begin{vmatrix}2 & -1 & 1\\3 & 1 & -2\\1 & 4 & -1\end{vmatrix}$ (v) $\begin{vmatrix}1 & 2 & -1\\3 & 1 & -4\\1 & 0 & -2\end{vmatrix}$

4. Solve for x:

(i) $\begin{vmatrix}1 & 2 & 3\\2 & x & 3\\3 & 4 & 3\end{vmatrix} = 0$ (ii) $\begin{vmatrix}1 & 2 & 5\\2 & x & 10\\3 & 0 & -2\end{vmatrix} = 0$ (iii) $\begin{vmatrix}x & 0 & 1\\0 & x & 0\\1 & 0 & x\end{vmatrix} = 0$

(iv) $\begin{vmatrix}7 & 4 & 11\\-3 & 5 & x\\-x & 3 & 1\end{vmatrix} = 0$ (v) $\begin{vmatrix}x & 0 & 2\\2 & 0 & x\\0 & x & 0\end{vmatrix} = 0$

ANSWERS 1.3

I. 1. (i) 11 (ii) 10 (iii) -1 (iv) $x^2 - y^2$ (v) $2ab + b^2 = a^2$

2. (i) ± 7 (ii) ± 2 (iii) $\frac{1}{4}$ (iv) $\frac{17}{4}$ (v) ± 1

3. (i) 39 (ii) 58 (iii) 40 (iv) 24 (v) 3

4. (i) 3 (ii) 4 (iii) $0, \pm 1$ (iv) $2, \frac{13}{2}$ (v) 0 ± 2

1.7 Properties of Determinants

Property 1: The value of the determinant remains the same if the rows and columns are interchanged.

Proof : We shall prove the property for a third order determinant.

Let $\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Now $\Delta_1 = a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1)$

Let Δ_2 be the determinant obtained from the determinant Δ_1 by interchanging the rows and columns.

ie. $\Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$

$= \Delta_1$

Hence the result.

Note: Since the value determinant is not altered when its rows and columns are interchanged all the properties which are true for rows will also be true for columns.

Property 2: Determinant changes its sign when any two of its rows (or columns) are interchanged.

Proof : Let $\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$\Delta_1 = a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1)$

$\Delta_1 = a_1\ b_2c_3 - a_1b_3\ c_2 - a_2\ b_1\ c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$

Let Δ_2 be the determinant obtained from Δ_1 by interchanging the first and second rows.

ie. $\Delta_2 = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$\Delta_2 = b_1(a_2c_3 - a_3c_2) - b_2\ (a_1c_3 - a_3c_1) + b_3\ (a_1c_2 - a_2c_1)$

$\Delta_2 = a_2\ b_1\ c_3 - a_3b_1\ c_2 - a_1\ b_2c_3 + a_3b_2c_1 + a_1\ b_3c_2 - a_2b_3c_1$

$\Delta_2 = [a_1\ b_2\ c_3 - a_1b_3\ c_2 - a_2\ b_1\ c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1]$

$\Delta_2 = -\Delta_1$

Hence the result.

Property 3: The value of determinant is zero if any two of its rows (or columns) are identical.

Proof : Let $\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, here the elements of 1st & 2nd row are Identical.

$\Delta_1 = a_1\ (a_2c_3 - a_3c_2) - a_2\ (a_1c_3 - a_3c_1) + a_3\ (a_1c_2 - a_2c_1)$

$\Delta_1 = a_1\ a_2c_3 - a_1a_3\ c_2 - a_1\ a_2\ c_3 + a_2a_3c_1 + a_1a_3c_2 - a_2a_3c_1$

$0_1 = 0$

Hence the result.

Property 4: If in a determinant the elements of any row (or column) are multiplied by the same scalar, say k, then the value of the new determinant is K times the given determinant.

Proof : Let $\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Now $\Delta_1 = a_1\ (b_2c_3 - b_3c_2) - a_2\ (b_1c_3 - b_3c_1) + a_3\ (b_1c_2 - b_2c_1)$

Let Δ_2 be the determinant obtained from the determinant Δ_1 by multiplying the elements of the first row by the scalar K.

ie. $\Delta_2 = \begin{vmatrix} Ka_1 & Ka_2 & Ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

ie. $\Delta_2 = Ka_1 (b_2c_3 - b_3c_2) - Ka_2 (b_1c_3 - b_3c_1) + Ka_3 (b_1c_2 - b_2c_1)$

$\Delta_2 = K[a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1)]$

$\Delta_2 = K\Delta_1$

Note: The property - 4 implies that, if there is a common factor among the elements of a row or a column, it can be taken outside the determinant.

Property 5: If each element of any row (or column) of a determinant is the sum of two terms, then the determinant can be expressed as the sum of two determinants.

Proof : Let $\Delta = \begin{vmatrix} a_1 + x & a_2 + y & a_3 + z \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= (a_1 + x) (b_2c_3 - b_3c_2) - (a_2 + y) (b_1c_3 - b_3c_1) + (a_3 + z) (b_1c_2 - b_2c_1)$

$= a_1(b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1)$

$+ x (b_2c_3 - b_3c_2) - y (b_1c_3 - b_3c_1) + z (b_1c_2 - b_2c_1)$

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Hence the result.

Property 6: If each element of a row (or column) is multiplied by a scalar and added to the corresponding elements of any other row (or column), then the value of the determinant is unaltered.

Proof : Let $\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Now, let Δ_2 be the determinant obtained by multiplying the second row elements of Δ_1 by K and adding to the corresponding elements of the first row elements.

i.e. $\Delta_2 = \begin{vmatrix} a_1 + Kb_1 & a_2 + Kb_2 & a_3 + Kb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\Delta_2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} Kb_1 & Kb_2 & Kb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ [by property –5]

$$\Delta_2 = \Delta_1 + K\begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \text{[by property –4]}$$

$\Delta_2 = \Delta_1 + K(0)$ [by property –3]

$\Delta_2 = \Delta_1$

Hence the result.

Property 7: If all the elements in a row (or column) of a determinant are zeroes, then the value of the determinant is zero.

Proof : Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{vmatrix}$ here the elements & 3rd row are zeroes

Now, $\Delta = a_1(0-0) - a_2(0-0) + a_3(0-0)$

$\Delta = 0$

Property 8: If all the elements on any one side or both side of the principal diagonal are zeroes, then the value of the determinant is equal to the product of the elements in the principal diagonal.

Proof : Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix}$

Now, $\Delta = a_1(b_2c_3 - 0) - a_2(0-0) + a_3(0-0)$

$\Delta = a_1b_2c_3$

Example: (i) $\begin{vmatrix} 2 & 3 & -4 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{vmatrix} = 2 \times 5 \times 8 = 80$

(ii) $\begin{vmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & -2 \end{vmatrix} = 4 \times 2 \times -2 = -16$

(iii) $\begin{vmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{vmatrix} = 7 \times 5 \times -3 = -105$

Worked Examples

Example 1 :

Evaluate: $\begin{vmatrix} \mathbf{2011} & \mathbf{2012} \\ \mathbf{2013} & \mathbf{2014} \end{vmatrix} \begin{matrix} \mathbf{R_1} \\ \mathbf{R_2} \end{matrix}$

Solution :

Taking $R_2^1 \to R_2 - R_1$

$$\Rightarrow \begin{vmatrix} 2011 & 2012 \\ 2 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2011 & 2012 \\ 1 & 1 \end{vmatrix}$$

$$= 2\,[2011 - 2012]$$

$$= 2\,(-1)$$

$$= -2$$

Example 2 :

Find the value of $\begin{vmatrix} \mathbf{5431} & \mathbf{5433} \\ \mathbf{5435} & \mathbf{5437} \end{vmatrix}$

Solution :

Taking $R_2^1 \to R_2 - R_1$

$$\begin{vmatrix} 5431 & 5433 \\ 4 & 4 \end{vmatrix} = 4 \begin{vmatrix} 5431 & 5433 \\ 1 & 1 \end{vmatrix}$$

$$= 4\,[5431 - 5433]$$

$$= 4\,(-2)$$

$$= -8$$

Example 3 :

Evaluate: $\begin{vmatrix} \mathbf{81} & \mathbf{82} & \mathbf{83} \\ \mathbf{84} & \mathbf{85} & \mathbf{86} \\ \mathbf{87} & \mathbf{88} & \mathbf{89} \end{vmatrix}$

Solution : Taking $R_2^1 \to R_2 - R_1$

$R_3^1 \to R_3 - R_1$

$$\therefore \begin{vmatrix} 81 & 82 & 83 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 2 \begin{vmatrix} 81 & 82 & 83 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} = 0 \ (\because \text{two rows are identical})$$

Example 4 :

Evaluate : $\begin{vmatrix} 441 & 442 & 443 \\ 445 & 446 & 447 \\ 449 & 450 & 451 \end{vmatrix}$

Solution :

Taking $R_2^1 \to R_2 - R_1$

$R_3^1 \to R_3 - R_1$

$$\therefore \begin{vmatrix} 441 & 442 & 443 \\ 4 & 4 & 4 \\ 8 & 8 & 8 \end{vmatrix} = 2 \begin{vmatrix} 441 & 442 & 443 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{vmatrix}$$

$= 4\,(0)$ ($\because$ two rows are identical)

$= 0$

Example 5 :

Show (without actual expansion) that each of the following determinants is zero.

(a) $\begin{vmatrix} 1 & 5 & 7 \\ 5 & 25 & 35 \\ 3 & -1 & 0 \end{vmatrix}$ **(b)** $\begin{vmatrix} 1 & 10 & 2 \\ 2 & 15 & 4 \\ 3 & 16 & 6 \end{vmatrix}$ **(c)** $\begin{vmatrix} 8 & 2 & 1 \\ 12 & 3 & -5 \\ 16 & 4 & 2 \end{vmatrix}$

(d) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ **(e)** $\begin{vmatrix} 3 & 3^2 & 3^3 \\ 3^2 & 3^3 & 3^4 \\ 3^4 & 3^5 & 3^6 \end{vmatrix}$

Solution :

(a) In the given determinant, the elements of the second row has a common factor 5. Thus we can write.

$$\begin{vmatrix} 1 & 5 & 7 \\ 5 & 25 & 35 \\ 3 & -1 & 0 \end{vmatrix} = 5 \begin{vmatrix} 1 & 5 & 7 \\ 1 & 5 & 7 \\ 3 & -1 & 0 \end{vmatrix} = 0$$ ($\because$ two rows are identical)

(b) In the given determinant, the elements of the third column has a common factor 2. Thus we can write.

$$\begin{vmatrix} 1 & 10 & 2 \\ 2 & 15 & 4 \\ 3 & 16 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & 10 & 1 \\ 2 & 15 & 2 \\ 3 & 16 & 3 \end{vmatrix}$$ $= 2(0) = 0$ ($\because$ two columns are identical)

(c) In the given determinant, the elements of the first column has a common factor 4. Thus we can write.

$$\begin{vmatrix} 8 & 2 & 1 \\ 12 & 3 & -5 \\ 16 & 4 & 2 \end{vmatrix} = 4 \begin{vmatrix} 2 & 2 & 1 \\ 3 & 3 & -5 \\ 4 & 4 & 2 \end{vmatrix} = 4(0) = 0 \qquad (\because \text{ two rows are identical})$$

(d) Taking $C_1^1 \to C_1 + C_2 + C_3$

$$\text{Det} = \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix}$$

$$= 0$$

(e) In the given determinant, the elements of the first row has a common factor 3 and second row has a common factor 3^2

$$\begin{vmatrix} 3 & 3^2 & 3^3 \\ 3^2 & 3^3 & 3^4 \\ 3^4 & 3^5 & 3^6 \end{vmatrix} = 3 \times 3^2 \begin{vmatrix} 1 & 3 & 3^2 \\ 1 & 3 & 3^2 \\ 3^4 & 3^5 & 3^6 \end{vmatrix} = 3 \times 3^2 \times (0) = 0 \quad (\because \text{ two rows are identical})$$

Example 6 : Prove that $= \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{a^2} & \boldsymbol{b^2} & \boldsymbol{c^2} \end{vmatrix} = \boldsymbol{(a-b)\,(b-c)\,(c-a)}$

Solution :

$$\text{L.H.S} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \qquad \text{Take} \quad \begin{matrix} C_2^1 \to C_2 - C_1 \\ C_3^1 \to C_3 - C_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} + & - & + \\ 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

Taking $(b-a)$ common from C_2 and $(c-a)$ common form C_3

$= (b-a)(c-a)[1\{(c+a)-(b+a)\} - 0 + 0]$

$= (b-a)(c-a)\left[c+\cancel{a}-b-\cancel{a}\right]$

$= (b-a)(c-a)(c-b)$

$= (-1)(a-b)(c-a)\ (-1)(b-c)$

$= (a-b)(b-c)(c-a)$

$= \text{R.H.S}$

Example 7 :

Prove that $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution :

$$\text{L.H.S} = \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$ Taking $\begin{matrix} R_2^1 \to R_2 - R_1 \\ R_3^1 \to R_3 - R_1 \end{matrix}$

$$= \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & a-b & a^2-b^2 \\ 0 & a-c & a^2-c^2 \end{vmatrix} = \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & a-b & (a-b)(a+b) \\ 0 & a-c & (a-c)(a+c) \end{vmatrix}$$

Taking $(a-b)$ from R_2 and $(a-c)$ from R_3

$$= (a-b)(a-c)\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & 1 & a+b \\ 0 & 1 & a+c \end{vmatrix}$$

$= (a-b)(a-c)[1\{(a+c)-(a+b)\} - (b-c)\{0-0\} + (b^2+c^2)\{0-0\}]$

$= (a-b)(a-c)[c-b]$

$= (a-b)(-1)(c-a)(-1)(b-c)$

$= (a-b)(b-c)(c-a)$

$= \text{R.H.S}$

Example 8 :

Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Solution :

$$\text{L.H.S} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} \quad \text{Take} \quad \begin{matrix} C_1^1 \to C_1 - C_2 \\ C_2^1 \to C_2 - C_3 \end{matrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

$= (a-b)(b-c)[1(b^2+bc+c^2-(a^2+ab+b^2))]$

$= (a-b)(b-c)[c^2-a^2+bc-ab]$

$= (a-b)(b-c)[(c-a)(c+a)+b(c-a)]$

$= (a-b)(b-c)(c-a)[c+a+b]$

$= (a-b)(b-c)(c-a)(a+b+c)$

$=$ R.H.S

Example 9 :

Show that $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

Solution :

$$\text{L. H. S} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} \quad \text{Take} \quad \begin{matrix} C_1^1 \to C_1 - C_2 \\ C_2^1 \to C_2 - C_3 \end{matrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ x^2-y^2 & y^2-z^2 & z^2 \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ (x-y)(x+y) & (y-z)(y+z) & z^2 \\ (x-y)(x^2+xy+y^2) & (y-z)(y^2+yz+z^2) & z^3 \end{vmatrix}$$

$$= (x-y)\,(y-z) \begin{vmatrix} 0 & 0 & 1 \\ x+y & y+z & z^2 \\ x^2+xy+y^2 & y^2+yz+z^2 & z^3 \end{vmatrix}$$

$= (x-y)\,(y-z)\,[1\,(x+z)\,(y^2+yz+z^2) - (y+z)\,(x^2+xy+y^2)]$

$= (x-y)\,(y-z)\,[\cancel{xy^2} + \cancel{xyz} + xz^2 + \cancel{y^3} + \cancel{y^2z} + yz^2 - x^2y - \cancel{xy^2} - \cancel{y^3} - zx^2 - \cancel{xyz} - \cancel{zy^2}]$

$= (x-y)\,(y-z)\,[xz^2 - zx^2 + yz^2 - x^2y)]$

$= (x-y)\,(y-z)\,[xz\,(z-x) + y\,(z-x)\,(z+x)]$

$= (x-y)\,(y-z)\,(z-x)\,[(xz + y\,(z+x)]$

$= (x-y)\,(y-z)\,(z-x)\,[xz + yz + yx)]$

$=$ R.H.S

Example 10 :

Prove that $\begin{vmatrix} \mathbf{1+a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a} & \mathbf{1+b} & \mathbf{c} \\ \mathbf{a} & \mathbf{b} & \mathbf{1+c} \end{vmatrix} = \mathbf{1+a+b+c}$

Solution : L.H.S $= \begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix}$ Take $C_1^1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 1+a+b+c & b & c \\ a+1+b+c & 1+b & c \\ a+b+1+c & b & 1+c \end{vmatrix}$$

$$= (1+a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & 1+b & c \\ 1 & b & 1+c \end{vmatrix} \quad \begin{matrix} R_2^1 \to R_2 - R_1 \\ R_3^1 \to R_3 - R_1 \end{matrix}$$

$$= (1+a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (1 + a + b + c)\,(1) = 1 + a + b + c$

$=$ R.H.S

Example 11 :

Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Solution :

$$\text{L.H.S} = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad \text{Taking } R_1^1 \to R_1 + R_2 + R_3$$

$$= \begin{vmatrix} a-b-c+2b+2c & 2a+b-c-a+2c & 2a+2b+c-a-b \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad \begin{matrix} C_2^1 \to C_2 - C_1 \\ C_3^1 \to C_3 - C_1 \end{matrix}$$

$$\Rightarrow (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c)\left[-(b+c+a) \times -(c+a+b)\right]$$

Since all the element on one side of the principal diagonal are zeroes, the value of the determinant is equal to the product of the elements of the principal diagonal (property).

$$= (a+b+c)\ (a+b+c)^2$$

$$= (a+b+c)^3$$

Example 12 :

Show that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

Solution : $\text{L.H.S} = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$ Taking $C_1^1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} a+b+2c+a+b & a & b \\ c+b+c+2a+2c & b+c+2a & b \\ c+a+c+a+2b & a & c+a+2b \end{vmatrix}$$

$$= \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$$

$$= (2a+2b+2c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix} \quad \text{Taking } \begin{matrix} R_2^1 \to R_2 - R_1 \\ R_3^1 \to R_3 - R_1 \end{matrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$= 2\,(a+b+c)\,(a+b+c)^2$

$= 2\,(a+b+c)^3$

= R.H.S

Example 13 :

Show that $\begin{vmatrix} x & y & y \\ y & x & y \\ y & y & x \end{vmatrix} = (x+2y)(x-y)^2$

Solution :

$$\text{L.H.S} = \begin{vmatrix} x & y & y \\ y & x & y \\ y & y & x \end{vmatrix} \quad \text{Taking } C_1^1 \to C_1 + C_2 + C_3$$

$$= \begin{vmatrix} x+y+y & y & y \\ y+x+y & x & y \\ y+y+x & y & x \end{vmatrix}$$

$$= \begin{vmatrix} x+2y & y & y \\ x+2y & x & y \\ x+2y & y & x \end{vmatrix}$$

$$= (x+2y) \begin{vmatrix} 1 & y & y \\ 1 & x & y \\ 1 & y & x \end{vmatrix} \quad \text{Taking } \begin{matrix} R_2^1 \to R_2 - R_1 \\ R_3^1 \to R_3 - R_1 \end{matrix}$$

$$= (x+2y)\begin{vmatrix} 1 & y & y \\ 0 & x-y & 0 \\ 0 & 0 & x-y \end{vmatrix}$$

$$= (x + 2y)\,(x-y)^2$$

$$= \text{R.H.S}$$

Example 14 :

Prove that $\begin{vmatrix} x & p & q \\ p & x & q \\ p & q & x \end{vmatrix} = (x-p)(x-q)(x+p+q)$

Solution :

$$\text{L.H.S} = \begin{vmatrix} x & p & q \\ p & x & q \\ p & q & x \end{vmatrix} = \text{ Taking } C_1^1 \to C_1 + C_2 + C_3$$

$$= \begin{vmatrix} x+p+q & p & q \\ p+x+q & x & q \\ p+q+x & q & x \end{vmatrix}$$

$$= (x+p+q)\begin{vmatrix} 1 & p & q \\ 1 & x & q \\ 1 & q & x \end{vmatrix} \text{ Taking } \begin{matrix} R_2^1 \to R_2 - R_1 \\ R_3^1 \to R_3 - R_1 \end{matrix}$$

$$= (x+p+q)\begin{vmatrix} 1 & p & q \\ 0 & x-p & 0 \\ 0 & q-p & x-q \end{vmatrix}$$

$$= (x + p + q)\,[1\,(x-p)\,(x-q) - 0]$$

$$= (x + p + q)\,(x-p)\,(x-q)$$

$$= \text{R.H.S}$$

Example 15 : Show that $\begin{vmatrix} a^2+2a & 2a & 1 \\ 2a+1 & a+1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = (a-1)^3$

Solution: $\text{L.H.S} = \begin{vmatrix} a^2+2a & 2a & 1 \\ 2a+1 & a+1 & 1 \\ 3 & 2 & 1 \end{vmatrix}$ Taking $\begin{matrix} R_1^1 \to R_1 - R_2 \\ R_2^1 \to R_2 - R_3 \end{matrix}$

$$= \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (a-1)(a+1) & a-1 & 0 \\ 2(a-1) & a-1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$$ Take $(a-1)$ is common is R_1^1, $(a-1)$ is common is R_2^1

$$= (a-1)\,(a-1) \begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \; R_1^1 \to R_1 - R_2$$

$$= (a-1)^2 \begin{vmatrix} a-1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix}$$

$= (a-1)^2\,(a-1) = (a-1)^3$

= R.H.S

Example 16 :

Show that $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

Solution: L.H.S $= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$ Taking *a, b* and *c* common from R_1, R_2 and R_3 respectively

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$ Taking a, b & c common from C_1, C_2 & C_3 respectively

$$= (abc)\,(abc) \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$, expand

$= a^2b^2c^2\,[-1\,(1-1) - 1\,(-1-1) + 1\,(1+1)]$

$= a^2b^2c^2\,[0 + 2 + 2] = 4\,a^2b^2c^2$

= R.H.S

Example 17 :

Show that $\begin{vmatrix} \mathbf{0} & \mathbf{ab^2} & \mathbf{ac^2} \\ \mathbf{a^2b} & \mathbf{0} & \mathbf{bc^2} \\ \mathbf{a^2c} & \mathbf{b^2c} & \mathbf{0} \end{vmatrix} = \mathbf{2a^3b^3c^3}$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix}$$ Taking a^2, b^2 and c^2 as common from C_1, C_2 and C_3

$$= a^2b^2c^2 \begin{vmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{vmatrix} = (a^2b^2c^2)(a\ b\ c) \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= a^3b^3c^3[1+1]$$

$$= 2\ a^3b^3c^3$$

$$= \text{R.H.S}$$

Example 18 :

Prove that $\begin{vmatrix} \mathbf{a^2} & \mathbf{bc} & \mathbf{ac+c^2} \\ \mathbf{a^2+ab} & \mathbf{b^2} & \mathbf{ac} \\ \mathbf{ab} & \mathbf{b^2+bc} & \mathbf{c^2} \end{vmatrix} = \mathbf{4a^2b^2c^2}$

Solution :

$$\text{L.H.S} = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$ Taking common a, b, c from C_1, C_2 & C_3

$$= abc \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$ Take $R_2^1 \to R_2 - (R_1 + R_3)$

$$= a\ b\ c \begin{vmatrix} a & c & a+c \\ 0 & -2c & -2c \\ b & b+c & c \end{vmatrix}$$ Take '–2c' common from R_2

$$= -2abc^2 \begin{vmatrix} a & c & a+c \\ 0 & 1 & 1 \\ b & b+c & c \end{vmatrix}$$ Take $R_3^1 \to R_3 - R_1$

$$= -2abc^2 \begin{vmatrix} a & c & a+c \\ 0 & 1 & 1 \\ b-a & b & -a \end{vmatrix} \text{ Take } C_2^1 \to C_2 - C_3$$

$$= -2abc^2 \begin{vmatrix} a & -a & a+c \\ 0 & 0 & 1 \\ b-a & b+a & -a \end{vmatrix}$$

$$= -2abc^2[a\,(0-b-a) + a\,(0-b+a)+(a+c)\,(0)]$$

$$= -2\,abc^2[-ab\, - a^2 - ab + a^2]$$

$$= -2\,abc^2[-2ab]$$

$$= 4a^2b^2c^2$$

$$= \text{R.H.S}$$

Example 19 :

Prove that if $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$ **where** $x \neq 0, y \neq 0$ **&** $z \neq 0$ **then show that**

$1 + \Sigma \frac{1}{x} = 0$.

Solution :

$$xyz \begin{vmatrix} \frac{1}{x}+1 & \frac{1}{x} & \frac{1}{x} \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix} = 0 \qquad \begin{matrix} \text{multiply \&} \div \ R_1 \text{ by } x \\ R_2 \text{ by } y \\ R_3 \text{ by } z \end{matrix}$$

$$\text{i.e.} \quad \Rightarrow \begin{vmatrix} \frac{1}{x}+1 & \frac{1}{x} & \frac{1}{x} \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix} = 0 \ (\because x \neq 0, y \neq 0 \ \& \ z \neq 0)$$

Taking $R_1^1 \to R_1 + R_2 + R_3$ we get

$$= \begin{vmatrix} 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix} = 0$$

$$= \left(1+\Sigma\frac{1}{x}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix} = 0 \quad \text{Taking } \begin{matrix} C_1^1 \to C_1 - C_2 \\ C_2^1 \to C_2 - C_3 \end{matrix}$$

$$= \left(1+\Sigma\frac{1}{x}\right)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{y} \\ 0 & -1 & \frac{1}{z}+1 \end{vmatrix} = 0$$

$$\left(1+\Sigma\frac{1}{x}\right)(1) = 0$$

$$1+\Sigma\frac{1}{x} = 0$$

Example 20 :

If x, y and z are all different and if $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}$ then show that $1+xyz = 0$.

Solution :

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$$ [using the property of determinants]

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$ [by taking x,y,z from R_1, R_2, R_3 of the second determinant]

$$(-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$$(1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$ [expanding the determinant by making use the properties]

we get $(1 + xyz)(x - y)(y - z)(z - x) = 0$

But $x - y \neq 0$, $y - z \neq 0$ and $z - x \neq 0$ because x,y & z are all different

$\therefore\ 1 + xyz = 0$

Example 21 : Solve $\begin{vmatrix} \mathbf{1} & \mathbf{-2} & \boldsymbol{x+3} \\ \mathbf{1} & \boldsymbol{x-2} & \mathbf{3} \\ \boldsymbol{x+1} & \mathbf{-2} & \mathbf{3} \end{vmatrix} = \mathbf{0}$

Solution :

Consider $C_1^1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} 1-2+x+3 & -2 & x+3 \\ 1+x-2+3 & x-2 & 3 \\ x+1-2+3 & -2 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x+2 & -2 & x+3 \\ x+2 & x-2 & 3 \\ x+2 & -2 & 3 \end{vmatrix} = 0$$

$$(x+2) \begin{vmatrix} 1 & -2 & x+3 \\ 1 & x-2 & 3 \\ 1 & -2 & 3 \end{vmatrix} = 0$$

Taking $R_2^1 \to R_2 - R_1$ and $R_3^1 \to R_3 - R_1$

$$(x+2) \begin{vmatrix} 1 & -2 & x+3 \\ 0 & x & -x \\ 0 & 0 & -x \end{vmatrix} = 0$$

$(x + 2)\ [1(-x^2)] = 0$

$x + 2 = 0 \qquad -x^2 = 0$

$x = -2 \qquad x = 0$

Example 22 :

Solve $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$

Solution :

Consider $R_1^1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} 3x-8+3+3 & 3+3x-8+3 & 3+3+3x-8 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3x-2 & 3x-2 & 3x-2 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

$$(3x-2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

Consider $C_2^1 \to C_2 - C_1$ and $C_3^1 \to C_3 - C_1$

$$(3x-2)\begin{vmatrix} 1 & 0 & 0 \\ 3 & 3x-11 & 0 \\ 3 & 0 & 3x-11 \end{vmatrix} = 0$$

$(3x - 2)\ [1(3x - 11)\ (3x - 11)] = 0$

$3x - 2 = 0$, $3x - 11 = 0$, $3x - 11 = 0$

$3x = 2$, $3x = 11$ $3x = 11$

$x = \frac{2}{3}$ $x = \frac{11}{3}$ $x = \frac{11}{3}$

Example 23 :

Solve $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$

Solution: Consider $C_1^1 \to C_1 + C_3$

$$\begin{vmatrix} x-1 & -6 & -1 \\ x-1 & -3x & x-3 \\ x-1 & 2x & x+2 \end{vmatrix} = 0$$

$$(x-1)\begin{vmatrix}1 & -6 & -1\\ 1 & -3x & x-3\\ 1 & 2x & x+2\end{vmatrix}=0$$

Consider $R_2^1 \to R_2 - R_1$ and $R_3^1 \to R_3 - R_1$

$$(x-1)\begin{vmatrix}1 & -6 & -1\\ 0 & -3x+6 & x-2\\ 0 & 2x+6 & x+3\end{vmatrix}=0$$

Take $(x-2)$ comman from R_2
$(x+3)$ comman from R_3

$$(x-1)(x-2)(x+3)\begin{vmatrix}1 & -6 & -1\\ 0 & -3 & 1\\ 0 & 2 & 1\end{vmatrix}=0$$

$(x-1)\ (x-2)\ (x+3)\ (-5)=0$

$x-1=0, \qquad x-2=0, \qquad x+3=0$

$x=1, \qquad x=2 \qquad x=-3$

Example 24 :

Solve $\begin{vmatrix}\boldsymbol{x+1} & \boldsymbol{x+2} & \boldsymbol{3}\\ \boldsymbol{3} & \boldsymbol{x+2} & \boldsymbol{x+1}\\ \boldsymbol{x+1} & \boldsymbol{2} & \boldsymbol{x+3}\end{vmatrix}=\boldsymbol{0}$

Solution :

Consider $C_1^1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix}2x+6 & x+2 & 3\\ 2x+6 & x+2 & x+1\\ 2x+6 & 2 & x+3\end{vmatrix}=0$$

$$(2x+6)\begin{vmatrix}1 & x+2 & 3\\ 1 & x+2 & x+1\\ 1 & 2 & x+3\end{vmatrix}=0$$

Consider $R_2^1 \to R_2 - R_1$ and $R_3^1 \to R_3 - R_1$

$$(2x+6)\begin{vmatrix}1 & x+2 & 3\\ 0 & 0 & x-2\\ 0 & -x & x\end{vmatrix}=0$$

$(2x+6)\ [(1x(x-2))]=0$

$2x+6=0, \qquad x=0, \qquad x-2=0$

$2x=-6$

$x=-3, \qquad x=0 \qquad x=2$

EXERCISE 1.4

I. One marks questions:

1. Evaluate

(i) $\begin{vmatrix} 3200 & 3201 \\ 3202 & 3203 \end{vmatrix}$ (ii) $\begin{vmatrix} 4423 & 4425 \\ 4428 & 4430 \end{vmatrix}$ (iii) $\begin{vmatrix} 400 & 404 \\ 408 & 412 \end{vmatrix}$

(iv) $\begin{vmatrix} 440 & 441 & 442 \\ 443 & 444 & 445 \\ 446 & 447 & 448 \end{vmatrix}$ (v) $\begin{vmatrix} 85 & 88 & 91 \\ 86 & 89 & 92 \\ 87 & 90 & 93 \end{vmatrix}$

II. Two marks questions:

2. Without actual expansion show that

(i) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = 0$ (ii) $\begin{vmatrix} x & 3 & y+z \\ y & 3 & z+x \\ z & 3 & x+y \end{vmatrix} = 0$

(iii) $\begin{vmatrix} x-y & y-z & z-x \\ a-b & b-c & c-a \\ p-q & q-r & r-p \end{vmatrix} = 0$ (iv) $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$

(v) $\begin{vmatrix} 243 & 156 & 300 \\ 81 & 52 & 100 \\ -3 & 0 & 4 \end{vmatrix} = 0$ (vi) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

(vii) $\begin{vmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{vmatrix} = 0$ (viii) $\begin{vmatrix} x+y & y+z & z+x \\ x & y & z \\ x-y & y-z & z-x \end{vmatrix} = 0$

(ix) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix} = ab$ (x) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-a & 1 \\ 1 & 1 & 1-b \end{vmatrix} = ab$

III. Three marks questions:

3. Prove that

(i) $\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ b^2 & c^2 & a^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & a+b & a^2+b^2 \\ 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ b^3 & c^3 & a^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ b^2 & c^2 & a^2 \\ b^3 & c^3 & a^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(v) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

(vi) $\begin{vmatrix} a^2+bc & a & 1 \\ b^2+ca & b & 1 \\ c^2+ab & c & 1 \end{vmatrix} = -2(a-b)(b-c)(c-a)$

4. Prove that

(i) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

(ii) $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

(iii) $\begin{vmatrix} a-3b-3c & 4b & 4c \\ 4a & b-3c-3a & 4c \\ 4a & 4b & c-3a-3b \end{vmatrix} = 9(a+b+c)^3$

(iv) $\begin{vmatrix} a-2b-2c & 3b & 3c \\ 3a & b-2c-2a & 3c \\ 3a & 3b & c-2a-2b \end{vmatrix} = 4(a+b+c)^3$

5. Prove that

(i) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

(ii) $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

(iii) $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$

(iv) $\begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc$

IV. Three marks questions:

6. Solve

(i) $\begin{vmatrix} x+2 & 3 & 4 \\ 2 & x+3 & 4 \\ 2 & 3 & x+4 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} x+2 & 1 & 3 \\ 2 & x+1 & 3 \\ 2 & 1 & x+3 \end{vmatrix} = 0$

(iii) $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$

(iv) $\begin{vmatrix} 2+x & 3 & -4 \\ 2 & 3+x & -4 \\ 2 & 3 & -4+x \end{vmatrix} = 0$

(v) $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$

(vi) $\begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$

(vii) $\begin{vmatrix} 1+x & 1-x & 1-x \\ 1-x & 1+x & 1-x \\ 1-x & 1-x & 1+x \end{vmatrix} = 0$ (viii) $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$

ANSWERS 1.4

I. 1. (i) -2 (ii) -10 (iii) -32 (iv) 0 (v) 0

6. (i) $0, -9$ (ii) $0, -6$ (iii) $\frac{2}{3}, \frac{11}{3}, \frac{11}{3}$ (iv) $0, -1$ (v) $-1, 2, 2$

(vi) $0, 9, -9$ (vii) $3, 0$ (viii) $1, 2, -3$.

Solution of system of Linear Equations by:

1.8 Cramer's rule (or Determinant method)

To solve simultaneous linear equations using determinants. Consider the equations.

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\therefore \Delta x = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$ Apply $C_1 \to C_1 + C_2y + C_3z$

$\Delta x = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$

$\Delta x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \Delta_1 (say)$

Similarly, $\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = \Delta_2$

$$\Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \Delta_3$$

Hence if $\Delta \neq 0$, $\boxed{x = \frac{\Delta_1}{\Delta},\quad y = \frac{\Delta_2}{\Delta},\quad z = \frac{\Delta_3}{\Delta}}$

This is known as **Cramer's Rule** in honour of the swiss mathematician G. Cramer.

Worked Examples

Example 1 :

Solve by cramer's rule : $3x + 4y = 7$

$7x - y = 6$

Solution :

$$\Delta = \begin{vmatrix} 3 & 4 \\ 7 & -1 \end{vmatrix} = -3 - 28 = -31$$

$$\Delta_1 = \begin{vmatrix} 7 & 4 \\ 6 & -1 \end{vmatrix} = -7 - 24 = -31$$

$$\Delta_2 = \begin{vmatrix} 3 & 7 \\ 7 & 6 \end{vmatrix} = 18 - 49 = -31$$

$$\therefore\ x = \frac{\Delta_1}{\Delta} = \frac{-31}{-31} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-31}{-31} = 1$$

Example 2 :

Solve by using cramer's rule : $5x + 7y - 3 = 0$

$7x + 5y - 9 = 0$

Solution:

$5x + 7y - 3 = 0$

$7x + 5y - 9 = 0$

$$\Delta = \begin{vmatrix} 5 & +7 \\ 7 & +5 \end{vmatrix} = 25 - 49 = -24$$

$$\Delta_1 = \begin{vmatrix} 3 & 7 \\ 9 & 5 \end{vmatrix} = 15 - 63 = -48$$

$$\Delta_2 = \begin{vmatrix} 5 & 3 \\ 7 & 9 \end{vmatrix} = 45 - 21 = 24$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-48}{-24} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{24}{-24} = -1$$

Example 3 :

Solve by using cramer's rule :

$$x + y + z = 7; \quad 2x + 3y + 2z = 17; \quad 4x + 9y + z = 37$$

Solution:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 9 & 1 \end{vmatrix} = 1\,(3 - 18) - 1\,(2 - 8) + 1\,(18 - 12) = -3$$

$$\Delta_1 = \begin{vmatrix} 7 & 1 & 1 \\ 17 & 3 & 2 \\ 37 & 9 & 1 \end{vmatrix} = 7(3 - 18) - 1\,(17 - 74) + 1\,(153 - 111) = -6$$

$$\Delta_2 = \begin{vmatrix} 1 & 7 & 1 \\ 2 & 17 & 2 \\ 4 & 37 & 1 \end{vmatrix} = 1(17 - 74) - 7\,(2 - 8) + 1\,(74 - 68) = -9$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 7 \\ 2 & 3 & 17 \\ 4 & 9 & 37 \end{vmatrix} = 1(111 - 153) - 1\,(74 - 68) + 7\,(18 - 12) = -6$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-6}{-3} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-9}{-3} = 3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-6}{-3} = 2$$

Example 4 : Solve by Cramer's rule : $2x+5y+z=-1$

$x+7y-6z=-18$

$y+2z=3$

Solution : $\Delta=\begin{vmatrix}2 & 5 & 1\\ 1 & 7 & -6\\ 0 & 1 & 2\end{vmatrix}=2\,(14+6)-5\,(2-0)+1\,(1-0)=31$

$\Delta_1=\begin{vmatrix}-1 & 5 & 1\\ -18 & 7 & -6\\ 3 & 1 & 2\end{vmatrix}=-1\,(14+6)-5\,(-36+18)+1\,(-18-21)=31$

$\Delta_2=\begin{vmatrix}2 & -1 & 1\\ 1 & -18 & -6\\ 0 & 3 & 2\end{vmatrix}=-2\,(-36+18)+1\,(2+0)+1\,(3-0)=-31$

$\Delta_3=\begin{vmatrix}2 & 5 & -1\\ 1 & 7 & -18\\ 0 & 1 & 3\end{vmatrix}=2\,(21+18)-5\,(3-0)-1\,(1-0)=62$

$x=\frac{\Delta_1}{\Delta}=\frac{31}{31}=1$

$y=\frac{\Delta_2}{\Delta}=\frac{-31}{31}=-1$

$z=\frac{\Delta_3}{\Delta}=\frac{62}{31}=2$

EXERCISE 1.5

1. Solve the following by cramer's rule

(i) $2x+3y=11\ ;\ x-y-3=0$

(ii) $3x+4y=7\ ;\ 7x=y+6$

(iii) $3x+2y=8\ ;\ 4x-3y=5$

(iv) $2x+y=1\ ;\ x-3y=4$

2. Solve the following by cramer's rule

(i) $4x+y=7$, $3y+4z=5$, $3z+5x=2$

(ii) $x+y+z=7$, $2x+3y+2z=17$, $4x+9y+z=37$

(iii) $3x+2y+5z=32$, $2x+5y+3z=31$, $5x+3y+2z=27$

(iv) $3x+y+5z=10$, $x+y+z=0$, $2x-y+3z=9$

ANSWERS 1.5

1. (i) $x = 4,\ y = 1$ (ii) $x = 1,\ y = 1$
   (iii) $x = 2,\ y = 1$ (iv) $x = 1,\ y = -1$
2. (i) $x = 1,\ y = 3,\ z = -1$ (ii) $x = 2,\ y = 3,\ z = 2$
   (iii) $x = 2,\ y = 3,\ z = 4$ (iv) $x = -1,\ y = -2,\ z = 3$

1.9 Minor, cofactor and Adjoint of a square matrix

If A is any square matrix then the minor of an element a_{ij} of A is the determinant obtained by deleting the row and column of $|A|$ to which the element a_{ij} belongs.

The cofactor of a_{ij} is denoted by A_{ij} and $A_{ij} = (-1)^{i+j}$. (Minor of a_{ij}).

Hence the cofactor of a_{ij} is the minor with proper sign. The signs of the cofactors of the element of the determinants of 3rd order and second order are given here.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \text{ and } \begin{vmatrix} + & - \\ - & + \end{vmatrix}$$

For example, if $A = \begin{vmatrix} -1 & 2 & 0 \\ 3 & -4 & -2 \\ 5 & -3 & 4 \end{vmatrix}$ then

Minor of $3 = \begin{vmatrix} 2 & 0 \\ -3 & 4 \end{vmatrix} = 8 - 0 = 8$

Minor of $0 = \begin{vmatrix} 3 & -4 \\ 5 & -3 \end{vmatrix} = -9 + 20 = 11$

Minor of $2 = \begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix} = 12 + 10 = 22$

Cofactor of $3 = -\begin{vmatrix} 2 & 0 \\ -3 & 4 \end{vmatrix} = -(8 - 0) = -8$

Cofactor of $0 = +\begin{vmatrix} 3 & -4 \\ 5 & -3 \end{vmatrix} = +(-9 + 20) = 11$

Cofactor of $2 = -\begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix} = -(12 + 10) = -22$

Adjoint of a square matrix

If A is any square matrix, then its adjoint is the transpose of the matrix obtained by replacing every element of A by its cofactor and is denoted by adjoint A or adj A.

Hence if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

where A_{11} = cofactor of $a_{11} = + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\ a_{33} - a_{23}\ a_{32}$

Similarly A_{12}, A_{13}, A_{21} ..... can be calculated

Example (i) if $\mathbf{A} = \begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{vmatrix}$**, then**

Cofactor of $a = + |d| = d$

Cofactor of $b = - |c| = -c$

Cofactor of $c = - |b| = -b$

Cofactor of $d = + |a| = a$

$$\therefore \textit{adj } A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note: Hence, the adjoint of a 2×2 matrix is obtained from A by

(i) interchanging the elements in the leading diagonal and

(ii) changing the signs of the elements in the other diagonal

Example (ii) **If** $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{3} & \mathbf{-2} & \mathbf{0} \\ \mathbf{4} & \mathbf{5} & \mathbf{-3} \end{bmatrix}$

$$\text{Adj } A = \begin{bmatrix} + \begin{vmatrix} -2 & 0 \\ 5 & -3 \end{vmatrix} & - \begin{vmatrix} 3 & 0 \\ 4 & -3 \end{vmatrix} & + \begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} \\ - \begin{vmatrix} -1 & 2 \\ 5 & -3 \end{vmatrix} & + \begin{vmatrix} 1 & 2 \\ 4 & -3 \end{vmatrix} & - \begin{vmatrix} 1 & -1 \\ 4 & 5 \end{vmatrix} \\ + \begin{vmatrix} -1 & 2 \\ -2 & 0 \end{vmatrix} & - \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} & + \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 6 & 9 & 23 \\ 7 & -11 & -9 \\ 4 & 6 & 1 \end{bmatrix}^T = \begin{bmatrix} 6 & 7 & 4 \\ 9 & -11 & 6 \\ 23 & -9 & 1 \end{bmatrix}$$

Theorem 1 : Let A be any square matrix. Then

(a) the sum of the products of the elements of any row (or column), of a determinant by the corresponding cofactors is the value of the determinant.

(b) the sum of the products of the element of any row (or column) of a determinant by the corresponding cofactors of some other row (or column) is zero.

Proof : (a) Let $|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\ A_{11} + a_{12}\ A_{12} + a_{13}\ A_{13}$$

Similarly, $|A| = a_{11}\ A_{11} + a_{21}\ A_{21} + a_{31}\ A_{31}$

$= a_{21}\ A_{21} + a_{22}\ A_{22} + a_{23}\ A_{23}$ etc

(b) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Consider $a_{11}\ A_{12} + a_{12}\ A_{22} + a_{13}\ A_{23}$

$$= -a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{13}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -a_{11}\ (a_{12}\ a_{33} - a_{32}\ a_{13}) + a_{12}\ (a_{11}\ a_{33} - a_{31}\ a_{13}) - a_{13}\ (a_{11}\ a_{32} - a_{31}\ a_{12})$$

$= 0$ (by cancelling the terms)

Similarly, $a_{11}\ A_{13} + a_{21}\ A_{23} + a_{31}\ A_{33} = 0$

$a_{12}\ A_{11} + a_{22}\ A_{21} + a_{32}\ A_{31} = 0$ etc.

Theorem 2 : If A is any square matrix then

$$\mathbf{A \cdot adjA = adjA \cdot A = |A| \cdot I \cdot}$$

where I is the unit matrix of same order as that of A.

Proof : We prove this result for a matrix of order 3×3.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\therefore \text{ adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$A\,(\text{adj}A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} & a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} \\ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} & a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} & a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} \\ a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= |A| \cdot I \cdot$$

Similarly, $(\text{adj}A)\,A = |A|I$

Note : The result is true for a square matrix of any order. If A is a square matrix of order n, then

$$|(adjA)A| = ||A|I|$$

$$|adjA||A| = |A|^n \qquad \because |AB| = |A|\,|B|$$

$$\therefore |adjA| = |A|^{n-1} \qquad \text{provided } |A| \neq 0$$

1.10 Singular and non-singular matrics:

If A is a square matrix, then it is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

for example, if (i) $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$, then

$$|A| = \begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} = 12 - 12 = 0 \Rightarrow A \text{ is singular.}$$

for example, if (ii) $A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$, then

$|A| = 8 + 12 = 20 \neq 0 \Rightarrow A$ is non-singular.

WORKED EXAMPLES

Example 1: Write the adjoint of $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

Solution :

$$\text{Cofactor of } A = \begin{bmatrix} +(-1) & -(1) \\ -(1) & +(-1) \end{bmatrix}$$

$$\therefore \text{ adj}A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Example 2 : If $A = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$ **write adjA**

Solution :

$$\text{adj}A = \begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}$$

Example 3 : If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **find adj(AB)**

Solution :

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+0 & 0+2 \\ -3+0 & 0+4 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$\text{adj}(AB) = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$$

Example 4 :

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. **Find adj(AB)**

Solution :

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a+0 & 0+b \\ c+0 & 0+d \end{bmatrix}$$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\therefore \text{adj}(AB) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 5 :

If $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$, prove that $A.\text{adj}A = \text{adj}A.A = |A| \cdot I$.

Solution :

$$\text{adj}A = \begin{bmatrix} 5 & -3 \\ -4 & 1 \end{bmatrix}$$

$$A\ \text{adj}A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 5-12 & -3+3 \\ 20-20 & -12+5 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} .....(i)$$

$$\text{adj}A.A = \begin{bmatrix} 5 & -3 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 5-12 & 15-15 \\ -4+4 & -12+5 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} .....(ii)$$

$$|A| = \begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix} = 5 - 12 = -7$$

$$|A| \cdot I = -7 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} .....(3)$$

from i ii & iii

$$\boxed{A\ \text{adj}A = \text{adj}A\ A = |A| \cdot I}$$

Example 6 :

If $A = \begin{bmatrix} 5 & 3 \\ -2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -3 \\ 4 & -8 \end{bmatrix}$ verify that $(\text{adj}A) \cdot (\text{adj}B) = \text{adj}(BA)$
or adj (AB) = (adj B) . (adj A)

Solution :

$$\text{adj}A = \begin{bmatrix} 4 & -3 \\ 2 & 5 \end{bmatrix} \qquad \text{adj}B = \begin{bmatrix} -8 & 3 \\ -4 & 2 \end{bmatrix}$$

$$\therefore\ \text{adj}A \cdot \text{adj}B = \begin{bmatrix} 4 & -3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -8 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -32+12 & 12-6 \\ -16-20 & 6+10 \end{bmatrix} = \begin{bmatrix} -20 & 6 \\ -36 & 16 \end{bmatrix} .....(1)$$

$$BA = \begin{bmatrix} 2 & -3 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 10+6 & 6-12 \\ 20+16 & 12-32 \end{bmatrix} = \begin{bmatrix} 16 & -6 \\ 36 & -20 \end{bmatrix}$$

$$\therefore \text{ adj } BA = \begin{bmatrix} -20 & 6 \\ -36 & 16 \end{bmatrix} \text{.....(ii)}$$

from (i) and (ii)

$$\boxed{\text{adjA} \cdot \text{adjB} = \text{adj(BA)}}$$

Example 7 :

Find the adjoint of $\begin{bmatrix} \mathbf{1} & \mathbf{4} & \mathbf{-2} \\ \mathbf{-2} & \mathbf{-5} & \mathbf{4} \\ \mathbf{1} & \mathbf{-2} & \mathbf{1} \end{bmatrix}$

Solution :

$$\text{Let } A = \begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\text{adjA} = \begin{bmatrix} +\begin{vmatrix} -5 & 4 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & 4 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} -2 & -5 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix} \\ +\begin{vmatrix} 4 & -2 \\ -5 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ -2 & -5 \end{vmatrix} \end{bmatrix}^T$$

$$\text{adjA} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 6 \\ 6 & 0 & 3 \end{bmatrix}^T$$

$$\text{adjA} = \begin{bmatrix} 3 & 0 & 6 \\ 6 & 3 & 0 \\ 9 & 6 & 3 \end{bmatrix}$$

Example 8 :

If $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{5} & \mathbf{0} & \mathbf{2} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$**, then verify** $A \cdot \text{adj}A = \text{adj}A \cdot A = |A| \cdot I$.

Solution : $\text{adj}A = \begin{bmatrix} +\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 5 & 2 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 5 & 0 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & -3 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 5 & 2 \end{vmatrix} & +\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} \end{bmatrix}^T$

$$= \begin{bmatrix} 2 & -3 & -5 \\ 1 & 4 & 3 \\ 4 & -17 & -10 \end{bmatrix}^T$$

$$\text{adj}A = \begin{bmatrix} 2 & 1 & 4 \\ -3 & 4 & -17 \\ -5 & 3 & -10 \end{bmatrix}$$

$$A\,\text{adj}A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 \\ -3 & 4 & -17 \\ -5 & 3 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 2-6+15 & 1+8-9 & 4-34+30 \\ 10-0-10 & 5+0+6 & 20-0-20 \\ 2+3-5 & 1-4+3 & 4+17-10 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \text{.....(i)}$$

$$\text{adj}A\,A = \begin{bmatrix} 2 & 1 & 4 \\ -3 & 4 & -17 \\ -5 & 3 & -10 \end{bmatrix}\begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+5+4 & 4+0-4 & -6+2+4 \\ -3+20-17 & -6+0+17 & 9+8-17 \\ -5+15-10 & -10+0+10 & 15+6-10 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \text{....(ii)}$$

$|A| = 1(0+2) - 2(5-2) - 3(-5-0)$

$= 2 - 6 + 15$

$|A| = 11$

$$|A|\, I = 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \text{.....(iii)}$$

From (i), (ii), (iii)

$$\boxed{A\,\text{adj}A = \text{adj}\,A.A = |A|\, I}$$

Example 9 :

If $A = \begin{bmatrix} 2 & -1 & 3 \\ -2 & 4 & 2 \\ 0 & -3 & 1 \end{bmatrix}$ **and** $B = [-1 \quad 2 \quad -1]$. **Find B(adjA)**

Solution :

$$\text{adj}A = \begin{bmatrix} +\begin{vmatrix} 4 & 2 \\ -3 & 1 \end{vmatrix} & -\begin{vmatrix} -2 & 2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} -2 & 4 \\ 0 & -3 \end{vmatrix} \\ -\begin{vmatrix} -1 & 3 \\ -3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} \\ +\begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ -2 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 6 & 2 & 6 \\ -8 & 2 & 6 \\ -14 & -10 & 6 \end{bmatrix}^T$$

$$\text{adj}A = \begin{bmatrix} 10 & -8 & -14 \\ 2 & 2 & -10 \\ 6 & 6 & 6 \end{bmatrix}$$

$$B(\text{adj}A) = [-1 \quad 2 \quad -1] \begin{bmatrix} 10 & -8 & -14 \\ 2 & 2 & -10 \\ 6 & 6 & 6 \end{bmatrix}$$

$$= [-10+4-6 \quad 8+4-6 \quad 14-20-6]$$

$$= [-12 \quad 6 \quad -12]$$

Example 10 : Find x, if the following matrices are singular

a) $\begin{bmatrix} 2x & -9 \\ -4 & 5x \end{bmatrix}$ b) $\begin{bmatrix} 3 & 2 & x \\ 4 & 1 & -1 \\ 0 & 3 & 4 \end{bmatrix}$

Solution :

(a) $A = \begin{bmatrix} 2x & -9 \\ -4 & 5x \end{bmatrix}$ is singular

$$\Rightarrow |A| = \begin{vmatrix} 2x & -9 \\ -4 & 5x \end{vmatrix} = 0$$

$$10x^2 - 36 = 0$$

$$5x^2 = 18$$

$$x = \pm \sqrt{\frac{18}{5}}$$

(b) Given matrix is singular & hence

$$\begin{vmatrix} 3 & 2 & x \\ 4 & 1 & -1 \\ 0 & 3 & 4 \end{vmatrix} = 0$$

$$3(4+3) - 2(16-0) + x(12-0) = 0$$

$$21 - 32 + 12x = 0$$

$$12x = 11$$

$$\therefore x = 11/12$$

1.11 Inverse of a square matrix:

If A and B are square matrices of the same order such that AB = BA = I, then B is called the inverse of A (and A is called the inverse of B) and is denoted by $B = A^{-1}$

Hence $AA^{-1} = A^{-1}A = I$

Formula for A^{-1}

We know that $A(adjA) = (adjA)A = |A| I \div |A|$

$$\Rightarrow A\left[\frac{adjA}{|A|}\right] = \left[\frac{adjA}{|A|}\right]A = I$$

compare this with $A \cdot A^{-1} = A^{-1} \cdot A = I$

Hence $\boxed{A^{-1} = \frac{adjA}{|A|}}$ It $|A| \neq 0$

Note:

(i) A^{-1} exists if $|A| \neq 0$, i.e. A is non-singular
i.e. A^{-1} is unique

(ii) Since $AA^{-1} = A^{-1}A = I$, if A is non-singular then
A^{-1} is non-singular

(iii) Again $|A||A^{-1}| = |I| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}$

(iv) If A and B are square matrices of same order, then $(AB)^{-1} = B^{-1} A^{-1}$
[The reversal law of inverses]

(v) $A \cdot adjA = adj\ A \cdot A = |A| I$

(vi) adj (AB) = (adj B) (adj A)

WORKED EXAMPLES

Example 1 :

Find the inverse of $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$

Solution :

$$|A| = -4 + 3 = -1$$

$$adjA = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adjA}{|A|} = \frac{1}{|A|} \cdot adjA = \frac{1}{-1}\begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

Example 2 :

If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$**, find** A^{-1}

Solution :

$$|A| = 4 - 1 = 3$$

$$adjA = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \cdot adjA$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Example 3 :

If $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$**. Verify that** $(AB)^{-1} = B^{-1} A^{-1}$

Solution :

$$\text{LHS : } AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix} = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

$$AB = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

$$|AB| = 67 \times 61 - 47 \times 87 = -2$$

$$\text{adjAB} = \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|} \cdot \text{adjAB} = \frac{1}{-2} \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

$$\text{RHS : Again } A^{-1} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}, \quad B^{-1} = \frac{1}{-2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$$

$$\therefore B^{-1} A^{-1} = \frac{1}{-2} \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} \quad \therefore \text{ LHS} = \text{RHS}$$

Thus $\boxed{(AB)^{-1} = B^{-1} A^{-1}}$

Example 4 : Fin the inverse of $A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 2 \\ 0 & -3 & 1 \end{bmatrix}$

Solution : $|A| = 2 \begin{vmatrix} 4 & 2 \\ -3 & 1 \end{vmatrix} + 1 \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} -1 & 4 \\ 0 & -3 \end{vmatrix}$

$$= 2\,[4 + 6] + 1\,[-1 - 0] + 3\,[3 - 0]$$

$$= 20 - 1 + 9 = 28$$

$$\text{adjA} = \begin{bmatrix} +\begin{vmatrix} 4 & 2 \\ -3 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} -1 & 4 \\ 0 & -3 \end{vmatrix} \\ -\begin{vmatrix} -1 & 3 \\ -3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} \\ +\begin{vmatrix} -1 & 3 \\ 4 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$\text{adjA} = \begin{bmatrix} 10 & 1 & 3 \\ -8 & 2 & 6 \\ -14 & -7 & 9 \end{bmatrix}^T \qquad \text{adjA} = \begin{bmatrix} 10 & -8 & -14 \\ 1 & 2 & -7 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\,\text{adjA}$$

$$= \frac{1}{28}\begin{bmatrix} 10 & -8 & -14 \\ 1 & 2 & -7 \\ 3 & 6 & 9 \end{bmatrix}$$

Example 5 :

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix}$

Solution :

$$|A| = 1\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= 1\,(6-2) - 1\,(0+2) + 2\,(0+2)$$

$$= 1\,(4) - 1\,(2) + 2\,(2)$$

$$= 4 - 2 + 4 = 6$$

$$\text{adjA} = \begin{bmatrix} +1\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$\text{adjA} = \begin{bmatrix} 4 & -2 & 2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix}^T \qquad \text{adjA} = \begin{bmatrix} 4 & -1 & -2 \\ -2 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\,\text{adjA}$$

$$= \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 \\ -2 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

Example 6 : If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$ **and** $\mathbf{B} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$**. Find** $\mathbf{A^{-1}B}$

Solution :

$$|A| = 1\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} + 2\begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix}$$

$$= 1\,(8 - 6) + 2\,(0 + 9) + 2\,(0 - 6)$$

$$= 1\,(2) + 2\,(9) + 2\,(-6)$$

$$= 2 + 18 - 12 = 8$$

$$\text{adjA} = \begin{bmatrix} +\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} \\ -\begin{vmatrix} -2 & 2 \\ -2 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 3 & -2 \end{vmatrix} \\ +\begin{vmatrix} -2 & 2 \\ 2 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & -9 & -6 \\ 4 & -2 & -4 \\ 2 & 3 & 2 \end{bmatrix}^T$$

$$\text{adjA} = \begin{bmatrix} 2 & 4 & 2 \\ -9 & -2 & 3 \\ -6 & -4 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\,\text{adj}A$$

$$= \frac{1}{8}\begin{bmatrix} 2 & 4 & 2 \\ -9 & -2 & 3 \\ -6 & -4 & 2 \end{bmatrix}$$

$$A^{-1}B = \frac{1}{8}\begin{bmatrix} 2 & 4 & 2 \\ -9 & -2 & 3 \\ -6 & -4 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -4+8+0 \\ 18-4+0 \\ 12-8+0 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 4 \\ 14 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{7}{4} \\ \frac{1}{2} \end{bmatrix}$$

Example 7 :

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. Show that $A^2 - 5A + 7I = 0$. Hence find A^{-1}.

Solution :

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore\ A^2 - 5A + 7I = 0$

Now, $A^2 - 5A + 7I = 0$

Multiply by A^{-1}

$A^{-1}A^2 - 5A^{-1}A + 7A^{-1}I = 0$

$(A^{-1}A)A - 5(A^{-1}A) + 7A^{-1}I = 0$

$IA - 5I + 7A^{-1} = 0$ $\quad \because A^{-1}A = I$ and

$A - 5I + 7A^{-1} = 0$ $\quad A^{-1}I = A^{-1}$

$7A^{-1} = 5I - A$ $\quad IA = A$

$$7A^{-1} = 5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$$

$$7A^{-1} = \begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix} - \begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$$

$$7A^{-1} = \begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$$

Exercise 1.6

1. Find the adjoint of the following matrices

(i) $\begin{bmatrix}1 & 2\\ 3 & -4\end{bmatrix}$ (ii) $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ (iii) $\begin{bmatrix}-4 & -3\\ -2 & -1\end{bmatrix}$

(iv) $\begin{bmatrix}1 & -2 & 3\\ 0 & 2 & 1\\ -4 & 5 & 2\end{bmatrix}$ (v) $\begin{bmatrix}1 & -1 & 2\\ 2 & 3 & 5\\ -2 & 0 & 1\end{bmatrix}$ (vi) $\begin{bmatrix}1 & 3 & -2\\ 2 & 1 & 3\\ 4 & 1 & 2\end{bmatrix}$

2. Verify $A \cdot \text{adj}A = \text{adj}A \cdot A = |A| \cdot I$ for the following matrices.

(i) $\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}$ (ii) $\begin{bmatrix}-1 & -4\\ 2 & 3\end{bmatrix}$ (iii) $\begin{bmatrix}1 & -5\\ 2 & -4\end{bmatrix}$

(iv) $\begin{bmatrix}1 & 5 & 7\\ 2 & 3 & 0\\ -2 & 1 & 0\end{bmatrix}$ (v) $\begin{bmatrix}1 & 2 & 3\\ 2 & -3 & 1\\ 3 & 1 & -2\end{bmatrix}$ (vi) $\begin{bmatrix}1 & 1 & 1\\ 3 & 4 & 7\\ 1 & -1 & 1\end{bmatrix}$

3. Verify adj (A·B) = adjB· adjA

(i) $A = \begin{bmatrix} 2 & -3 \\ 1 & -6 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$

(ii) $A = \begin{bmatrix} -5 & -1 \\ 4 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 \\ -3 & -2 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 4 & 3 \\ -2 & -6 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$

4. If $A = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 4 & 1 \\ 5 & -4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ then find $(\text{adj } A) \cdot B$

5. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$. Show that $A \cdot \text{adj} A = 0$

6. Find the inverse of the following matrics.

(i) $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 1 \\ -3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$

(vii) $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ (viii) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (ix) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(x) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

7. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$

8. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$. Verify that $A^2 - 5A - 14I = 0$ and hence find A^{-1}.

9. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$. Show that $A^2 - 4A + 5I = 0$. Hence obtain A^{-1}.

10. If $\begin{bmatrix} 5x & 2 \\ -10 & 1 \end{bmatrix}$ is a singular matrix. Find x.

Answers 1.6

1. (i) $\begin{bmatrix} -4 & -2 \\ -3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}$

(iv) $\begin{bmatrix} -1 & 19 & -8 \\ -4 & 14 & -1 \\ 8 & 3 & 2 \end{bmatrix}$ (v) $\begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$ (vi) $\begin{bmatrix} -1 & -8 & 11 \\ 8 & 10 & -7 \\ -2 & 11 & -5 \end{bmatrix}$

6. (i) $\begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{bmatrix}$ (ii) $\begin{bmatrix} \frac{4}{23} & \frac{-1}{23} \\ \frac{3}{23} & \frac{5}{23} \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{3}{8} & \frac{1}{4} \\ \frac{-1}{8} & \frac{1}{4} \end{bmatrix}$

(iv) $\begin{bmatrix} \frac{1}{17} & \frac{-5}{17} \\ \frac{3}{17} & \frac{2}{17} \end{bmatrix}$ (v) $\begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix}$ (vi) $\begin{bmatrix} \frac{1}{17} & \frac{5}{17} & \frac{1}{17} \\ \frac{8}{17} & \frac{6}{17} & \frac{-9}{17} \\ \frac{10}{17} & \frac{-1}{17} & \frac{-7}{17} \end{bmatrix}$

(vii) It self (viii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (ix) $\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{4}{3} & \frac{1}{4} \\ \frac{-1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$

(x) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (8) $\begin{bmatrix} -\frac{1}{7} & -\frac{5}{14} \\ -\frac{2}{7} & \frac{-3}{14} \end{bmatrix}$ (9) $\begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{3} & \frac{1}{5} \end{bmatrix}$ (10) $x = -4$

1.12 Solution of system of linear equations using matrix method.

Consider $a_1x+b_1y+c_1z=d_1$

$a_2x+b_2y+c_2z=d_2$

$a_3x+b_3y+c_3z=d_3$

Let $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

The given set of equations can be written as AX = B .....(1)

Assume that $|A| \neq 0$

Then, A^{-1} exists

Multiply on both the sides of (1) by A^{-1}

$(A^{-1}A)\ X = A^{-1}B$

$I\ X = A^{-1}B$ $\quad \because A^{-1}A = I$

$X = A^{-1}B$ $\quad \because I\ X = X$

The solution is given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B$

Worked Examples

Example 1:

Solve using matrix method $5x+2y=4,\ 7x+3y=5$

Solution : The system of equations can be written in the form AX = B where

$$A=\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix},\ X=\begin{bmatrix} x \\ y \end{bmatrix},\ B=\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$|A|=\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix}=15-14=1\neq 0 \qquad \therefore A^{-1} \text{ exists}$$

$$A^{-1}=\frac{1}{|A|}\cdot \text{adj}A = \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

Solution is $X = A^{-1}B$

$$= \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$X = \begin{bmatrix} 12-10 \\ -28+25 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$\therefore x = 2,\ y = -3$

Example 2 :

Solve by matrix method $x + 2y + 3 = 0,\ 2x - 3y - 8 = 0$

Solution :

$x + 2y = -3$

$2x - 3y = 8$

The system of equation can be written in the form AX = B where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$$

$|A| = -3 - 4 = -7 \neq 0$ $\quad\quad\quad\quad$ $\therefore$ A^{-1} exists

$$\text{adj}A = \begin{bmatrix} -3 & -2 \\ -2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}A$$

$$= \frac{1}{-7}\begin{bmatrix} -3 & -2 \\ -2 & 1 \end{bmatrix}$$

Solution is $X = A^{-1}B$

$$= \frac{1}{-7}\begin{bmatrix} -3 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 8 \end{bmatrix}$$

$$= \frac{1}{-7}\begin{bmatrix} 9-16 \\ 6+8 \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} -7 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \therefore \quad x = 1,\ y = -2$$

Example 3 : Solve by matrix method. $2x + 3y - 5 = 0\ ;\ x - 2y + 1 = 0$

Solution : $2x + 3y = 5$

$x - 2y = -1$

The system of equation can be written in form AX = B where

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$|A| = -4 - 3 = -1 \neq 0$ $\quad\therefore\ A^{-1}$ exists

$$\text{adj}A = \begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\therefore\quad A^{-1} = \frac{1}{|A|} \cdot \text{adj}A$$

$$= \frac{1}{-7}\begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix}$$

Solution is $X = A^{-1}B$

$$= \frac{1}{-7}\begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$= \frac{1}{-7}\begin{bmatrix} -10+3 \\ -5-2 \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} -7 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore\ x = 1,\ y = 1$

Example 4 :

Solve the following system of equation by matrix method

$3x - y + 2z = 13,\ \ 2x + y - z = 3,\ \ x + 3y - 5z = -8$

Solution :

The system of equations can be written in the form AX = B where

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & -1 \\ 1 & 3 & -5 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 13 \\ 3 \\ -8 \end{bmatrix}$$

$$|A| = 3(-5+3) + 1(-10+1) + 2(6-1)$$

$$= 3(-2) + 1\,(-9) + 2(5)$$

$$= -6 - 9 + 10$$

$= -5 \neq 0$ $\quad\therefore\ A^{-1}$ exists

$$\text{adjA} = \begin{bmatrix} +\begin{vmatrix}1 & -1\\ 3 & -5\end{vmatrix} & -\begin{vmatrix}2 & -1\\ 1 & -5\end{vmatrix} & +\begin{vmatrix}2 & 1\\ 1 & 3\end{vmatrix} \\ -\begin{vmatrix}-1 & 2\\ 3 & -5\end{vmatrix} & +\begin{vmatrix}3 & 2\\ 1 & -5\end{vmatrix} & -\begin{vmatrix}3 & -1\\ 1 & 3\end{vmatrix} \\ +\begin{vmatrix}-1 & 2\\ 1 & -1\end{vmatrix} & -\begin{vmatrix}3 & 2\\ 2 & -1\end{vmatrix} & +\begin{vmatrix}3 & -1\\ 2 & 1\end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 9 & 5 \\ 1 & -17 & -10 \\ -1 & 7 & 5 \end{bmatrix}^T$$

$$\text{adjA} = \begin{bmatrix} -2 & 1 & -1 \\ 9 & -17 & 7 \\ 5 & -10 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \cdot \text{adjA}$$

$$= \frac{1}{-5}\begin{bmatrix} -2 & 1 & -1 \\ 9 & -17 & 7 \\ 5 & -10 & 5 \end{bmatrix}$$

Solution is $X = A^{-1}B$

$$= \frac{1}{-5}\begin{bmatrix} -2 & 1 & -1 \\ 9 & -17 & 7 \\ 5 & -10 & 5 \end{bmatrix}\begin{bmatrix} 13 \\ 3 \\ -8 \end{bmatrix}$$

$$= \frac{1}{-5}\begin{bmatrix} -26+3+8 \\ 117-51-56 \\ 65-30-40 \end{bmatrix} = \frac{1}{-5}\begin{bmatrix} -15 \\ 10 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore \quad x = 3, \quad y = -2, \quad z = 1$

Example 5 :

Solve by matrix method

$x - y + 2z = 3;\ \ 2x + z = 1,\ \ 3x + 2y + z = 4$

Solution :

The system of equations can be written in matrix form AX = B where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{aligned} |A| &= 1(0-2) + 1(2-3) + 2(4-0) \\ &= 1(-2) + 1(-1) + 2(4) \\ &= -2 - 1 + 8 \\ &= 5 \neq 0 \qquad\qquad \therefore\ A^{-1} \text{ exists} \end{aligned}$$

$$\text{adj}A = \begin{bmatrix} +\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} \\ +\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 1 & 4 \\ 5 & -5 & -5 \\ -1 & 3 & 2 \end{bmatrix}^T$$

$$\text{adj}A = \begin{bmatrix} -2 & 5 & -1 \\ 1 & -5 & 3 \\ 4 & -5 & 2 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|} \cdot \text{adj}A$$

$$= \frac{1}{5}\begin{bmatrix} -2 & 5 & -1 \\ 1 & -5 & 3 \\ 4 & -5 & 2 \end{bmatrix}$$

Solution is $X = A^{-1}B$

$$= \frac{1}{5}\begin{bmatrix} -2 & 5 & -1 \\ 1 & -5 & 3 \\ 4 & -5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -6+5-4 \\ 3-5+12 \\ 12-5+8 \end{bmatrix} \qquad X = \frac{1}{5}\begin{bmatrix} -5 \\ 10 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore \quad x = -1, \quad y = 2, \quad z = 3$

Exercise 1.7

Solve the following equations using matrix method.

2 marks questions:

1) $2x + 5y = 1, \quad 3x + 2y = 7$
2) $2x - y = -2, \quad 3x + 4y = 3$
3) $4x - 3y = 3, \quad 3x - 5y = 7$
4) $2x + 3y + 1 = 0, \quad x + 2y - 2 = 0$
5) $3x + y = 19, \quad 3x - y = 23$

5 marks questions:

6) $x + y + z = 5, \quad 2x + y - z = 2, \quad 2x - y + z = 2$
7) $x + y - z = 1, \quad 3x + y - 2z = 3, \quad x - y - z = -1$
8) $2x - z = 0, \quad 5x + y = 4, \quad y + 3z = 5$
9) $6x - 12y + 25z = 4, \quad 4x + 15y - 20z = 3, \quad 2x + 18y + 15z = 10$
10) $x - y + z = 2, \quad 2x - y = 0, \quad 2y - z = 1$
11) $x - y - 2z = 3, \quad 2x + y + z = 5, \quad 4x - y - 2z = 1$
12) $3x + y + 2z = 3, \quad 2x - 3y - z = -3, \quad x + 2y + z = 4$

Answers 1.7

1) $x = 3, \ y = -1$
2) $x = \frac{-5}{11}, \ y = \frac{12}{11}$
3) $x = \frac{-6}{11}, \ y = \frac{-19}{11}$
4) $x = -8, \ y = 5$
5) $x = 7, \ y = -2$
6) $x = 1, \ y = 2, \ z = 2$

7) $x = 2,\ y = 1,\ z = 2$ 8) $x = 1,\ y = -1,\ z = 1$ 9) $x = \frac{1}{2},\ y = \frac{1}{3},\ z = \frac{1}{5}$

10) $x = 1,\ y = 2,\ z = 3$ 11) $x = \frac{-2}{3},\ y = \frac{49}{3},\ z = -10$

12) $x = 1,\ y = 2,\ z = -1$

1.13. Application problems to Commerce:

Example 1:

There are two families A and B. There are 2 men 3 women and 1 child in the family A and 1 man, 1 woman and 2 children in family B. The recommended daily allowance of calories is men 2400, women 1900 and children 1800, represent the above data in matrix form.

Solution : Lex X denote members of family A and B

$$X = \begin{matrix} \\ A \\ B \end{matrix}\begin{matrix} \begin{matrix} M & W & C \end{matrix} \\ \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \end{matrix}$$

Y denote the daily allowance of calories

$$Y = \begin{bmatrix} 2400 \\ 1500 \\ 1800 \end{bmatrix}\begin{matrix} M \\ W \\ C \end{matrix}$$

Example 2:

A biscuit company has two distribution agents at the beginning of a week, the stock in hand of the three varieties A,B,C of the biscuits with the agents are given in matrix M. The quantities of A,B and C sold during the week are given in matrix N. What are the stocks in hand at the end of the week?

$\mathbf{M} = \begin{bmatrix} 50 & 55 & 75 \\ 40 & 50 & 60 \end{bmatrix}$ **denotes stock at the begining of a week**

$\mathbf{N} = \begin{bmatrix} 40 & 40 & 60 \\ 30 & 40 & 60 \end{bmatrix}$ **denotes the stock sold during the week**

Solution : Stock in hand at end of the week is $M-N = \begin{bmatrix} 10 & 15 & 5 \\ 10 & 10 & 0 \end{bmatrix}$

Example 3 : A Business organisation has two distribution outlets. At the beginning of a month, the stock on hand of products A,B and C are given in matrix M and the quantities of A,B & C sold during the month are given in matrix N. What are the stocks on hand (in matrix form) at the end of the month?

$M = \begin{bmatrix} 100 & 170 & 250 \\ 200 & 300 & 150 \end{bmatrix}$ **denotes the stock on hand at the begining of the month**

$N = \begin{bmatrix} 80 & 170 & 200 \\ 150 & 280 & 150 \end{bmatrix}$ **denotes the sales during the month**

Solution : $M-N = \begin{bmatrix} 20 & 0 & 50 \\ 50 & 20 & 0 \end{bmatrix}$ denotes the stock on hand at the end of the month.

Example 4 :

The following matrix shows the belongings of three friends Amar, Akbar and Antony

	Shirt	Trousers
Amar	**3**	**3**
Akbar	**5**	**5**
Anthony	**6**	**8**

Write down the row matrix which represents the belongings of Anthony.

Solution : [6 8] row matrix of Anthony's belongings.

Example 5:

Company sold 22 scooters, 15 cars and 10 buses in June and 20, 21 and 11 respectively in October. Represent the data in the matrix form.

Solution : $M = \begin{matrix} S & C & B \\ \end{matrix}$

$$\begin{bmatrix} 22 & 15 & 10 \\ 20 & 21 & 11 \end{bmatrix} \begin{matrix} \text{June} \\ \text{October} \end{matrix}$$

First row denotes sales in the month of June and second row denotes the sales in the month of October. The three colums are the sales of scooters, cards and buses respectively.

Example 6 :

A company sold 40 metal chairs, 20 wooden charis and 10 plastic charis in April and 60, 70 and 80 chairs respectively in May. Represent the sales of the company in matrix form

Solution : $N = \begin{bmatrix} 40 & 20 & 10 \\ 60 & 70 & 80 \end{bmatrix}$

Example 7 :

Ramesh buys 3 kgs of dal, 2 kgs of Rice, 5 kgs of oil. If the cost of each is ₹35, ₹20 and ₹75 respectively per kg. Find the total cost by matrix method.

Solution : Let the quantity = [3 2 5] = Q

$$\text{cost / kg} = \begin{bmatrix} 35 \\ 20 \\ 75 \end{bmatrix} = P \qquad \therefore \text{ Total cost} = Q \times P$$

$$\begin{bmatrix}3 & 2 & 5\end{bmatrix}\begin{bmatrix}35\\20\\75\end{bmatrix}=[3\times35+2\times20+5\times75]$$

$$= [105 + 40 + 375] = [520]$$

$\therefore$ Total cost = ₹520

Example 8 :

A company sold 40 metal chairs, 20 wooden chairs and 10 plastic chairs in April and 60, 70, 80 respectively in May. The selling price of a metal chair is ₹150, that of a wooden chair is ₹500 and plastic chair is ₹300. Find the total revenue in April and May using matrix method.

Solution : Let $M=\begin{bmatrix}40 & 20 & 10\\60 & 70 & 80\end{bmatrix}$

where Ist row indicates the sales in April and 2nd row indicates the sales in May.

1st column indicates metal chairs

2nd column indicates wooden chairs

3rd column indicates plastic chairs

The price matrix $P=\begin{bmatrix}150\\500\\300\end{bmatrix}$

$\therefore$ Total revenue is given by $MP=\begin{bmatrix}40 & 20 & 10\\60 & 70 & 80\end{bmatrix}\begin{bmatrix}150\\500\\300\end{bmatrix}$

$$= \begin{bmatrix}6000+10000+3000\\7000+35000+24000\end{bmatrix} = \begin{bmatrix}19000\\68000\end{bmatrix}$$

₹19000 is the total revenue is April and

₹68000 is the total revenue is May

Total revenue of April & May is ₹87000

Example 9 :

A man buys 8 kg of apples, 4kg of Bananas and 6 kg of Oranges. Apples cost ₹30 per kg, Bananas cost ₹15 per kg and Oranges cost ₹24 per kg. Obtain the total cost using matrices.

Solution : Total cost = quantity × price, Let $Q=\begin{matrix}A & B & O\end{matrix}\begin{bmatrix}8 & 4 & 6\end{bmatrix}$, $P=\begin{bmatrix}30\\15\\24\end{bmatrix}\begin{matrix}App\\Ban\\Orang\end{matrix}$

$$= \begin{bmatrix} 8 & 4 & 6 \end{bmatrix} \begin{bmatrix} 30 \\ 15 \\ 24 \end{bmatrix}$$

$$= [18 \times 30 + 4 \times 15 + 6 \times 24]$$

Total cost = ₹444

6 marks question:

Example 10 :

A Sales person Samarth has the following record of sales for the month of January, February and March 1996 for three products A, B and C. He is paid a commission at fixed rate per unit but at varying rates for products A, B and C.

Months	Sales in units			Commission
	A	B	C	in ₹
January	9	10	2	800
February	15	5	4	900
March	6	10	3	850

Find the rate of commission payable on A,B & C per unit sold using matrix method.

Solution:

Let the rates of commission per unit of sales be ₹x, ₹y, and ₹z for A, B and C respectively.

Now we can form as system of simultaneous equations as follows:

$$9x + 10y + 2z = 800$$
$$15x + 5y + 4z = 900$$
$$6x + 10y + 3z = 850$$

If M is the matrix containing the co-efficients of unknown variables in the equations, X is the matrix of unknown variables and N is the matrix of commission received then

$$M \cdot X = N \cdot$$

$$M = \begin{bmatrix} 9 & 10 & 2 \\ 15 & 5 & 4 \\ 6 & 10 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad N = \begin{bmatrix} 800 \\ 900 \\ 850 \end{bmatrix}$$

$$\Rightarrow X = M^{-1} \cdot N \qquad |M| = 9(15 - 40) - 10(45 - 24) + 2(150 - 30) = -195$$

$$\therefore \ M^{-1} \text{ exists}$$

$$\text{adjM}=\begin{bmatrix} +\begin{vmatrix}5 & 4\\ 10 & 3\end{vmatrix} & -\begin{vmatrix}15 & 4\\ 6 & 3\end{vmatrix} & +\begin{vmatrix}15 & 5\\ 6 & 10\end{vmatrix} \\ -\begin{vmatrix}10 & 2\\ 10 & 3\end{vmatrix} & +\begin{vmatrix}9 & 2\\ 6 & 3\end{vmatrix} & -\begin{vmatrix}9 & 10\\ 6 & 10\end{vmatrix} \\ +\begin{vmatrix}10 & 2\\ 5 & 4\end{vmatrix} & -\begin{vmatrix}9 & 2\\ 15 & 4\end{vmatrix} & +\begin{vmatrix}9 & 10\\ 15 & 5\end{vmatrix} \end{bmatrix}^{T}$$

$$\text{adjM}=\begin{bmatrix} -25 & -21 & 120 \\ -10 & 15 & -30 \\ 30 & -6 & -105 \end{bmatrix} \qquad \text{adjM}=\begin{bmatrix} -25 & -10 & 30 \\ -21 & 15 & -6 \\ 120 & -30 & -105 \end{bmatrix}$$

$$\boxed{M^{-1}=\frac{1}{|A|}\cdot \text{adjM}}$$

$$= \frac{1}{-195}\begin{bmatrix} -25 & -10 & 30 \\ -21 & 15 & -6 \\ 120 & -30 & -105 \end{bmatrix}$$

Solution is $\mathbf{X = M^{-1}\cdot N}$

$$X = \frac{1}{-195}\begin{bmatrix} -25 & -10 & 30 \\ -21 & 15 & -6 \\ 120 & -30 & -105 \end{bmatrix}\begin{bmatrix} 800 \\ 900 \\ 850 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17.95 \\ 43.05 \\ 103.85 \end{bmatrix}$$

$\therefore x = ₹17.95, \qquad y = ₹43.05, \qquad z = ₹103.85$

Example 11 :

A manufacturer Nandhini produces three products *x, y, z* which she sells in two markets. Annual sales are indicated below.

Market	**Products**		
I	**10000**	**2000**	**18000**
II	**6000**	**20000**	**8000**

(a) If unit sale prices of x, y and z are ₹ 2.50, ₹ 1.50 and ₹ 1.00, respectively. Find the total revenue in each market with the help of matrix algebra.

(b) If the unit costs of the above three commodities are ₹ 2.00, ₹ 1.00 and 50 paise respectively.

Solution :

$$\text{Let } A = \begin{bmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{bmatrix},\ B = \begin{bmatrix} 2.50 \\ 1.50 \\ 1.00 \end{bmatrix},\ C = \begin{bmatrix} 2.00 \\ 1.00 \\ 0.50 \end{bmatrix}$$

respectively represents annual sales, sales price and cost price of the three products.

$$\text{Now } AB = \begin{bmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{bmatrix} \begin{bmatrix} 2.5 \\ 1.5 \\ 1.0 \end{bmatrix}$$

$$= \begin{bmatrix} 25000 + 3000 + 18000 \\ 15000 + 30000 + 8000 \end{bmatrix} \begin{bmatrix} 46000 \\ 53000 \end{bmatrix}$$

$\Rightarrow$ Total sale in Market I is ₹ 46000 and that in Market II is ₹ 53000

$\therefore$ Total revenue = 46000 + 53000 = ₹ 99000

$$AC = \begin{bmatrix} 10000 & 2000 & 18000 \\ 6000 & 20000 & 8000 \end{bmatrix} \begin{bmatrix} 2.0 \\ 1.0 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 20000 + 2000 + 9000 \\ 12000 + 20000 + 4000 \end{bmatrix} \begin{bmatrix} 31000 \\ 36000 \end{bmatrix}$$

$\therefore$ Total cost = 31000 + 36000 = ₹ 67000

$$AB - AC = \begin{bmatrix} 15000 \\ 17000 \end{bmatrix} \text{ gives the profit}$$

Example 12 :

A school wants to award its students for the values of punctuality, Good Behaviour, Hardwark with a total cash award ₹6000. Three times the award money for Hard work added to that given for punctuality amounts to ₹11,000. The award money for punctuality and Hardwork together is double the one given for Good Behaviour. Represent the above situation algebraically and also find the award money for each value, using matrix method.

Solution :

Let the values of punctuality, Good Behaviour and Hard work be denoted by x, y and z respectively.

$$x + y + z = 6000 \quad \text{.....(1)}$$
$$\therefore \quad x + 0y + 3z = 11000 \quad \text{.....(2)}$$
$$x + z = 2y \qquad \therefore\ x - 2y + z = 0 \quad \text{.....(3)}$$

Its matrix form is $AX = B$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 1(0+6) - 1(1-3) + 1(-2-0) = 6 \neq 0 \qquad \therefore\ A^{-1} \text{ exists}$$

$$\text{adj}A = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}A$$

$$= \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$AX = B$$

$$\boxed{X = A^{-1}B}$$

$$= \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

Thus $x = ₹500$, $y = ₹2000$ and $z = ₹3500$

EXERCISE 1.8

2 and 3 marks questions:

1. A man buys 5 apple, 6 mangoes and 7 oranges. The cost of each apple is ₹4, each mango is ₹3 and each orange is ₹2. Represent the above in the matrix form and find the total cost.
2. Bhavana buys 8 dozen of mangoes, 10 dozen of apples and 4 dozen of bananas. Mangoes cost ₹18 per dozen, apples cost ₹9 per dozen and bananas cost ₹6 per dozen. Find the total cost by matrix method.
3. A manufacturer Renuka produces 3 products A, B and C which are sold in Delhi and Kolkata. The annual sales of these products are given below:

	A	B	C
Delhi	5000	7500	1500
Kolkata	9000	1200	8700

It the sale price of the products A, B and C per unit be ₹2, ₹3 and ₹4 respectively. Calculate the total revenue in each centre by using matrix.

6 marks questions:

4. The monthly expenditure in an office for 3 months is given in the table.

Month	No. of Clerks	No. of Peons	Typists	Total Monthly Salary (₹)
July	8	4	6	3750
August	9	9	6	5000
September	12	9	12	8850

Assuming that the salary for the different categories of the staff did not vary from month to month. Calculate the salary for each type of staff per month using matrix method.

3 marks questions:

5. Two farmers Preethi and Nithya cultivate only three varieties of rice namely. Basmati, permal & Naura. The sale (in rupees) of these varieties of rice by both farmers in the month of September and October are given by the following matrics A and B. September sale (in rupees).

$$A = \begin{bmatrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ 10000 & 20000 & 30000 \\ 50000 & 30000 & 10000 \end{bmatrix} \begin{matrix} \\ \text{Preethi} \\ \text{Nithya} \end{matrix}$$

October sales (in Rupees)

$$B = \begin{bmatrix} \text{Basmati} & \text{Permal} & \text{Naura} \\ 5000 & 10000 & 6000 \\ 20000 & 10000 & 10000 \end{bmatrix} \begin{matrix} \\ \text{Preethi} \\ \text{Nithya} \end{matrix}$$

(a) Find the combined sales in September and October for each farmer in each variety.

(b) Find the decrease in sales from September to October.

6. In a legislative assembly election, a political party hired a public relations firm to promote its candidate in three ways: telephone, housecalls and letters. The cost per contact (in paise) is given in matrix A as:

$$A = \begin{bmatrix} \text{cost per contact} \\ 40 \\ 100 \\ 50 \end{bmatrix} \begin{matrix} \\ \text{Telephone} \\ \text{House call} \\ \text{Letter} \end{matrix}$$

The number of contacts of each type made in two cities X and Y is given by

$$B = \begin{bmatrix} \text{Telephone} & \text{House call} & \text{Letter} \\ 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \begin{matrix} \text{X} \\ \text{Y} \end{matrix}$$

Find the total amount spent by the group in the two cities X and Y.

7. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹80, ₹60 and ₹40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

ANSWERS 1.8

1. Sale matrix = [5 6 7] price matrix = $\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ Total cost = ₹52
2. ₹258
3. ₹38500 and ₹56400
4. ₹928.57, ₹381.75, ₹772.63
5. (a) $\begin{bmatrix} 15000 & 30000 & 36000 \\ 70000 & 40000 & 20000 \end{bmatrix}$ (b) $\begin{bmatrix} 5000 & 10000 & 24000 \\ 30000 & 20000 & 0 \end{bmatrix}$
6. ₹3400, ₹7200 7. ₹20160

* * * *

Chapter

2

PERMUTATION AND COMBINATION

2.1 Introduction:

A permutation is a way to arrange items or numbers if order matters. To know if order matters think of this problem. If you are trying to visit three cities, does it matter which one you visit first? If it is a business trip, then the order probably matters because your boss wants you to spend the least amount of money possible. It is possible to fly certain routes more cheaply than others. However, if this were for your own vacation, perhaps you would not care in what order you visit these cities. But when the order matters it is a permutation problem.

"MY FRUIT SALAD IS A COMBINATION OF APPLES, GRAPES AND BANANAS". We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas" its the same fruit salad.

"THE COMBINATION TO THE SAFE LOCK IS 298". Now we do care about the order "982" would not work nor would "829". It has to be exactly 2-9-8.

Then this combination lock should be really called as "permutation lock"!!

In Mathematics we use more precise language:

* **If the order doesn't matter, it is a combination**
* **If the order does matter it is a permutation**

In other words

* A permutation is an ordered combination.
* To help you to remember, think "permutation.... position".

If n distinct objects have to be arranged in a row then first place can be filled in n ways, second place in (n –1) ways, third in (n –2) ways and so on... Hence total arrangements is $n \times (n-1) \times (n-2) \times .... 3 \times 2 \times 1$. But how do we write that mathematically?? For this we use the factorial notation.

2.2 The factorial:

The product of first n natural numbers is denoted by n! or $\lfloor n$ and is read as factorial n.

i.e. $n! = 1 \times 2 \times 3 .... n = n \times (n-1) \times (n-2) .... 3 \times 2 \times 1$ for eg. $4! = 4 \times 3 \times 2 \times 1 = 24$.

According to this definition 0! is meaningless. However we make a convention that $0! = 1$

2.3 Fundamental principle of counting

(a) **Multiplication :** If one operation can be performed in 'm' ways and corresponding to

each way of performing the first operation a second operation can be performed in 'n' ways, then the two operations together can be performed in (m × n) ways.

Example 1 :

If a die is cast and then a coin is tossed, find the number of all possible outcomes.

Solution :

A die can fall in 6 different ways showing any one of six numbers and a coin can fall in two ways showing Head (H) or Tail (T).

∴ Total possible outcomes is = 6 × 2 = 12

i.e., (1,H) (2,H) (3,H) (4,H) (5,H) (6,H)

(1,T) (2,T) (3,T) (4,T) (5,T) (6,T)

are the 12 possible outcomes.

(b) Addition : If an operation can be performed in m different ways and if another operation can be performed in n different ways and the two operations are mutually exclusive (Two operations are said to be mutually exclusive if the performance of one does not affect the performance of the other) then either of the two operations can be performed in (m + n) ways.

Example 2 :

In how many ways can you choose a number from 1 to 20 which is a multiple of 5 or 7.

Solution: Multiples of 5 are 5, 10, 15, 20 multiples of 7 are 7, 14

∴ no. of ways of choosing multiple of 5 or 7 is 4 + 2 = 6

2.4 Permutations vs. Combinations:

(a) Consider three letters a, b, c. Taking all the three letters at a time, we can form one group abc. But it can be arranged in six different ways {abc, acb, bac, bca, cab, cba} In this case one group gives rise to six arrangements.

(b) Let us consider 3 letters a, b and c. Taking 2 letters out of these 3 letters we can form 3 groups namely ab, ac, bc. Since each of these groups give rise to 2 arrangements, the total arrangements that can be made by taking 2 out of the 3 letters are six i.e. {ab, ba, ac, ca, bc, cb}

Hence we can say that the number of groups or **combinations** of three letters, taking two at a time is ${}^3c_2 = 3$ and the number of arrangements or **permutations** of three letters taken two at a time is ${}^3p_2 = 6$.

Thus we have two types of classification of things a) groups and b) arrangements. Groups are called combinations and arrangements are known as permutations.

Note : Whenever order of the things matter it is permutation, wherever order of the selections do not matter it is combinations.

2.5 Permutations

The number of permutations of 'n' different things taken 'r' at a time is the same as different ways in which 'r' places can be filled up with n given things. It is denoted by ${}^{n}p_{r}$ or p(n, r) and is given by $\boxed{{}^{n}p_{r} = \frac{n!}{(n-r)!}}$

Also ${}^{n}p_{r} = n\,(n-1)\,(n-2)\,\ldots\,(n-r+1)$

where the product has exactly r factors.

eg: ${}^{8}p_{3} = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{40320}{120} = 336$

or

${}^{8}p_{3} = 8 \times 7 \times 6$ (3 factors)

$= 336$

Note: (1) ${}^{n}p_{n} = n!$

(2) ${}^{n}p_{1} = n$

Example 1 :

Find n if ${}^{n}p_{3} = 210$

Solution : ${}^{n}p_{3} = 210$

$$\frac{n!}{(n-3)!} = 210$$

$$\frac{n(n-1)(n-2)(n-3)!}{(n-3)!} = 210$$

$n(n-1)\,(n-2) = 210$

Assume $n = 7$

$7 \times 6 \times 5 = 210$

$\therefore \; n = 7$

Example 2 : **If ${}^{n}p_{3} : {}^{n}p_{2} : 3:1$ find 'n'**

Solution : $\frac{{}^{n}p_{3}}{{}^{n}p_{2}} = \frac{3}{1}$

$$\frac{n(n-1)(n-2)}{n(n-1)} = \frac{3}{1}$$

$n - 2 = 3$

$\therefore\ n = 5$

Example 3 :

In how many ways can 5 people stand in a queue.

Solution :

The number of ways 5 people can stand in a queue is the permutation of 5 things taken 5 at a time i.e. ${}^{5}p_{5} = 5! = 120$

Example 4 :

How many 3 letter words can be formed using only consonants but each only once.

Solution:

There are 21 consonants. Hence the number of 3 letter words is the permutations of 21 things taken 3 at a time i.e. ${}^{21}p_{3} = 21 \times 20 \times 19 = 7980$

2.6 Various types of permutations

Type I : When a certain number of things always occur together.

Example 5 :

In how many ways can 7 people be seated in a row for a photograph if two particular people always want to be together.

Solution :

Consider the two people who want to be together as one unit. Then number of permutations of remaining five people + one unit i.e. six units is 6!

For each of these the two people considered as one unit can be arranged in 2! ways

$\therefore$ Total permutations $= 6! \times 2!$

$= 720 \times 2$

$= 1440$

Example 6 :

In how many ways can 7 maths books, 8 statistics books and 4 Accountancy books can be arranged in a shelf if the maths books are always together.

Solution :

Consider the 7 maths books as one. No of permutations of 1 unit (7 maths books) + 8 statistics books + 4 Accountancy books i.e. total 13 items is 13!

For each of these arrangements the 7 books can be arranged in 7! ways.

$\therefore$ Total ways = $13! \times 7!$

Type II : when certain number of things occuring together is more than one type.

Example 7 :

In example 6, in how many ways can the books be arranged if books of the same subject are always together.

Solution :

Consider all 7 maths books as one unit, 8 statistics books as 1 unit and 4 accountancy books as 1 unit .

$\therefore$ Total three units can be arranged in 3! ways. For each of these the maths, statistics and accountancy books can be arranged in 7!, 8!, 4! ways respectively.

$\therefore$ Total permutations = 3! 7! 8! 4!

Type III : When certain things of one type do not occur together

Example 8:

In how many ways can 10 examination papers be arranged so that the best and worst papers never come together.

Solution : Total ways in which best and worst are never together = Total Arrangements - Two papers are always together.

Total ways = $10! - (9! \times 2!)$

Type IV : When no two of a certain type occur together.

Example 9 :

In how many ways can 9 boys and 6 girls be arranged in a row if no two girls are together.

Note **:** No two girls are together is different from all the girls are not together. All the girls are not together means few girls can be together.

Solution :

If no two girls can be together we first arrange the boys. This can be done in 9! ways.
In the gaps we have to arrange the girls. Considering the left most and right most positions and the gaps in between the boys.

B_1 B_2 B_3 B_4 B_5 B_6 B_7 B_8 B_9

1 2 3 4 5 6 7 8 9 10

$\therefore$ In 10 places 6 girls can be arranged in ${}^{10}p_6$ ways.

$\therefore$ Total ways = $9! \times {}^{10}p_6$

Type V : Formation of numbers with digits (digits are not repeated)

Case 1 : When there is no restriction.

Example 10 :

How many different 4 digit numbers can be formed using the digits 1, 2, 4, 5, 7, 8, 9 no digit being repeated in any number.

Solution :

Th	H	T	U
7	6	5	4

The Thousand place can be filled in 7 ways. For each of these the hundred, tenth, units place can be filled in 6, 5, 4 ways respectively.

$\therefore$ Total ways = $7 \times 6 \times 5 \times 4 = 840$ nos.

Case 2 : Numbers divisible by a particular number or even numbers or odd numbers.

Example 11 :

How many 4 digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 no digit being repeated.

Solution :

Th	H	T	U
5	4	3	3

If the number is even it has to end with even number. Hence units place has 3 choices and thousandth, hundredth and tenth place can be filled in 5, 4, 3 ways respectively.

$\therefore$ Total 4 digit even numbers = $5 \times 4 \times 3 \times 3 = 180$

Case 3 : Numbers which are greater than or lesser than a certain number.

Example 12 :

How many 5 digit numbers can be formed with the digits 2, 3, 5, 7, 9 which are a) less than 30,000 b) greater than 70,000 c) between 50,000 & 90,000 d) greater than 75,000

Solution :

a) If the number has to be less than 30,000 then Ten thousandth place can be filled by 2 in one way. The remaining places can be filled in 4, 3, 2 and 1 ways. Hence total numbers lessthan 30,000 is $1 \times 4 \times 3 \times 2 \times 1 = 24$

Ten Th	Th	H	T	U
1	4	3	2	1

(b) Greater than 70,000, Ten thousandth place can be filled in 2 ways with 7 or 9.

Ten Th	Th	H	T	U
7,9				

$2 \times 4 \times 3 \times 2 \times 1 = 48$

(c) Between 50,000 and 90,000: Ten Thousandth place can be filled by 5 or 7 in 2 ways.

Ten Th	Th	H	T	U
5 , 7				

$2 \times 4 \times 3 \times 2 \times 1 = 48$

d) **Greater than 75,000 :** when the the thousandth place is filled by 7. The thousandth place can be filled by 5 or 9 but when ten thousandth place is filled by 9 there is no restriction in thousandth place. Hence the two cases have to be considered separately.

Case (a) : 1st digit is 7

Ten Th	Th	H	T	U
7	5, 9			

$1 \times 2 \times 3 \times 2 \times 1 = 12$

Case (b) : 1st digit is 9

Ten Th	Th	H	T	U
9				

$1 \times 4 \times 3 \times 2 \times 1 = 24$

Total Numbers = 12 + 24 = 36

Case 4 : When one of the given digits is zero.

Example 13 :

How many 4 digit numbers can be formed with the digits 0, 1, 2, 3, 4, 5, 6 (Digits not repeated)

a) How many of them are odd.

b) How many of them are divisible by 5.

Solution :

Zero cannot be placed in the thousandth place because then it will be a 3 digit number. Hence it can be filled in 6 places.

Th	H	T	U
$\neq 0$			

$6 \times 6 \times 5 \times 4$

In the hundreds place, since zero can be included it can be filled in six ways. The other two places can be filled in 5 & 4 ways.

$\therefore$ Total ways = $6 \times 6 \times 5 \times 4 = 720$

a) odd numbers

Th	H	T	U
≠0			1,3,5

$5 \times 5 \times 4 \times 3$

Here we first fill the units place, then thousandth place.

The units place can be filled by odd numbers only i.e., 1, 3, 5 hence 3 ways. In thousandth place zero cannot be placed hence we have 5 choices. The other two places can be filled in 5 and 4 ways.

$\therefore$ Total odd numbers = $5 \times 5 \times 4 \times 3 = 300$

d) divisible by 5

≠0			5 or 0

Since the restriction on zero is there for two places. Take the two cases separately.

Case i) Last digit is 5

≠0 5

$5 \times 5 \times 4$

Units place can be filled in one way, then thousandth place can be filled in 5 ways and again the other two places in 5 and 4 ways.

$\therefore$ Total numbers = $5 \times 5 \times 4 = 100$

Case ii) Last digit is 0

			0

$6 \times 5 \times 4$

Since zero is in units place, the thousandth place can be filled by any one of the remaining 6 digits and the other two in 5 & 4 ways.

$\therefore$ Total numbers = $6 \times 5 \times 4 = 120$

Hence total numbers divisible by 5 is $100 + 120 = 220$

Type VI : Formation of numbers with digits (digits can be repeated)

Example 14 :

How many 3 digit numbers can be formed using digits 1, 2, 5, 6, 8

Solution :

Each place can be filled by any one of 5 digits. $\therefore$ Total numbers = $5 \times 5 \times 5 = 125$

Type VII : Word building (Alphabets not repeated)

Case I : When there is no restrictions.

Example 15 :

Find the number of words that can be formed by using all the letters of the word "MONKEY".

Solution :

The word MONKEY has 6 letters. The six different letters can be arranged in ${}^{6}p_{6} = 6! = 720$ ways.

Case II : When all the words Begin / End with particular letters.

Example 16 :

Find the number of words that can be formed by using all the letters of the word "RAINBOW".

a) **How many of them begin with R and end with W.**

b) **How many of them end with BOW.**

c) **How many of them begin with a ROW**

Solution :

The word RAINBOW has 7 letters therefore total words is = 7! = 5040

a) Words beginning with R and ending with W. Here R & W positions are fixed. Therefore only the remaining 5 letters AINBO have to be arranged. This can be done in 5! = 120 ways.

Ⓡ A I N B O Ⓦ

The circled letters are not considered for permutations.

b) Words ending with BOW:

RAIN ⒷⓄⓌ

The circled letters positions are fixed the remaining 4 letters can be arranged in 4! = 24 ways.

c) Words beginning with ROW

Ⓡ A I N BⓄⓌ

The circled letters positions are fixed to the first 3 places. The remaining 4 letters can be arranged in remaining 4 places in 4! = 24 ways.

Case III : When few letters / vowels / consonants occur together.

Example 17 :

In how many ways can the letters of word "LAUGHTER" be arranged so that the vowels are always together.

Solution : The word LAUGHTER has 3 vowels and 5 consonants. Consider the 3 vowels as 1

unit. Hence 1 unit + 5 consonants, Total = 6 can be arranged in 6! ways. For each of these, the 3 vowels can be arranged in 3! ways.

$\therefore$ Total words $= 6! \times 3!$

$= 720 \times 6 = 4320$

Case IV : When vowels / consonants occupy odd / even places.

Example 18 :

In how many ways can the letters of the word HEXAGON be arranged so that the vowels are always in even places.

Solution :

The word HEXAGON has 7 letters of which 3 are vowels.

Since total letters is 7, there are 4 odd places and 3 even places.

The 3 vowels can be arranged in the 3 odd places in 3! ways. For each of these the 4 consonants can be arranged in the 4 odd places in 4! ways. Hence total ways = $3! \times 4!$

$= 6 \times 24 = 144$

Note : Problems based on type III and type IV can be given in word problems also.

Type VIII : Permutations of like things

The number of permutations of 'n' things taken all at a time, where *'x'* of the things are alike and of one kind, '*y*' others are alike of another kind, 'z' others are alike and of another kind and so on is

$$\text{Total No. of ways} = \frac{n!}{x!y!z!}$$

Example 19 :

Find the number of permutations of the letters of the word ENGINEERING. How many of these.

a) **begin with E and end with E**

b) **have all the 3 E's together**

c) **begin with GIN and end with GRIN.**

d) **all the vowels are not together.**

e) **no two vowels are together.**

Solution :

The word ENGINEERING has 11 letters out of which E is repeated thrice, N is repeated thrice, G is repeated twice, I is repeated twice.

Total — 11
E — 3
N — 3
G — 2
I — 2

$\therefore$ Total permutations $= \frac{11!}{3!\ 3!\ 2!\ 2!}$

a) **Beginning with E and ending with E:**

(E)N G I N(E)E R I N G

Circled letter's position is fixed. Hence permutations of remaining letters is $\frac{9!}{3!\ 2!\ 2!}$

Total — 9
N — 3
G — 2
I — 2

b) **All 3 E's are together:**

Consider the 3 E's as one unit

Hence we have to arrange 1 unit (3 E's) & remaining letters

(E E E) & NGINRING

1 + 8

Total — 9
N — 3
G — 2
I — 2

Total permutations is $\frac{9!}{3!\ 2!\ 2!}$

The 3 E's considered as one can be arranged in only 1 way as all the 3 letters are the same.

Hence total permutations is $\frac{9!}{3!\ 2!\ 2!}$

c) Begin with GIN and end with GRIN:

E N (G) (I) (N) E E (R I N G)

The circled letters positions are fixed. Hence permutations of the remaining letters is $\frac{4!}{3!}$

Total — 4

E — 3

d) All the vowels are not together:

All vowels are not together

= Total Arrangements — All vowels are together

$$\text{Total Arrangements} = \frac{11!}{3!\,3!\,2!\,2!}$$

All vowels are together: The word ENGINEERING has 5 vowels which can be considered as one unit. Therefore letters for permutations are

[E I E E I] + N G N R N G

1 + 6

Total — 7

N — 3

G — 2

$$\text{No. of permutations} = \frac{7!}{3!\,2!}$$

For each of these, the vowels taken as one unit can be arranged in $\frac{5!}{3!\,2!}$

Total — 5

E — 3

I — 2

Hence total words with vowels together is $\frac{7!}{3!\,2!} \times \frac{5!}{3!\,2!}$

$$\therefore \text{ Total words with all vowels not together} = \frac{11!}{3!\,3!\,2!\,2!} - \frac{7!}{3!\,2!} \times \frac{5!}{3!\,2!}$$

e) No two vowels are together:

First arrange the six consonants

N G N R N G

Total — 6

N — 3

G — 2

This can be done in $\frac{6!}{3!\ 2!}$ ways.

If there are six letters then there are 7 gaps. In the 7 gaps we have to arrange the 5 vowels. But in these 5 vowels E is repeated thrice and I is repeated twice.

$\therefore$ So total ways is $\frac{^{7}P_{5}}{3!\ 2!}$

$\therefore$ Total words with no two vowels are together is $\frac{6!}{3!\ 2!} \times \frac{^{7}P_{5}}{3!\ 2!}$

EXERCISE 2.1

I. One mark questions:

1. Find the value of

   a) $^{12}p_4$ b) $^{8}p_6$ c) $^{10}p_5$ d) $^{10}p_2$

2. In how many ways can 9 soldiers stand in a queue.
3. How many different signals can be made by taking 3 different coloured flags at a time from 7 different coloured flags.
4. If $^{5}p_r = 60$ find the value of r.
5. In how many ways can 5 letters be posted in 5 letter boxes if each box has one letter.
6. How many different arrangements can be made with the letters of the word "MONDAY'.
7. How many six digit numbers can be formed with the digits 2, 7, 6, 1, 9, 8
8. In how many ways can 4 people occupy 6 vacant chairs.
9. There are 4 routes to go from A to B, and 3 routes to go from B to C. In how many ways can you go from A to C via B.
10. In how many ways can you go from A to B and return to A from a different route, if there are 10 ways of travelling from A to B.

II. Two marks questions.

1. Find the value of n if ${}^{n}p_{5} = 20\ {}^{n}p_{3}$
2. In how many ways can 7 persons be seated in a row if two persons always occupy the end seats.
3. In how many ways can 3 boys and 4 girls be arranged in a row so that all the three boys are together.
4. In how many ways the word " CARROM" be arranged such that the 2 R's are always together.
5. How many words can be formed from the letters of the word "FRIEND" which begin with F and end with D.
6. If ${}^{n}p_{3} : {}^{n}p_{2} = 3 : 1$ find n.
7. How many 4 digit numbers can be formed with the 0, 2, 3, 5, 7 such that no digits are repeated.
8. How many 3 digit numbers ending with 7 can be formed using the digits 1, 2, 5, 7, 8 digits cannot be repeated.
9. In how many ways can the letters of the word 'HOPPER' be arranged.
10. How many 6 digit numbers can be formed from the digits 1, 2, 3, 4, 5, 6 (no digit being repeated) which are divisible by 5.

III. Three marks questions.

1. In how many ways can 7 students and 4 teachers be seated in a row such that no two teachers are together.
2. In how many ways can 7 books on art, 5 books on fiction, 4 books on adventure be arranged so that books of the same category are always together.
3. 5 men and 4 women are to be seated in a row so that women occupythe even places. How many such arrangements are possible .....
4. Find the number of ways in which 4 boys and 5 girls can be arranged in a row so that no two boys are together and no two girls are together.
5. Find the number of permutations of the letters of the word "MATHEMATICS".
6. A family of 4 brothers and 3 sisters is to be arranged for a photograph in one row. In how many ways can they be seated if i) all the sisters sit together ii) No two sisters sit together
7. A code word is to consist of two distinct english alphabets followed by 2 distinct numbers between 1 and 9. How many such codes are possible.
8. When a group photograph is taken all the 10 teachers should be in the first row with principal and vice principal in the middle. The 18 students will be in second row with the two leaders at the extreme. How many arrangements are possible.

9. How many 5 digit even numbers can be formed using the digits 0, 1, 2, 3, 4, 5
10. How many numbers between 300 and 3000 can be formed with the digits 0, 1, 2, 3, 4, 5
11. In how many ways the letters of the word "ACCOUNTANT" be arranged? In how many of them vowels are always together.

IV 3 Marks questions.

1. Find the number of permutations of the letters of the word MISSISSIPPI. In how many of these
   a) the 4 s's are together
   b) the 4 s's are not together
   c) begin with MISS
   d) begin with SIP
2. Find the number of permutations of the letters of the word "COMMITTEE"
   a) How many of them begin with T and end with T.
   b) In how many all the vowels are together.
   c) In how many no two vowels are together.
   d) In how many of them end with MITE
3. Find the number of permutations of the letters of the word ASSASSINATION. In how many of these.
   a) the vowels are in even places.
   b) vowels are in odd places.
   c) the word NATION is always present together.
   d) begins with 'AS' and ends with 'AS'
4. In how many ways can 7 english books, 8 Kannada books and 5 Hindi books be arranged in a shelf. In how many of these
   a) All Kannada books are together
   b) No two English books are together
   c) Books of the same language are together
   d) All Kannada books are together and all Hindi books are together
5. How many 4 digit numbers can be formed using digits 0, 1, 2, 3, 4, 5, 6 (No digit can be repeated) How many of them are
   a) even
   b) odd

c) greater than 4000

d) greater than 3,500

6. How many 5 digit numbers can be formed using the digits 1, 2, 3, 5, 7, 8, 9 (no. digit being repeated)

   How many of these are

   a) divisible by 5

   b) ending with 25

   c) less than 50,000

   d) greater than 75,000

7. How many even numbers greater than 300 can be formed with the digits 1, 2, 3, 4, 5 no digit being repeated.

   [Note: The numbers could be 3 digit, 4 digit or 5 digit]

8. In how many ways can 6 boys and 6 girls be arranged in a row so that

   a) All girls are together

   b) All boys are not together

   c) No two girls are together

   d) Boys and girls are arranged alternatively

   e) Two particular girls are always together.

9. How many numbers can be formed with digits 2, 4, 6, 8 without repitition

   [Hint : numbers could be of 1 digit to 4 digits]

10. How many four digit numbers can be formed using digits 0, 2, 3,5, 7, 8

    a) How many of them are even

    b) How many are divisible by 5

    c) How many are greater than 5300

    d) How many end with 7

    [Hint : Here digits can be repeated]

11. Find the number of permutations of the letters of the word UNIQUE

    a) How many of them end with QUE

    b) How many begin with U and end with E

    c) In how many all vowels are together

12. How many 4 digit even numbers can be formed using the digits 0, 1, 2, 3, 4, 5, 6
    a) when digits can be repeated
    b) when digits are not repeated

Answers 2.1

I. 1. a) 11880 b) 20160 c) 30240 d) 90
2. 9! 3. 210 4. 3 5. 120
6. 720 7. 720 8. 360 9. 12
10. 90

II. 1. 8 2. 240 3. 720 4. 120 5. 24 6. 5
7. 96 8. 12 9. 360 10. 120

III. 1. $7!\ {}^{8}p_{4}$ 2. 3! 7! 5! 4! 3. 2880 4. 2880
5. $\frac{11!}{2!2!2!}$ 6. (i) 720 (ii) 1440 7. 27300 8. 16! 8! 2! 2! 9. 3240
10. 540 11. 2,26,800, 7,560

IV. 1. $\frac{11!}{4!4!2!}$ a) $\frac{8!}{4!2!}$ b) $\frac{11!}{4!4!2!} - \frac{8!}{4!2!}$ c) $\frac{7!}{3!2!2!}$ d) $\frac{8!}{3!3!}$

2. $\frac{9!}{2!2!2!}$ a) $\frac{7!}{2!2!}$ b) $\frac{6!}{2!2!} \times \frac{4!}{2!}$ c) $\frac{5!}{2!2!} \times \frac{{}^{6}p_{4}}{2!}$ d) 5!

3. $\frac{13!}{3!4!2!2!}$ a) $\frac{6!}{3!2!} \times \frac{7!}{4!2!}$ b) $\frac{{}^{7}p_{6}}{3!2!} \times \frac{7!}{4!2!}$ c) $\frac{8!}{4!2!}$ d) $\frac{9!}{2!2!2!}$

4. 20! a) 13! 8! b) $13!\ {}^{14}p_{7}$ c) 3! 7! 8! 5! d) 9! × 8! 5!

5. 720 a) 420 b) 300 c) 360 d) 400

6. 2520 a) 360 b) 60 c) 1080 d) 900

7. 276 8) a) 7! 6! b) 12! – 7! 6! c) $6! \times {}^{7}p_{6}$ d) 6! × 6! × 2
e) 11! 2!

9. 64 10. 1080 a) 540 b) 360 c) 576 d) 180

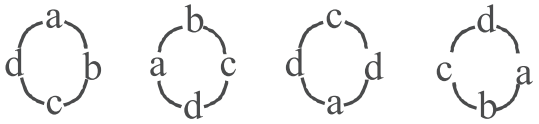
11. 360 a) 6 b) 24 c) 72

12. a) 1176 b) 420

2.7 Circular Permutations

When objects are arranged along a closed curve like a circle or ring the permutations are known as **circular permutations.**

It is important to note that in a circular arrangement there is neither a beginning (first term) nor an end (last term). For example if we have to arrange 4 letters A B C D. Incase of linear permutation ABCD, BCDA, CDAB, DABC are different arrangements. But in case of circular arrangements



are the same. So every 4 arrangements are considered as one. Hence if linear permutations are 4! circular permutations are $\frac{4!}{4} = 3!$ Hence in general, n objects can be arranged in a circle in $\frac{n!}{n} = (n-1)!$ Hence the number of circular permutations of n objects is (n-1)! Another explanation is, as the number of circular permutations depends on the relative position of the objects, we fix the position of one object and then arrange the remaining (n-1) objects in all possible ways. This can be done in (n-1)! ways.

Note : If the clockwise and anti-clockwise orders are not distinguishable, then the number of circular permutations is $\frac{(n-1)!}{2}$.

Example 20 :

In how many ways can 8 persons be seated at a round table.

Solution:

This is circular permutations. Hence number of ways 8 persons can be seated at a round table is (8–1)! = 7!

Example 21 : .

In how many ways can 10 different precious stones be set to form necklace.

Solution :

In case of a necklace there is no distinction between the clockwise and anti clockwise arrangement. Hence no. of ways = $\frac{1}{2} \times 9!$

Example 22 :

In how many ways can six boys be arranged in a circle if 2 particular boys always want to be together.

Solution :

Consider the 2 boys as one unit then on unit + remaining 4 boys = total 5 have to be arranged in a circle. This can be done in 4! ways.

For each of these the two boys can be arranged in 2! ways.

$\therefore$ Total ways $= 4! \times 2!$

$= 24 \times 2 = 48$

Example 23 :

In how many ways can 5 boys and 5 girls be arranged in a circle if no two girls are together .

Solution :

First arrange the 5 boys in a circle. This can be done in 4! ways. When 5 boys are in circle a the number of gaps is 5. In these 5 gaps the 5 girls can be arranged in 5! ways. Note here since the positions are fixed it is linear permutation and not circular permutation when the arrangement is in between gaps.

$\therefore$ Total ways = 4! 5!

EXERCISE 2.2

I. One mark questions:

1. Find the total number of ways in which 8 different coloured beads can be strung together to form a necklace.
2. In how many ways can 9 flowers of different colours be strung together to form a garland.
3. In how many ways can 10 people be seated around a table.

II. Two marks questions:

1. Find the number of ways in which 8 men be arranged round a table so that 2 particular men may not be next to each other.
2. In how many ways can 6 gentlemen and 4 ladies be seated round a table so that no two ladies are together.
3. In how many ways can 10 beads of different colours be strung into a necklace if the red, green and yellow beads are always together.
4. In how many ways can 6 boys and 6 girls be arranged in a circle so that no two boys are together.

5. In how many ways can 7 gentlemen and 5 ladies be arranged in a circle if no two ladies are together.
6. Find the number of ways in which 10 flowers can be strung into a garland if 3 particular flowers are always together.
7. Find the number of ways in which 15 staff members can be seated around a circular table for a meeting if the vice-principal and dean have to be on either side of the principal.

ANSWERS 2.2

I. 1) 2520 2) $\frac{8!}{2}$ 3) 9!

II. 1) 7! — 6! 2! 2) 5! × ${}^{6}p_{4}$ 3) 7! 3! 4) 5! 6! 5) 6! ${}^{7}p_{5}$
5) 7! 3! 7) 12! × 2!

2.8 Combinations:

The different groups or collections or selections that can be formed out of a given set of things, by taking some or all of them at a time (without considering the order of the arrangements) is called their **combination.**

The number of all combinations of n distinct things taken r at a time is

$$\boxed{{}^{n}C_{r} = \frac{n!}{(n-r)!\,r!}} \quad (r \le n)$$

The word combination is generally used for selection of things and the word permutation is used for arrangement of things.

Properties of ${}^{n}C_{r}$ (without proof)

1. ${}^{n}C_{r} = \frac{{}^{n}p_{r}}{r!}$
2. ${}^{n}C_{o} = {}^{n}C_{n} = 1$
3. ${}^{n}C_{n-r} = {}^{n}C_{r}$
4. if ${}^{n}C_{r} = {}^{n}C_{k}$ then either r = k or r + k = n
5. ${}^{n+1}C_{r} = {}^{n}C_{r} + {}^{n}C_{r-1}$
6. ${}^{n}C_{0} + {}^{n}C_{1} + \ldots\ldots {}^{n}C_{n} = 2^{n}$
7. ${}^{n}C_{1} + {}^{n}C_{2} + \ldots\ldots {}^{n}C_{n} = 2^{n} - 1$

Some useful formulae.

1. If there are n points in a plane out of which k points are collinear then
   a) number of straight lines that can be formed using these points is ${}^{n}C_2 - {}^{k}C_2 + 1$
   b) number of triangles that can be formed using these points is ${}^{n}C_3 - {}^{k}C_3$
2. If there are 'm' parallel lines intersecting another set of 'n' parallel lines then the number of parallelograms that can be formed is ${}^{m}C_2 \times {}^{n}C_2$
3. If a polygon has 'n' sides or 'n' vertices then number of diagonals = ${}^{n}C_2 - n = \frac{n(n-3)}{2}$
4. Relation between ${}^{n}P_r$ and ${}^{n}C_r$ is $\boxed{\frac{{}^{n}P_r}{{}^{n}C_r} = r!}$

Example 24 :

If ${}^{n}C_4 = {}^{n}C_5$ find the value of n.

Solution :

If ${}^{n}C_r = {}^{n}C_k$ then r = k or r + k = n

Here since r ≠ k, r + k = n $\therefore$ 4 + 5 = n $\therefore$ n = 9

Example 25 :

Find *x* if ${}^{99}C_{40} + {}^{99}C_x = {}^{100}C_{59}$

Solution : ${}^{99}C_{40} = {}^{99}C_{99-40} = {}^{99}C_{59}$

$\therefore$ ${}^{99}C_{59} + {}^{99}C_x = {}^{100}C_{59}$

now using ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

we get $x = 59 - 1 = 58$

Example 26 :

If ${}^{n}p_r = 3024$, ${}^{n}C_r = 126$ find r

Solution : $\frac{{}^{n}P_r}{{}^{n}C_r} = r!$

$$\frac{3024}{126} = 24$$

$r! = 24$

$1 \times 2 \times 3 \times 4 = 24$

$\therefore$ 4! = 24, hence r = 4

Example 27 :

A person has 9 friends . In how many ways can he invite one or more of them to a dinner.

Solution :

He can invite only one friend or two friends, or 3 friends, maximum upto 9.

$\therefore$ different ways of inviting his friends is ${}^9C_1 + {}^9C_2 = {}^9C_3 + \ldots\ldots {}^9C_9$

$= 2^9 - 1$

$= 512 - 1 = 511$ ways

Example 28 :

There are 12 points in a plane of which 5 are collinear. Find the number of a) straight lines b) triangles that can be formed by joining these points.

Solution : a) By joining any two points we get a straight line.

$\therefore$ Total lines from 12 points $= {}^{12}C_2$

But 5 points are collinear, if both points are selected from these 5 points they give rise to one single line

$\therefore$ Total lines $= {}^{12}C_2 - {}^5C_2 + 1$

$= 66 - 10 + 1 = 57$

b) By joining any 3 non collinear points we get a triangle. But if all 3 points are chosen from 5 collinear points then we get a straight line and not a triangle.

$\therefore$ Total number of triangles $= {}^{12}C_3 - {}^5C_3$

$= 220 - 10 = 210$

Example 29 :

Find the total number of parallelograms that can be formed from a set of five parallel lines intersecting another set of 6 parallel lines.

Solution : number of parallelograms

$= {}^mC_2 \times {}^nC_2$

$= {}^5C_2 \times {}^6C_2$

$= 10 \times 15 = 150$

Example 30 : Find the number of diagonals in a decagon.

Solution : A decagon has 10 vertices. By joining any 2 vertices we get a diagonal or side.

Hence ${}^{10}C_2$ gives total number of diagonals and sides

Number of diagonals $= {}^{10}C_2 - 10$

$= 45 - 10 = 35$

Example 31 :

In how many ways can 6 people be selected out of 12 people so that

a) two particular members must be included

b) two particular members must be excluded.

Solution :

a) Two people are already selected. From the remaining 10 we have to choose 4 more. This can be done in $^{10}C_4$ ways = 210 ways.

b) Two people are rejected from the remaining 10 we have to choose 6. This can be done in $^{10}C_6$ ways = 210 ways.

Example 32:

A team of 11 is to be choosen from 18 cricketers of whom 6 are bowlers and 3 are wicket keepers. In how may ways can a team be choosen so that

a) There are exactly 4 bowlers and one wicket keeper.

b) There are atleast 4 bowlers and atleast 2 wicket keepers.

Solution : a)

Bowlers (6)	Wicket keeper (3)	Others (9)
4	1	6

$^6C_4 \times {}^3C_1 \times {}^9C_6 \times 15 \times 3 \times 84 = 3780$

b)

Bowlers (6)	Wicket keeper (3)	Others 9
4	2	5
4	3	4
5	2	4
5	3	3
6	2	3
6	3	2

$^6C_4 \times {}^3C_2 \times {}^9C_5 = 15 \times 3 \times 126 = 5670$

$^6C_4 \times {}^3C_3 \times {}^9C_4 = 15 \times 1 \times 126 = 1890$

$^6C_5 \times {}^3C_2 \times {}^9C_4 = 6 \times 3 \times 126 = 2268$

$^6C_5 \times {}^3C_3 \times {}^9C_3 = 6 \times 1 \times 84 = 504$

$^6C_6 \times {}^3C_2 \times {}^9C_3 = 1 \times 3 \times 84 = 252$

$^6C_6 \times {}^3C_3 \times {}^9C_2 = 1 \times 1 \times 36 = 36$

10620 ways

EXERCISE 2.3

I. One mark question:

1. In how many ways a committee of 5 can be chosen from 10 students.
2. Find n if ${}^{n}C_{10} = {}^{n}C_{15}$
3. Find x if ${}^{9}C_{x} + {}^{9}C_{7} = {}^{10}C_{7}$
4. How many straight lines can be formed from 10 points if no three of them are collinear.
5. How many triangles can be formed from 8 non collinear points.

II. Two marks questions:

1. If ${}^{n}C_{8} = {}^{n}C_{12}$ find ${}^{n}C_{5}$
2. If ${}^{n}P_{r} = 30240$ ${}^{n}C_{r} = 252$ find r
3. Find the number of ways in which a committee of 4 Students and 2 Lecturers can be chosen out of 10 students and 8 lecturers.
4. Find r if ${}^{25}C_{r+3} = {}^{25}C_{2r-3}$
5. Find the number of straight lines and triangles that can be formed out of 20 points of which 8 are collinear.
6. Find the number of parallelograms that can be formed from a set of 6 parallel lines intersecting another set of 4 parallel lines.
7. Find the number of diagonals of a polygon of 20 sides.
8. In a party each person shakes hands with everyone else. If there are 25 members in the party. Calculate the number of handshakes.
9. In how many ways can 6 red and 4 white marbles be chosen from a bag containing 10 Red and 6 White marbles.
10. In how many ways can 6 people be chosen out of 10 people if one particular person is always included.

III. Three marks questions.

1. If a convex polygon has 170 diagonals. Find the number of sides of the polygon.
2. A team of conveners consisting of 4 lecturers, 4 boys and 2 girls has to be selected from 15 lecturers, 10 boys and 12 girs. In how many ways the selection can be made if the 2 lecturers who were conveners last year have to be included and 3 boys who failed in the examination cannot be chosen.
3. A committee of 4 has to be chosen from 10 boys and 8 girls. In how many ways can this be done if girls are in a majority.

4. If ${}^{n}P_{5} = 24 \times {}^{n}C_{4}$ find n.
5. If ${}^{n}P_{r} = 240$, ${}^{n}C_{r} = 120$ find n and r
6. A team of 8 players has to be selected from 14 players. In how many ways the selections can be made if
   a) Two particular players are always included
   b) Two particular players are always excluded

IV. Three marks questions.

1. From a class of 9 boys and 7 girls, 12 students are to be chosen for a competition which includes atleast 6 boys and atleast 4 girls. In how many ways can this be done if a particular boy is always chosen.
2. A man has 10 relatives, 4 of them are ladies 3 gentlemen and 3 children. In how many ways can he invite 7 relatives to a dinner party so that
   1) there are exactly 2 ladies, 3 gentlemen and 2 children.
   2) there are exactly 2 gentlemen and atleast 3 ladies.
3. From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition, including atleast 4 boys and atleast 4 girls. The 2 girls who won the prize last year should be included. In how many ways can the selections be made.
4. A box contains 5 red, 4 black and 3 white balls. How many selections of 8 balls can be made if the selection contains
   1) exactly 4 red, 2 black and 2 white balls.
   2) atleast 3 red, atleast 3 black and atleast 1 white ball.
5. An examination paper consists of 12 questions divided into parts A and B contains 7 questions in part A and part B contains 5 questions. A candidate is required to answer 8 questions selecting atleast 3 from each part. In how many ways can the candidate select the questions.
6. A man has 10 relatives, 4 of them all ladies, 3 gentlemen and 3 children. In how many ways can he invite 7 relatives to dinner party so that (i) there are exactly 2 ladies, 3 gentlemen and 2 children (ii) there are exactly 2 gentlemen and atleast 3 ladies (iii) there are exactly 3 children, atleast 1 lady and atleast 2 gentlemen.
7. A team of eleven is to be chosen out of 16 cricket players of whom 4 are bowlers and 2 wicket keepers. In how many ways can the team be chosen so that (i) there are exactly 3 bowlers and one wicket keeper (ii) there are atleast 3 bowlers and atleast one wicket keeper.
8. Out of 4 officers and 10 clerks in an office a committee consisting of 2 officers and 3 clerks is to be formed. In how many ways can this be done if (i) any officer and any clerk can be

included (ii) one particular clerk must be on committee (iii) one particular officer cannot be on the committee.

9. A committee of 5 are to be formed from 8 Americans and 5 Anglo Indians. In how many ways can this be done when the committee contains (i) exactly two Anglo - Indians (ii) atleast two anglo Indians.

10. A candidate is required to answer 6 out of 12 questions which are divided into two groups containing 6 questions in each group. Find the number of choices he has if he cannot attempt more than five questions from any group.

EXERCISE 2.3

I. 1. 252 2. 25 3. 6 4. 45 5. 56

II. 1. 15504 2. 5 3. 5880 4. 6 5. 163, 1084
6. 90 7. 170 8. 300 9. 3150 10. 126

III. 1. 20 2. $^{13}C_2 \times {}^{7}C_4 \times {}^{12}C_2$ 3. 630 4. 5
5. n = 16, r = 2 6. a) $^{12}C_6$ b) $^{12}C_8$

IV. 1. 1260 2. 18, 45 3. 104874 4. 90, 210 5. 420
6. 18,45,22 7. 960,2472 8. 720,432,360 9. 560, 881 10. 922

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Chapter

3

PROBABILITY

3.1 Introduction:

What does it mean to say that a probability of a fair coin is one half or that the chances one passes in class is 80 percent or that the probability of RCB winning the IPL this season is 1?

First think of some event where the outcome is uncertain examples of such outcomes would be the roll of a die, the amount of rain that we get tomorrow, the state of the economy in one month, who will be the prime minister in the next elections. In each case, we don't know for sure what will happen. For example we don't know exactly how much rain we will get tomorrow.

Life is full of random events. Many events cannot be predicted with total certainty. Is everything on this planet determined by randomness? This question is open to philosophical debate. What is certain is that every day thousands and thousands of engineers, scientists, business persons, manufacturers and others are using tools from probability.

A Gambler's dispute in 1654 led to the creation of a mathematical theory of probability by two famous French Mathematicians Blaise Pascal and Pierre de Fermat. Understanding the nature of simple probability is key to understanding all gambling related matters. In gambling terms we define probability of a winning event as follows. **"The number of ways of winning divided by total number of possible outcomes".**

The study of probability helps us to figure out the likelihood of something happening. For instance when you roll a pair of dice, you might ask how likely you are to roll a seven. In math, we call the **"something happening"** an **"event".**

The probability of the occurrence of an event can be expressed as a fraction or a decimal from 0 to event tat are unlikely to happen will have a probability near 0 and events that are likely to happen have probabilities near 1.

In any probability problem, it is very important to identify all the different outcomes that could occur for instance, in the question about the dice, you must figure out all the different ways the dice could land and different ways you could roll a seven.

Reasoning based on probability and statistics give modern societies the ability to cope with uncertainty. It has astonishing powers to improve decision making, accuracy and test new ideas. Today probability theory is a well establishing branch of mathematics that funds applications in every area of scholarly activity from music to physics and in daily experience from weather prediction to predicting the risks of new medical treatments.

3.2. Definitions and some important terms

(A) Experiment : An operation which can produce some well defined outcomes is known as an experiment. There are two types of experiments

(1) Deterministic experiment, when repeated under identical conditions the results or outcomes is always the same.

For example science experiments

1) Blue litmus paper when dipped in acid always turns red and hence only one outcome.

(ii) **Random experiment :** Any experiment that results in two or more outcomes is called a random experiment.

Example : (1) tossing a fair coin has 2 outcomes – head or tail

(2) drawing a card randomly from a pack of 52 cards – any one of 52 cards.

(3) rolling of a dice has 6 outcomes – any one of the numbers from 1 to 6

(B) Sample space : The set of all possible outcomes of a random experiment is called as sample space and is generally denoted by S.

(a) In tossing of a fair coin, there are two possible outcomes head (H) and tail (T)

$\therefore$ S = {H, T}

(b) When two fair coins are tossed together, the possible outcomes of the experiment are HH, HT, TH & TT

S = {HH, HT, TH, TT}

(c) When 3 fair coins are tossed

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

(d) When a unbaised die is thrown

S = {1, 2, 3, 4, 5, 6}

(e) When two unbiased dice are rolled simultaneously there are total $6 \times 6 = 36$ possible outcomes.

$$S = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix} \right\}$$

(f) When a die and a coin are tossed simultaneously then there are 12 possible outcomes.

S = (1,H) (2,H), (3,H), (4,H), (5,H), (6,H)
(1,T) (2,T), (3,T), (4,T), (5,T), (6,T)

(C) Event : Any subset of a sample space is called an event.

For example:

1) In a single throw of a dice the sample space = {1, 2, 3, 4, 5, 6}

   E = {4, 5, 6} is a subset of S

   Hence the Event E can be described as getting a number greater than 3 in a single thow of a die.

2) When two coins are tossed

   S = {TT, HH, TH, HT}

   getting atleast one head E = {HH, TH, HT} is subset of S and hence E is an event.

(D) Null event (Impossible event) : An event which does not contain any outcome is called an null event. In is denoted by ϕ. Since $\phi \subseteq S$, it is an event also called as impossible event.

For example getting a number greater than 6 in a single throw of a dice is an impossible event.

(E) Sure event (Certain event) : An event which contains all the outcomes equal to sample space is called sure event of certain event.

Simple event (elementary event) : An event which has only one outcome is an simple event or elementary event.

Example : In a single throw of a dice if E is the event of getting an even prime number then E = {2} is a simple event.

(F) Compound event : An event which has more than one outcome is called as compound event.

Example: Consider a random experiment in which two dice are rolled simultaneously the sample space S = {(1,1) (1,2).... (6,5) (6,6)}

then E_1 = {(1, 5) (2, 4) (3,3) (4, 2) (5, 1)}

E_2 = {(1, 1) (2,2) (3,3) (4,4) (5,5) (6,6)}

are compound events

Here E_1 is the event of getting the sum of 6. Here E_2 is the event of getting identical results.

Equally likely events: Two or more events are equally likely if they have equal chance of occurence.

Example : If an unbiased die is rolled, then each of the six outcomes are equally likely to happen. If however the dice is so formed that a particular face occurs most often, then the dice is biased so in this case, the outcomes are not equally likely to happen.

(G) Exhaustive events : The total number of possible outcomes of an experiment is known as exhaustive events.

Eg. In drawing a card from a pack of 52 cards, all the 52 cards are exhaustive events.

(H) Mutually exclusive events (Disjoint events) : If two events cannot occur simultaneously in a random experiment then they are called mutually exclusive events.

Eg. Consider the random experiment of throwing a die. Let E_1, E_2, E_3 be three events such that

$E_1 = \{1, 3, 5\}$ the event of getting an odd number

$E_2 = \{2, 4, 6\}$ the event of getting an even number

$E_3 = \{2, 3, 5\}$ the event of getting a prime number

clearly $E_1 \cap E_2 = \phi$, $E_1 \cap E_3 \neq \phi$, $E_2 \cap E_3 \neq \phi$

Hence E_1 and E_2 are mutually exclusive events but E_1 and E_3, E_2 and E_3 are not mutually exclusive.

If A and B are mutually exclusive events then 1) $A \cap B = \phi$

2) $P(A \cap B) = \phi$

3) $P(A \cup B) = P(A) + P(B)$

If A and B are not mutually exclusive events then 1) $A \cap B \neq \phi$

2) $P(A \cap B) \neq \phi$

3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(I) Independent events : Events are said to be independent, if the occurence of one does not depend upon the occurence of the other.

Suppose an urn contains m red balls and n green balls. Two balls are drawn from the urn one after the other.

If the ball drawn in the first draw is not replaced back in the bag, then two events of drawing the ball are dependent because first draw of the ball determine the probability of drawing the second ball.

If the ball drawn in the first draw is replaced back in the bag, then two events are independent because first draw of a ball has no effect on the second draw.

Two events A and B associated with the same sample space of a random experiment are independent if and only if $P(A \cap B) = P(A)\ P(B)$.

Similarly if $A_1\ A_2\ A_3 \ldots\ldots A_n$ are independent events associated with a random experiment then

$P(A_1 \cap A_2 \cap \ldots\ldots\ldots \cap A_n) = P(A_1)\ P(A_2) \ldots\ldots\ldots P(A_n)$

(J) Complementary events : In a random experiment. Let S be the sample space and let E be an event. Then $E \subseteq S$. So E^C is also an event, called the complementary of E. It is also denoted by $\bar{E}$ of E^1 and also called as 'not E'.

Thus it is clear that $\bar{E}$ occurs only when E does not occur. Also in a trial one and only one of E and $\bar{E}$ must occur.

Examples in tossing a fair die

$S = \{1, 2, 3, 4, 5, 6\}$ If $E = \{2, 4, 6\}$ then $\overline{E} = \{1, 3, 5\}$

Note: It is clear that $A \cup \overline{A} = S$

(K) Union and intersection of two events: If A and B are two events then the occurence of either A or B is known as union of two events and occurence of both A and B is known as intersection of events.

When a die is tossed let the events A and B be $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$

then $A \cup B = \{1, 2, 3, 5\}$

$A \cap B = \{3\}$

Consider a fortune wheel which is divided into 7 equal sectors each sector coloured with one rainbow colour.

Terminologies	Examples
Experiment	spinning the fortune wheel
Outcome	getting any one of the seven coloures of VIBGYOR
Event	a) pointer landing on Red b) pointer landing on Red or Blue
Null event	getting brown colour
Sure event	getting a rainbow colour
Sample space	S = {violet, Indigo, blue, green, yellow, orange, red}
Simple event	pointer landing on yellow
Compound event	getting a six lettered colour i.e. {V, I, Y}
Mutually exclusive events	E_1 = {4 lettered colour} = {Blue} E_2 = {5 lettered colour} = {Green}
Independent events	occurence of any of the colours out of seven colours.

3.3 There are three systematic approaches to the study of probability

They are 1) classical or mathematical approach
2) frequency approach or empirical approach
3) axiomatic approach

In this chapter we confine ourselves to mathematical approach only.

3.4 Classical or Mathematical approach

Let a random experiment have n equally likely, mutually exclusive and exhaustive outcomes. Let m of these outcomes be favourable to an event A. Then, probability of A is

$$\boxed{P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}} = \frac{m}{n}}$$

Limitations:

1) The outcomes should be equally likely, mutually exclusive and exhaustive.
2) The number of outcomes n should be finite.

Note : 1) The least value of m is 0 and highest value of m is n. Hence P(A) is a value between 0 and 1

2) if P(A) = 1 the event is certain
3) if P(A) = 0, the event is impossible
4) P(A′) = 1 – P(A)

Example 1:

Find the probability of getting a head in a throw of a coin.

Solution :

When a coin is tossed we either get (H) or tail (T) upwards.

∴ Total number of cases = 2 = n, number of favourable cases to get H = 1 = m

$$\therefore P(H) = \frac{\text{No. of favourable cases}}{\text{Total no. of cases}} = \frac{m}{n} = \frac{1}{2}$$

Example 2 :

An unbiased die is rolled. Find the probability of

a) getting a multiple of 3 **b) getting a prime number**

Solution : When a die is rolled we can get any one of the 6 numbers 1, 2, 3, 4, 5 or 6

∴ Total number of cases = n = 6

a) Let event A be A : getting a multiple of 3. Then A = {3, 6}

∴ m = 2

$$P(A) = \frac{m}{n} = \frac{2}{6} = \frac{1}{3}$$

b) Let event B be B : getting a prime number

B = {2, 3, 5} ∴ m = 3

$$\therefore P(B) = \frac{m}{n} = \frac{3}{6} = \frac{1}{2}$$

Example 3 :

A card is drawn from a well-shuffled pack of 52 cards. Find the probability that

a) card drawn is red

b) card drawn is queen

c) card drawn is black and king

d) card drawn is red and number card

e) card drawn is either king or queen.

Note :

Before solving this problem let us recall the game of cards. One deck of cards contains totally 52 cards. Among them we have 13 spades and 13 clubs which are all black in colour, 13 diamonds and 13 hearts all of which are red. In the 13 spades, 9 are numbered cards numbered from 2 to 10, one ace card and 3 face cards namely Jack, Queen and King. Similarly for the clubs, diamonds and hearts also.

Solution :

Totally there are 52 cards and any one can be drawn

$\therefore$ Total number of cases = n = 52

a) There are 13 diamonds which are red and 13 hearts which are red

$\therefore$ number of red cards = m = 26

$$\therefore \text{P(getting red)} = \frac{26}{52} = \frac{1}{2}$$

b) There are 4 queens

$\therefore$ m = 4

$$\therefore \text{P (getting queen)} = \frac{m}{n} = \frac{4}{52} = \frac{1}{13}$$

c) The king of spade and clubs are black

$\therefore$ no. of cards which is king and black = m = 2

$$\therefore \text{P (king and black)} = \frac{m}{n} = \frac{2}{52} = \frac{1}{26}$$

d) The 9 number cards of hearts and 9 number cards of diamonds are red.

$\therefore$ no. of cards which are red and number cards = m = 18

$$\therefore \text{P(red and number)} = \frac{m}{n} = \frac{18}{52} = \frac{9}{26}$$

e) There are 4 queens and 4 kings

$\therefore$ no. of favourable case = m = 8

$\therefore$ P(queen or king) = $\frac{m}{n} = \frac{8}{52} = \frac{2}{13}$

Example 4 :

A bag contains 6 white beads and 4 red beads. A bead is drawn at random. What is the probability that the bead drawn is white.

Solution :

Total no. of beads in the bag = 6 + 4 = 10 $\therefore$ n = 10

Any one of the 6 white bead can be selected $\therefore$ m = 6

$\therefore$ P (getting white bead) = $\frac{m}{n} = \frac{6}{10} = \frac{3}{5}$

Example 5 :

A box contains 8 red marbles, 6 green marbles and 10 pink marbles. One marble is drawn at random from box. What is the probability that the marble drawn is either red or green.

Solution :

Total number of marbles = 8 + 6 + 10 = 24

$\therefore$ n = 24

There are 6 green and 8 red marbles

$\therefore$ number of favourable cases = 6 + 8 = 14

P(red or green) = $\frac{14}{24} = \frac{7}{12}$

Example 6 :

Two fair coins are tossed simultaneously. Find the probability of

a) getting two heads b) atleast one head c) exactly one head

Solution :

Sample space S = {TT, TH, HT, HH}

$\therefore$ no. of total cases = n = 4

a) getting two heads is only one case i.e. HH

$\therefore$ no. of favourable case = m = 1

$\therefore$ P(getting 2 heads) = $\frac{m}{n} = \frac{1}{4}$

b) atleast one head means one or more heads. In this case it is one or 2 heads, in 3 cases i.e. TH, HT, HH

$\therefore$ P(atleast one head) = $\frac{3}{4}$

OR

P (atleast one) = 1 — P(none)

P (atleast one head) = 1 — p (none of them are heads)

$= 1 - \frac{1}{4} = \frac{3}{4}$

c) exactly one head is two cases TH, HT

$\therefore$ P(exactly one head) = $\frac{2}{4} = \frac{1}{2}$

Example 7 :

Two dice are rolled simultaneously. Find the probability of

a) getting a total of 9 **b) getting a doublet**

c) getting a total of 9 or 11 **d) getting atleast one 5**

Solution :

Sample space

$$S = \left\{ \begin{array}{llll} (1,1) & (1,2) & \ldots\ldots\ldots\ldots & (1,6) \\ (2,1) & (2,2) & \ldots\ldots\ldots\ldots & (2,6) \\ (3,1) & (3,2) & \ldots\ldots\ldots\ldots & (3,6) \\ (4,1) & (4,2) & \ldots\ldots\ldots\ldots & (4,6) \\ (5,1) & (5,2) & \ldots\ldots\ldots\ldots & (5,6) \\ (6,1) & (6,2) & \ldots\ldots\ldots\ldots & (6,6) \end{array} \right\}$$

$\therefore$ Total number of cases = $6 \times 6 = 36$

a) we get a total of 9 in following case (3, 6) (6, 3) (5, 4) (4, 5)

$\therefore$ P(getting total of 9) = $\frac{4}{36} = \frac{1}{9}$

b) we get a doublet in following cases

(1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6)

$\therefore$ P(getting doublet) = $\frac{6}{36} = \frac{1}{6}$

c) we get a total of 9 or 11 in following cases

(3, 6) (6, 3) (5, 4) (4, 5) (6, 5) (5, 6)

$\therefore$ P(getting 9 or 11) $= \frac{6}{36} = \frac{1}{6}$

d) we get atleast one 5 in following cases

(1,5) (5, 1) (2, 5) (5, 2) (3, 5) (5, 3) (4,5) (5,4) (5,5) (6,5) (5,6)

$\therefore$ P(atleast one 5) $= \frac{11}{36}$

Example 8 :

Two cards are drawn from a well shuffled pack of playing cards find the probability that

a) both are same colour

b) both belong to same suit

c) both are aces

d) one Q and one K

Solution :

Different ways of selecting 2 cards from 52 cards in ${}^{52}C_2$

$\therefore$ Total no. of cases = n = ${}^{52}C_2$

a) Both are same colour implies both are red or both are black

$\therefore$ No. of favourable cases = $m = {}^{26}C_2 + {}^{26}C_2$

$\therefore$ P(same colour) $= \frac{m}{n} = \frac{{}^{26}C_2 + {}^{26}C_2}{{}^{52}C_2}$

$$= \frac{\frac{26\times 25}{2} + \frac{26\times 25}{2}}{\frac{52\times 51}{2}} = \frac{25}{21}$$

b) both belong to same suit implies both are spades or both are clubs or both are diamonds or both are hearts

$\therefore$ Total number of favourable cases

$= m = {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2$

$\therefore$ P(both same suit) $= \frac{m}{n} = \frac{{}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2}{{}^{52}C_2}$

$$= \frac{\frac{13\times12}{2}+\frac{13\times12}{2}+\frac{13\times12}{2}+\frac{13\times12}{2}}{\frac{52\times51}{2}} = \frac{4}{17}$$

c) Both aces can be selected in ${}^4C_2$ ways

$$\therefore \text{ P(both aces)} = \frac{{}^4C_2}{{}^{52}C_2} = \frac{\frac{4\times3}{2}}{\frac{52\times51}{2}} = \frac{1}{221}$$

d) One Q and one K can be chosen is ${}^4C_1 \times {}^4C_1$ ways

$$\therefore \quad \text{P(one Q and one K)} = \frac{{}^4C_1 \times {}^4C_1}{{}^{52}C_2}$$

$$= \frac{4\times4}{\frac{52\times51}{2}} = \frac{8}{663}$$

Example 9 :

Three fair coins are tossed simultaneously. Find the probability of

a) getting one head

b) getting atleast 2 heads

c) getting atmost 1 head

d) getting more heads than number of tails.

Solution :

When 3 coins are tossed total possible outcomes $2 \times 2 \times 2 = 8$

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

a) we get exactly one head in 3 cases i.e. HTT, THT, TTH

$$\therefore \text{ P(getting one head)} = \frac{3}{8}$$

b) Atleast 2 heads means 2 heads or 3 heads which is 4 cases namely HHH, HHT, HTH, THH

$$\therefore \text{ P(atleast 2 heads)} = \frac{4}{8} = \frac{1}{2}$$

c) Atmost 1 head implies no heads or 1 head which is 4 cases TTT, HTT, THT, TTH.

$$\text{P(atmost 1 head)} = \frac{4}{8} = \frac{1}{2}$$

d) more heads than number of tails is possible is 4 cases. HHH, HHT, HTH, THH

P(more heads than tails) = $\frac{4}{8} = \frac{1}{2}$

Example 10 :

A bag has 15 tickets numbered from 1 to 15. Two tickets are drawn at random from the bag. Find the probability that both the numbers drawn are a) odd b) prime

Solution :

Total ways of choosing 2 tickets from 15 tickets is ${}^{15}C_2$

$\therefore$ Total cases = n = ${}^{15}C_2$

a) out of 15, 8 are odd numbers

$\therefore$ favourable case = m = ${}^{8}C_2$

$$\therefore \text{P(both odd)} = \frac{m}{n} = \frac{{}^{8}C_2}{{}^{15}C_2} = \frac{\frac{8\times 7}{2}}{\frac{15\times 14}{2}} = \frac{4}{15}$$

b) There are 6 prime numbers. So different ways of choosing 2 numbers from 6 prime numbers is ${}^{6}C_2$

$\therefore$ favourable case = m = ${}^{6}C_2$

$$\therefore \text{P (both prime)} = \frac{m}{n} = \frac{{}^{6}C_2}{{}^{15}C_2} = \frac{\frac{6\times 5}{2}}{\frac{15\times 14}{2}} = \frac{1}{7}$$

Example 11 :

A bag contains 5 red, 7 yellow and 8 green marbles. Three marbles are drawn randomly from the bag. What is the probability that

a) all the three are red

b) all the three are same colour

c) they are of 3 different colours

Solution : There are totally 5 + 7 + 8 = 20 marbles.

Total ways of drawing 3 marbles from 20 marbles is = n = ${}^{20}C_3$

a) Total ways of choosing 3 marbles from 5 red marbles is ${}^{5}C_3$

$\therefore$ favourable case = ${}^{5}C_3$

$$\therefore \text{P(all 3 are red)} = \frac{{}^{5}C_3}{{}^{20}C_3} = \frac{1}{114}$$

b) All the three are same colour implies all the 3 are red or all 3 are yellow or all 3 are green.

Number of selections is ${}^5C_3 + {}^7C_3 + {}^8C_3$

$$\therefore \text{P(All 3 same colour)} = \frac{{}^5C_3 + {}^7C_3 + {}^8C_3}{{}^{20}C_3}$$

$$= \frac{10+35+56}{1140} = \frac{101}{1140}$$

c) 3 different colours implies one red and one yellow and one green

Total selections is ${}^5C_1 \times {}^7C_1 \times {}^8C_1$

$$\therefore \text{P(3 different colours)} = \frac{{}^5C_1 \times {}^7C_1 \times {}^8C_1}{{}^{20}C_3}$$

$$= \frac{5\times 7\times 8}{1140} = \frac{14}{57}$$

Note :

AND — Multiply
OR — Add

Theorems of Probability

There are two important theorems of probability

1) The Addition Theorem (Theorem of total probability)
2) Multiplication Theorem (Theorem of compound probability)

3.5 Addition Theorem

Case 1 : When the events are mutually exclusive:

If two events A and B are mutually exclusive, then probability of occurence of either A or B is P(A or B) = P(A) + P(B)

This can also be written as

$P(A \cup B) = P(A) + P(B)$
$P(A + B) = P(A) + P(B)$

or

Note : 1) The above theorem can be extended to any number of mutually exclusive events i.e. $P(A \cup B \cup C \cup D.....) = P(A) + P(B) + P(C) + ...$

2) Occurrence of either A or B also means occurrence of atleast one

Case 2: When the events are not mutually exclusive :

If two events A and B are not mutually exclusive then probability of occurrence of either A or B is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$$

Note :

1) $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

2) In case of 3 events A, B, C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

3.6 Conditional probability

Let A and B be two events. Then conditional probability of B given A, is the probability of happening of B when it it known that A has already happened. On the other hand, the probability of happening of B when nothing is known about happening of A is called unconditional probability of B.

The conditional probability of B given A is denoted by P(B/A) and $P(B/A) = \frac{P(A \cap B)}{P(A)}$

The conditional probability of A given B is denoted by P(A/B) and $P(A/B) = \frac{P(A \cap B)}{P(B)}$

3.7 Multiplication Theorem:

Case 1 : When the events are independent :

If two events A and B are independent then the probability that both A and B will occur is P(A and B) = P(A). P(B)

This can also be written as $P(A \cap B) = P(A) \cdot P(B)$

Note :

(1) The above theorem can be extended to any number of independent events. *i.e.* $P(A \cap B \cap C \cap D.....) = P(A) \cdot P(B) \cdot P(C).....$

(2) The above theorem can be treated as definition for independent events *i.e.* If two events A and B are independent then $P(A \cap B) = P(A).P(B)$

Case 2: When the events are dependent:

If two events A and B are dependent then the probability that both the events A and B will occur is given by P(A and B) = P(A) . P(B/A)

The theorem can also be written as $P(A \cap B) = P(A).P(B/A)$

Example 11 :

A bag contains 20 tickets numbered from 1 to 20. A ticket is drawn at random. What is the probability that the drawn ticket is

a) a multiple of 3 or 7

b) a multiple of 2 or 5

Solution :

Since the bag has 20 tickets. Total number of cases = n =20

a) Let A : getting a multiple of 3

B : getting a multiple of 7

event A has 6 favourable out comes A = {3, 6, 9, 12, 15, 18}

$\therefore P(A) = \frac{6}{20}$

event B has 2 favourable outcomes B = {7, 14}

$\therefore P(B) = \frac{2}{20}$

$A \cap B = 0$ as events A & B are muturally exclusive.

$\therefore$ P (multiple of 3 or 7)

$= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{6}{20} + \frac{2}{20} = \frac{8}{20} = \frac{2}{5}$

b) Let A : getting a multiple of 2

B : getting a multiple of 5

event A has 10 favourable outcomes

A = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20}

$\therefore P(A) = \frac{10}{20}$

event B has 4 favourable outcomes

B = {5, 10, 15, 20}

$\therefore P(B) = \frac{4}{20}$

$A \cap B = \{10, 20\}$

$\therefore P(A \cap B) = \frac{2}{20}$

$\therefore$ P(multiple of 2 or 5)

$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{10}{20} + \frac{4}{20} - \frac{2}{20} = \frac{12}{20} = \frac{3}{5}$

Example 12 :

Two dice are thrown at once, what is the probability of getting face upwards with
a) Sum equal to 4 or 5

b) Sum is multiple of 4 or 6

Solution :

When two dice are rolled total possible outcomes $= n = 6 \times 6 = 36$

a) Let A: sum equal to 4

b) B : sum equal to 5

A = {(1,3) (3,1) (2,2)}

$P(A) = \frac{3}{36}$

B = {(1,4) (4,1) (2,3) (3,2)}

$P(B) = \frac{4}{36}$

$A \cap B = 0 \qquad \therefore P(A \cap B) = 0$

P(sum of 4 or 5) $= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{3}{36} + \frac{4}{36} - 0 = \frac{7}{36}$

b) Let A : Sum if multiple of 4

B : Sum is multiple of 6

A = {(1,3) (3,1) (2,2) (2,6) (6,2) (3,5) (5,3) (4,4) (6,6)}

$P(A) = \frac{9}{36}$

B = {(1,5) (5,1) (2,4) (4,2) (3,3) (6,6)}

$P(B) = \frac{6}{36}$

$A \cap B = \{(6,6)\}$ $\quad P(A \cap B) = \frac{1}{36}$

$\therefore$ P(sum is multiple of 4 or 6)

$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{9}{36} + \frac{6}{36} - \frac{1}{36} = \frac{14}{36} = \frac{7}{18}$

Example 13:

What is the probability that a card drawn from a pack of playing cards is a) Diamond or a heart b) King or a club c) Spade or Jack d) Red colour or queen

Solution :

There are 52 cards

$\therefore$ Total number of outcomes = n = 52

a) Let A : card is diamond

B : card is heart

$P(A) = \frac{13}{52}$, $\quad P(B) = \frac{13}{52}$

$A \cap B = 0 \quad \therefore P(A \cap B) = 0$

$\therefore$ P (diamond or heart)

$= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{13}{52} + \frac{13}{52} - 0 = \frac{26}{52} = 1/2$

b) Let A : card is a king

B : card is a club

$P(A) = \frac{4}{52}$ $\quad P(B) = \frac{13}{52}$

$A \cap B = 1$ $\quad P(A \cap B) = \frac{1}{52}$

P(king or club) $= P(A \cup B)$

$= P(A) + P(B) - (A \cap B)$

$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

c) Let A: card is spade

B : card is Jack

$P(A) = \frac{13}{52}$ $P(B) = \frac{4}{52}$ $A \cap B = 1$ $P(A \cap B) = \frac{1}{52}$

P(spade or Jack) = $P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$

$= \frac{16}{52} = \frac{4}{13}$

d) Let A : card is red

B : card is gueen

$P(A) = \frac{26}{52}$ $P(B) = \frac{4}{52}$ $P(A \cap B) = \frac{2}{52}$

$\therefore$ P (red or gueen) = $P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52}$

$= \frac{28}{52} = \frac{7}{13}$

Example 16 :

The probability that a doctor gets job in Army is $\frac{1}{2}$ and the probability that he will not get a job in Navy is $\frac{2}{5}$. If the probability of getting atleast one job is $\frac{3}{4}$. What is the probability that he will get both jobs.

Solution :

A : doctor gets job in Army

B : doctor get job in Navy

given $P(A) = \frac{1}{2}$ $P(B') = \frac{2}{5}$ $\therefore P(B) = 1 - P(B')$

$= 1 - \frac{2}{5} = \frac{3}{5}$

$$P(\text{atleast one job}) = P(A \cup B) = \frac{3}{4}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\text{or } P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{1}{2} + \frac{3}{5} - \frac{3}{4}$$

$$= \frac{10+12-15}{20} = \frac{7}{20}$$

Example 17 :

According to a survey conducted on 500 women it was observed that 30% women like lakme products, 40% women liked loreal products and 12% women liked both the products. Find the probability that a woman randomly selected

a) likes atleast one of the products

b) does not like both the products

Solution :

Let A : number of women who like Lakme

B : number of women who like Loreal

$$\text{number of women who like Lakme is } \frac{30}{100} \times 500 = 150$$

$$P(A) = \frac{150}{500}$$

$$\text{number of women who like Loreal } = \frac{40}{100} \times 500 = 200$$

$$\therefore P(B) = \frac{200}{500}$$

$$\text{number of women who like both } = \frac{12}{100} \times 500 = 60$$

$$\therefore P(A \cap B) = \frac{60}{500}$$

a) P(atleast one product)

$= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$$= \frac{150}{500} + \frac{200}{500} - \frac{60}{500}$$

$$= \frac{290}{500} = \frac{29}{50}$$

b) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 150 + 200 - 60$

$= 290$

$\therefore$ number of women who do not like both products $= 500 - 290 = 210$

$\therefore$ P(do not like both) $= \frac{210}{500} = \frac{21}{50}$

Example 18 :

A couple appears in an interview for two vacancies in the same post. The probability of husbands section is $\frac{1}{7}$ and the probability of wife's selection is $\frac{1}{5}$. What is the probability that a) both of them will be selected b) only one of them will be selected c) none of them will be selected.

Solution : A : Husband selected

B : Wife selected

Then A' implies husband not selected, B' implies wife not selected

$P(A) = \frac{1}{7}$ $P(A') = 1 - \frac{1}{7} = \frac{6}{7}$

$P(B) = \frac{1}{5}$ $P(B') = 1 - \frac{1}{5} = \frac{4}{5}$

a) both of them are selected

implies Husband is selected and Wife is selected

$= P(A) \times P(B)$

$= \frac{1}{7} \times \frac{1}{5} = \frac{1}{35}$

b) Only one of them is selected implies Husband selected and Wife not selected OR Wife selected and Husband not selected.

$= P(A) \cdot P(B') + P(B) \cdot P(A')$

$= \frac{1}{7} \times \frac{4}{5} + \frac{1}{5} \times \frac{6}{7} = \frac{10}{35} = \frac{2}{7}$

c) None of them are selected implies Husband not selected and Wife not selected.

$$= \frac{6}{7} \times \frac{4}{5} = \frac{24}{35}$$

Example 19 :

There are 20 boys and 40 girls in a class. Half of the boys and half of the girls have brown eyes. Find the probability that a student chosen at random is a boy or a student with brown eyes.

Solution :

Let A : student is a boy

B : student has brown eyes.

Then $P(A) = \frac{20}{60} = \frac{1}{3}$

Number of students with brown eyes

= 10 boys + 20 girls = 30

$\therefore P(B) = \frac{30}{60} = \frac{1}{2}$

Number of students who are boys and having brown eyes = 10

$\therefore P(A \cap B) = \frac{10}{60} = \frac{1}{6}$

$\therefore$ P(boy or brown eyes)

$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3}$

Example 20 :

Student A can solve 35% of the problems. Students B can answer 80% of the problem and student C can answer 50% of the problem. Find the probability that the problem is solved, if they try independently.

Solution : A : A solves the problem

A' : A will not solve

B : B solves the problem

B' : B will not solve

C : C solves the problem

C' : C will not solve

$P(A) = \frac{35}{100}$ $P(A') = \frac{65}{100}$

$P(B) = \frac{80}{100}$ $P(B') = \frac{20}{100}$

$P(C) = \frac{50}{100}$ $P(C') = \frac{50}{100}$

The problem will be solved when atleast one of A,B,C solve the problem.

Note : **P(atleast one) = 1– P (none)**

P(atleast one of them solving)

$= 1 - P$ (none of them solving)

$= 1 - P$ (A not solving and B not solving and C not solving)

$= 1 - P(A') \cdot P(B') \cdot P(C')$

$= 1 - \frac{65}{100} \times \frac{20}{100} \times \frac{50}{100}$

$= \frac{187}{200}$

Example 21 :

Two cards are drawn one after the other from a pack of 52 playing cards. Find the probability that they are both aces if the first card is a) replaced b) not replaced

With replacement

Solution :

a) There are 52 cards out of which 4 are aces.

$\therefore$ Probability of getting an ace in the first draw $= \frac{4}{52}$

The card is replaced before before the second draw. Once again a card is drawn from 52 cards.

$\therefore$ Probability of getting an ace in the second draw $= \frac{4}{52}$

$\therefore$ Probability of getting two aces is

$=$ Ace in first draw and Ace in second draw

$= \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}$

Without replacement

b) When one card is not replaced there are totally 51 cards for the second draw and since one ace is already drawn from the remaining 3 aces one ace has to be selected.

$\therefore$ probability of getting ace in second draw $= \frac{3}{51}$

$\therefore$ P(both aces) = Ace in first draw and Ace in second draw

$$= \frac{4}{52} \times \frac{3}{51}$$

Note: drawing 2 cards without replacement is same as drawing 2 cards together.

Example 22 :

A box contains 8 white chalks and 9 pink chalks. Two chalks are taken at random from the box. Find the probability that both of them are pink, if

a) the two chalks are taken out together

b) the chalks are taken one after the other, without replacement

c) the chalks are taken out one after the other, with replacement

Solution :

a) Taken together: Two chalks can be selected from 17 chalks in ${}^{17}C_2$ ways

$\therefore$ Total cases ${}^{17}C_2$

Total ways of choosing 2 pink chalks from 9 is ${}^{9}C_2$

$\therefore$ P(both pink) $= \frac{{}^{9}C_2}{{}^{17}C_2} = \frac{9}{34}$

b) Without replacement: Probability of first chalk being pink is $\frac{9}{17}$

Probability of second chalk being pink is $\frac{8}{16}$

$\therefore$ P(both pink) = first pink chalk and second also pink chalk $= \frac{9}{17} \times \frac{8}{16} = \frac{9}{34}$

c) With replacement: Probability of first chalk being pink is $\frac{9}{17}$

Probability of second chalk being pink is $\frac{9}{17}$

$\therefore$ P(both pink) = first chalk is pink and second chalk is pink

$$= \frac{9}{17} \times \frac{9}{17} = \frac{81}{289}$$

Example 23 :

A bag contains 10 gold and 8 silver coins. Two successive drawings of 4 coins are made. Find the probability that the first drawing will get 4 gold and second 4 silver coins if a) coins are replaced before the second trial b) coins are not replaced before the second trial.

Solution :

a) Draw with replacement

4 coins can be drawn from 18 coins in ${}^{18}C_4$ ways. 4 gold coins can be drawn in ${}^{10}C_4$

$\therefore$ P (4 gold in first draw) $= \frac{{}^{10}C_4}{{}^{18}C_4}$

Since 4 coins are replaced. In the second draw again 4 coins are drawn from 18 coins in ${}^{18}C_4$ ways. 4 silver coins can be selected in ${}^{8}C_4$ ways

P (4 silver in second draw) $= \frac{{}^{8}C_4}{{}^{18}C_4}$

P (4 gold and 4 silver) $= \frac{{}^{10}C_4}{{}^{18}C_4} \times \frac{{}^{8}C_4}{{}^{18}C_4}$

b) Draw without replacement :

P(4 gold in first draw) $= \frac{{}^{10}C_4}{{}^{18}C_4}$

Since 4 coins are not replaced 4 coins have to be drawn from remaining 14 coins

$\therefore$ P (4 silver coins in second draw) $= \frac{{}^{8}C_4}{{}^{14}C_4}$

$\therefore$ P (4 gold and 4 silver) $= \frac{{}^{10}C_4}{{}^{18}C_4} \times \frac{{}^{8}C_4}{{}^{14}C_4}$

Example 24 :

What is the probability that there will be 53 mondays in a randomly selected a) leap year b) non-leap year.

Solution :

a) Every year has 52 complete weeks. $52 \times 7 = 364$ days. Hence 52 mondays, Leap year has 366 days. Now in the remaining two days we have to see what is the probability of getting a monday. The remaining two days may occur in any one of the following pattern.

(Sunday Monday) (Monday Tuesday) (Tuesday Wednesday) (Wednesday Thursday) (Thursday Friday) (Friday Saturday) (Saturday Sunday)

Out of the 7 possibilities, the first two cases have monday

$\therefore$ P (53 mondays in a leap year) = $\frac{2}{7}$

b) Non leap year has 365 day. So the remaining one day could be Sunday, Monday .... Saturday.

$\therefore$ P (53 mondays in a non-leap year) = $\frac{1}{7}$

Example 25 :

A card is drawn from a pack of 52 playing card. What is the probability that the card is king given that the card is red.

Solution :

Let A : card is red

B : card is king

$P(A) = \frac{26}{52}$ $P(B) = \frac{4}{52}$

The king cards of diamond and heart are red $\therefore P(A \cap B) = \frac{2}{52}$

$\therefore$ P(king given card is red)

$$= P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{2}{52}}{\frac{26}{52}} = \frac{1}{13}$$

Example 26 :

A fair die is rolled twice, what is the probability of getting a sum of 8 when it is known that the first number is prime.

Solution :

A : First number is prime

B : Sum is 8.

2, 3, 5 are prime numbers. For each of these the second number could be any one of the 6 numbers

$\therefore P(A) = \frac{18}{36}$

we get a sum of 8 in following cases (2,6) (6,2) (3,5) (5,3) (4,4)

$\therefore$ P(B) = $\frac{5}{36}$

we get a sum of 8 when first number is prime in following cases.

(2,6) (3,5) (5,3)

$\therefore$ P(A $\cap$ B) = $\frac{3}{36}$

$\therefore$ (getting sum of 8 given 1st is prime)

$$= \quad P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{3}{36}}{\frac{18}{36}} = \frac{3}{18}$$

Example 27 : A die is thrown, If E is the event "the number appearing is a multiple of 3" and F be the event "the number appearing is even" then find whether E and F are independent.

Solution :

S = {1, 2, 3, 4, 5, 6}, E = {3, 6}, F = {2, 4, 6}, E $\cap$ F = {6}

P(E) = $\frac{2}{6}$ P(F) = $\frac{3}{6}$ P(E) . P(F) = $\frac{2}{6} \times \frac{3}{6} = \frac{1}{6}$

P(E $\cap$ F) = $\frac{1}{6}$

P(E $\cap$ F) = P(E) . P(F)

Hence E and F are independent events.

Example 28 :

If P(A) = $\frac{2}{5}$, P(B) = $\frac{1}{5}$ find P(A $\cup$ B) if A and B are independent events.

Solution :

P(A $\cup$ B) = P(A) + P(B) – P(A $\cap$ B)

But A & B are independent events

Hence P(A $\cap$ B) = P(A) . P(B)

P(A $\cup$ B) = P(A) + P(B) – P(A) . P(B)

$$= \frac{2}{5} + \frac{1}{5} - \frac{2}{5} \times \frac{1}{5} = \frac{13}{25}$$

Example 29:

If $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{5}$, $P(A \cup B) = \frac{1}{2}$ **find P(A/B).**

Solution :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{1}{2} = \frac{1}{4} + \frac{2}{5} - p(A \cap B)$$

$$P(A \cap B) = \frac{1}{4} + \frac{2}{5} - \frac{1}{2} = \frac{3}{20}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{3}{20}}{\frac{2}{5}} = \frac{3}{8}$$

Example 30 :

If the letters of the word LOGARITHM are arranged at random in a row what is the probability that the word begins with LOG.

Solution :

Number of arrangements of wor LOGARITHM is 9!

Number of words which begin with LOG is 6!

$$\therefore P(\text{beginning with LOG}) = \frac{6!}{9!}$$

Example 31 :

6 boys and 6 girls have to be seated for a seminar. Find the probability that all girls are together when they are seated a) in a row b) in a circle.

Solution :

a) 6 boys and 6 girls can be arranged in a row in 12! ways. If all girls are together total arrangements = 7! × 6!

$$2(\text{girls together in a row}) = \frac{7! \times 6!}{12!}$$

b) 6 boys and 6 girls can be arranged in a circle in 11! ways. If all girls are together total arrangements = 6! × 6!

$$P(\text{girls together in a circle}) = \frac{6! \times 6!}{11!}$$

Example 32 :

If A and B are two events with probability 0.4 and 0.8 corresponding to A and $A \cup B$ find P(B) if A and B are mutually exclusive.

Solution :

If A and B are muturally exclusive then $P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

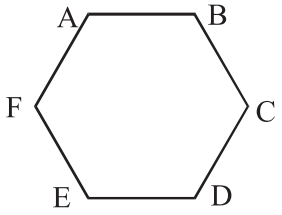
$0.8 = 0.4 + P(B)$

$P(B) = 0.8 - 0.4 = 0.4$

Example 33 :

Three of the six vertices of a regular hexagon are chosen at random. What is the probability that the triangle formed with these 3 vertices is equilateral.

Solution :



ABCDEF is a regular hexagon. Total number of triangles = ${}^6C_3 = 20$ (Since no three points are collinear) Out of these we get 2 equilateral triangles namely AEC and BDF

$\therefore$ P(equilateral triangles) $= \frac{2}{20} = \frac{1}{10}$

EXERCISE 3.1

I. One mark questions.

1) What is an experiment?
2) What is random experiment.
3) Define sample space.
4) What is the probability of sample space.
5) What is an event.
6) What is the range for probability.
7) What is probability of a certain event.
8) What is probability of an impossible event.
9) Define a) Mutually exclusive events b) Independent events c) Complimentary events.

10) If $P(A) = \frac{3}{5}$ Find P(A')

11) A coin is tossed thrice. Write the sample space.

12) Ved hits the target 7 times out of 10 shots. Find his probability of missing the target.

13) What is the probability of an event A if odds in favour are 3:5.

14) What is the probability of an event A if odds against are 2:7

15) A card is drawn from a pack of 52 playing cards. What is the probability that it is a queen.

16) What is the probability of getting two heads when a coin is tossed twice.

17) If $P(A) = \frac{3}{4}$, $P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{4}$, find $P(A/B)$

18) A bag contains 8 red and 4 green marbles. Find the probability that a marble selected at random is red.

19) What is the probability of getting a multiple of 3 when a die is thrown.

20) What is the probability that the child born is a boy?

21) A number chosen from 1 to 50. What is the probability that the number is a multiple of 8.

22) If A nd B are mutually exhaustive events, then what is $P(A \cup B)$

23) A die is thrown. What is the chance of getting a face with dots as multiple of 2.

24) If A and B are independent events find P(B/A)

25) A box contains 5 red balls, 8 green balls and 10 pink balls. A ball is drawn at random from the box. What is the probability that the ball drawn is either red or green.

II. Two marks questions.

1) What are independent events. Give one example.

2) Explain sample space with an example.

3) If A and B are mutually exclusive events with $P(A) = \frac{2}{5}$ $P(B) = \frac{1}{7}$ find $P(A \cup B)$

4) If $P(A) = \frac{1}{2}$ $P(B) = \frac{1}{3}$ $P(A \cup B) = \frac{7}{12}$ find P(B/A)

5) If $P(\overline{A}) = 0.65$, $P(A \cup B) = 0.65$ find P(B) if A and B are mutually exclusive events.

6) Two cards are drawn at random from a well-shuffled pack of 52 cards. What is the probability that either both are queens or both are king cards.

7) A die is thrown twice, what is the probability that atleast one of the two numbers is 6.

8) From a well - shuffled pack of 52 cards a card is drawn at random. Find probability that the drawn card is a king or a queen.

9) The probability of occurence of two events A and B are $\frac{1}{4}$ and $\frac{1}{2}$ respectively. The probability of their simultaneous occurence is $\frac{7}{50}$. What is the probability that neither A nor B occurs.

[Hint : find $1 - P(A \cup B)$

10) Two coins are tossed simultaneously. What is the probability of getting a) atmost 1 tail b) atleast 1 tail.

11) Three fair coins are tossed simultaneously. Find the probability of getting atleast one head and atleast one tail.

12) Two dice are rolled simultaneously. Find the probability of getting a doublet of even numbers.

13) If there are two children in a family. Find the probability that there is atleast one girl child in the family.

14) A box contains 5 defective and 15 non defective bulbs. Two bulbs are chosen at random. Find the probability that both the bulbs are non-defective.

15) A problem in a question paper is given to 3 students in a class to be solved. The probabilities of their solving the problem are 0.5, 0.7 and 0.8 respectively. Find the probability that the problem will be solved.

16) What is the probability that a randomly chosen two-digit positive integer is a multiple of 3.

17) Two cards are drawn from a pack of 52 cards. What is the probability that both are face cards.

18) Tickets are numbered from 1 to 18 are mixed up together and one ticket is drawn at random. What is the probability that the ticket has a number which is a multiple of 2 or 3.

19) If the letters of the word 'RAMLEELA' are arranged in random. What is the probability that it begins with REEL.

20) A committee of 4 has to be selected from 9 boys and 6 girls. What is the probability that the committee contains 2 boys and 2 girls.

III. 3 marks questions.

1) A box contains 4 defective and 6 non defective bulbs . Find the probability that atleast 3 bulbs are defective when 4 bulbs are selected at random.

2) A natural number is chosen at random from among the first 300. What is the probability that the number so chosen is divisible by 3 or 5.

3) The probability that a MBA aspirant will join IIM is $\frac{2}{5}$ and that he will join XLRI is $\frac{1}{3}$. Find the probability that a) he will join IIM or XLRI
   b) he will join neither IIM nor XLRI
4) A die is thrown twice and sum of the numbers appearing is observed to be 9. What is the conditional probability that the number 4 has appeared atleast once.
5) A die is rolled. If the outcome is an odd number. What is the probability that it is a number greater than 1.
6) A couple has two children. Find the probability that both are boys, if it is known that a) one of the children is a boy b) older child is a boy.
7) A die is rolled twice. If the sum of the numbers facing upwards is even find the probability that both are odd.
8) Seven persons are to be seated in a row. Find the probability that 2 particular persons sit next to each other.
9) There are 20 girls and 60 boys in a class. Half of the girls and half of the boys are first class students. A student is selected at random. What is the probability that the student is either a boy or a first class holder.
10) Among the members of a committee, there are 75% males and 25% females. The probability that a male member becomes the president is 0.25 and probability that a female member becomes the president is 0.4. Find the probability that the person selected at random becomes the president.

    [Hint : Person selected is Male and becomes President OR is female and becomes President]
11) Two cards are drawn from a pack of playing cards, one after the other. Find the probability of getting a heart in first draw and diamond in the second draw if the cards are drawn.
    a) without replacement
    b) with replacement
12) If three cards are drawn at random from a pack of 52 cards, what is the probability that atleast two of them are kings.

IV. 5 marks questions.

1) If the letters of the word INDEPENDENCE are arranged at random. Find the probability that
   a) 4 E's are together in the word.
   b) The 2 D's are together and 3 N's are together.
   c) All the 4 E's are not together.
   d) No two E's are together.

2) A bag contains 3 red and 4 black balls and another bag has 4 red and 2 block balls. One bag is selected at random and from the selected bag a ball is drawn. Let E be the event that the first bag is selected, F be the event that the second bag is selected, G be the event that ball drawn is red. Find

   a) P(E) b) P(F) c) P(G/E) d) P(G/F)

3) In a class 45% students read english, 30% read french and 20% read both english and french. One student is selected at random. Find the probability that a) he reads english, if it is known that he reads french b) he reads french if it is known that he reads english

4) A bag contains 8 red and 4 green balls. Find the probability that

   a) ball drawn is red when one ball is selected at random.

   b) all the 4 balls are green when four balls are drawn at random.

   c) Two balls are red and one ball is green when three balls are drawn at random.

   d) Three balls are drawn and none of them is green.

5) In case of 100 students, 60 drink tea, 50 drink coffee and 30 drink both tea and coffee. A student from this class is selected at random. What is the probability that the student takes

   a) atleast one of the two drinks b) only one of the drinks .

6) One card is drawn from a pack of 52 cards. Find the probability that

   a) Card is neither an ace nor a king b) Queen given the card is red

   c) Card is either black or a jack d) It is a face card.

7) Two dice are rolled simultaneously. Find the probability of

   a) getting a total of 11. b) getting a sum greater than 11.

   c) getting a multiple of 2 on one die and a multiple of 3 on the other.

8) Three fair coins are tossed simultaneously. Find the probability of

   a) getting one head b) getting atmost one head

   c) getting atleast two head

   d) getting 3 heads when it is known that first two are heads.

9) Two cards are drawn at random from a pack of cards. Find the probability that bothe are aces if the cards are drawn i) together ii) one after the other without replacement iii) one after the other with replacement.

10) A committee of 12 with atleast 5 women has to formed from 9 women and 8 men. What is the porbability that

   a) Women are in majority

   b) Men are in majority in the committee so formed.

11) From 8 gentlemen and 7 ladies a committee of 5 is to be formed. What is the probability that this committee consists of

a) exactly 2 ladies
b) atleast 3 gentlemen
c) atmost 2 ladies

12) There are six tickets numbered from 0 to 5. Two tickets are selected at random from the lot. What is the probability that the sum is

a) atmost 8 b) atleast 8 c) prime number

13) A bag contains 6 red, 4 white and 2 black balls. 2 balls are drawn at random. What is the probability that the balls drawn are

a) both red
b) 1 white and 1 black
c) same colour
d) different colour

14) An urn contains 4 white, 2 green and 4 black balls. 3 balls are drawn at random. What is the probability that the drawn balls are 1) 3 white 2) 1 white and 2 green 3) one of each colour.

15) Probability that Anirudh solves the given problem is ½ and probability that Akansh solves the given problem is $\frac{1}{4}$. If the problem is independently tried by them. What is the probability that

a) the problem is solved
b) both of them solve the problem
c) Anirudh alone solves the problem
d) Akansh alone solves the problem
e) Both do not solve the problem

16) An educational institute wants to select 3 teachers. If 5 ladies and 4 gents appear for the interview. What is the probability that the selected teachers are a) 3 gents b) 3 ladies c) 2 gents and 1 lady d) atleast one lady.

17) Two persons A and B climb a hill. The probability tht A climbs the hill is $\frac{1}{6}$ and that B climbs the hill is $\frac{1}{4}$. What is the probability that

a) Both of them climb the hill
b) Only one of them will climb the hill
c) None of them will climb the hill
d) Only A will climb the hill
e) Only B climbs the hill

18) The probability that India win a cricket test against Australia is $\frac{1}{3}$. If India and Australia play 3 tests what is the probability that

a) India will lose all the 3 matches.

b) India will win all the 3 matches.

c) India will atleast one match.

d) India will win exactly two matches.

19) A couple appears in an interview for two vacancies in the same post. The probability of husband's selection is $\frac{1}{7}$ and the probability of wife's selection is $\frac{1}{5}$. What is the probability that

a) both of them are selected

b) only one of them will be selected

c) none of them will be selected

d) Wife only is selected.

20) A bag contains 10 tickets numbered 0, 1, 2 .....9. Two tickets are drawn at random simultaneously. What is probability that

a) product of the numbers on the two tickets drawn is zero.

b) sum of the numbers on the two tickets is less than 4.

c) One ticket number is 3 times the other ticket number.

ANSWERS 3.1

I. 10) $\frac{2}{5}$ 11) S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

12) $\frac{3}{10}$ 13) $\frac{3}{8}$ 14) $\frac{7}{9}$ 15) $\frac{1}{13}$ 16) $\frac{1}{4}$

17) $\frac{1}{2}$ 18) $\frac{2}{3}$ 19) $\frac{1}{3}$ 20) $\frac{1}{2}$ 21) $\frac{3}{25}$

22) $P(A) + P(B) - P(A \cap B)$ 23) $\frac{1}{3}$ 24) P(B) 25) $\frac{13}{23}$

II. 3) $\frac{19}{35}$ 4) $\frac{1}{2}$ 5) 0.3 6) $\frac{2}{221}$ 7) $\frac{11}{36}$

8) $\frac{2}{13}$ 9) $\frac{39}{100}$ 10) a) $\frac{3}{4}$ b) $\frac{3}{4}$ 11) $\frac{3}{4}$

12) $\frac{1}{12}$ 13) $\frac{3}{4}$ 14) $\frac{21}{38}$ 15) 0.97 16) $\frac{1}{3}$

17) $\frac{11}{221}$ 18) $\frac{2}{3}$ 19) $\frac{1}{1680}$ 20) $\frac{36}{91}$

III. 1) $\frac{5}{42}$ 2) $\frac{7}{15}$ 3) a) $\frac{11}{15}$ b) $\frac{4}{15}$ 4) $\frac{1}{2}$

5) $\frac{2}{3}$ 6) a) $\frac{1}{3}$ b) $\frac{1}{2}$ 7) $\frac{1}{2}$ 8) $\frac{2}{7}$

9) $\frac{7}{8}$ 10) 0.2875 11) a) $\frac{13}{204}$ b) $\frac{1}{16}$ 12) 0.0132

IV. 1) a) 0.018 b) 0.009 c) 0.98 d) 0.2545

2) a) $\frac{1}{2}$ b) $\frac{1}{2}$ c) $\frac{3}{7}$ d) $\frac{4}{6}$

3) a) $\frac{20}{30}$ b) $\frac{20}{45}$ 4) a) $\frac{2}{3}$ b) $\frac{1}{495}$ c) $\frac{28}{55}$ d) $\frac{14}{55}$

5) a) $\frac{4}{5}$ b) $\frac{1}{2}$ 6) a) $\frac{11}{13}$ b) $\frac{1}{13}$ c) $\frac{7}{13}$ d) $\frac{3}{13}$

7) a) $\frac{1}{18}$ b) $\frac{1}{36}$ c) $\frac{11}{36}$ 8) a) $\frac{3}{8}$ b) $\frac{1}{2}$ c) $\frac{1}{2}$ d) $\frac{1}{2}$

9) a) $\frac{1}{221}$ b) $\frac{1}{221}$ c) $\frac{1}{169}$ 10) a) $\frac{193}{442}$ b) $\frac{36}{221}$

11) a) 0.3916 b) $\frac{1722}{3003}$ c) $\frac{1722}{3003}$

12) a) $\frac{14}{15}$ b) $\frac{2}{15}$ c) $\frac{8}{15}$

13) a) $\frac{5}{22}$ b) $\frac{4}{33}$ c) $\frac{1}{3}$ d) $\frac{2}{3}$

14) a) $\frac{1}{30}$ b) $\frac{1}{30}$ c) $\frac{4}{15}$

15) a) $\frac{5}{8}$ b) $\frac{1}{8}$ c) $\frac{3}{8}$ d) $\frac{1}{8}$ e) $\frac{3}{8}$

16) a) $\frac{4}{84}$ b) $\frac{10}{84}$ c) $\frac{30}{84}$ d) $\frac{80}{84}$

17) a) $\frac{1}{24}$ b) $\frac{1}{3}$ c) $\frac{5}{8}$ d) $\frac{1}{8}$ e) $\frac{5}{24}$

18) a) $\frac{8}{27}$ b) $\frac{1}{27}$ c) $\frac{19}{27}$ d) $\frac{2}{9}$

19) a) $\frac{1}{35}$ b) $\frac{2}{7}$ c) $\frac{24}{35}$ d) $\frac{6}{35}$

20) a) $\frac{1}{5}$ b) $\frac{4}{45}$ c) $\frac{1}{15}$

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Chapter

4

BINOMIAL THEOREM

4.1 Introduction :

In earlier classes, we have learnt how to find the squares and cubes of binomials like $(x + a)$ and $(x - a)$. Using them, we could evaluate the numerical values of numbers like $(97)^2 = (100 - 3)^2$, $(998)^3 = (1000 - 2)^3$ etc. However for higher power's like $(97)^5$, $(101)^6$ etc. the calculations become difficult by using repeated multiplication. This difficulty was overcome by a theorem known as **Binomial Theorem**, which gives us a formula for the n^{th} power of a binomial $(x + a)$. An expression consisting of two terms is called binomial. This theorem was first established by Sir Issac Newton. In this chapter we study binomial theorem for positive integral indices only.

4.2 Binomial Theorem for positive Integral Indices. Let us have a look.

$(x + a)^0 = 1 , \quad x + a \neq 0$

$(x + a)^1 = x + a$

$(x + a)^2 = x^2 + 2xa + a^2$

$(x + a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$

$(x + a)^4 = (x + a)^3 (x + a) = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$

In these expansions, we observe that

(i) **The total number of terms in the expansion is one more than the index.** For example in the expansion of $(x + a)^2$ number of terms is 3 whereas the index of $(x + a)^2$ is 2.

(ii) Powers of the first quantity 'x' go on decreasing by 1 whereas the powers of the second quantity a increase by 1, in the successive terms.

(iii) In each term of the expansion, the sum of the indices of x and a is the same and is equal to the index of $(x + a)$

We now arrange the co-efficients in these expansions as follows **(Fig 4.1)**

Index	**Co-efficients**
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1

Fig 4.1

Do we observe any pattern in this table that will help us to write the next row? Yes, we do. It can be seen that the addition of 1's in the row for index 1 gives rise to 2 in the row for index 2. The addition of 1,2 and 2,1 in the row for index 2, gives rise to 3 and 3 in the row for index 3 and so on. Also, 1 is present at the beginning and at the end of each row. This can be continued till any index of our interest.

We can extend the pattern given in fig 4.2 by writing a few more rows.

Index	Coefficients
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1

Fig 4.2

Pascal's triangle

The structure given in **fig 4:2** looks like a triangle with 1 at the top vertex and running down two slanting sides. This array of numbers is known as **pascal's triangle** after the name of french mathematician Blaise Pascal. It is also known as Meru Prastara by Pingla.

Expansions for the higher powers of a binomial are also possible by using Pascal's triangle.

Let us expand $(2x + 3y)^5$ by using pascal's triangle

The row of index 5 is

1 5 10 10 5 1

Using this row and our observatiosn (i), (ii) and (iii) we get

$$(2x + 3y)^5 = (2x)^5 + 5(2x)^4 3y + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 + 5(2x)(3y)^4 + (3y)^5$$
$$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$$

Now, if we want to find the expansion of $(2x + 3y)^{12}$, we are first required to get the row for index 12. This can be done by writing all the rows of the Pascal's triangle till index 12. This is a slightly lengthy process. The process, as you observe will become more difficult if we need the expansions involving still larger powers.

We thus try to find a rule that will help us to find the expansion of the binomial for any power without writing all the rows of the Pascals triangle, that come before the row of the desired index. For this, we make use of the concept of combinations studied earlier to rewrite the numbers in the Pascals triangle. We know that $\boxed{{}^nC_r = \frac{n!}{r!(n-r)!}}$ $0 \le r \le n$ and n is a non-negative integer. Also, ${}^nC_0 = 1 = {}^nC_n$

The pascal's triangle can now be rewritten as **fig 4.3**

Index	Co-efficients
0	${}^0C_0$ (=1)
1	${}^1C_0$ (=1) ${}^1C_1$ (=1)
2	${}^2C_0$ (=1) ${}^2C_1$ (−2) ${}^2C_2$ (=1)
3	${}^3C_0$ (=1) ${}^3C_1$ (=3) ${}^3C_2$ (=3) ${}^3C_3$ (=1)
4	${}^4C_0$ (=1) ${}^4C_1$ (=4) ${}^4C_2$ (=6) ${}^4C_3$ (=4) ${}^4C_4$ (=1)
5	${}^5C_0$ (=1) ${}^5C_1$ (=5) ${}^5C_2$ (=10) ${}^5C_3$ (=10) ${}^5C_4$ (=5) ${}^5C_5$ (=1)

Fig 4.3: Pascal's triangle

observing this pattern, we can now write the row of the Pascal's triangle for any index without writing the earlier rows. For example for the index 7, the row would be

$${}^7C_0 \quad {}^7C_1 \quad {}^7C_2 \quad {}^7C_3 \quad {}^7C_4 \quad {}^7C_5 \quad {}^7C_6 \quad {}^7C_7$$

Thus, using this row and the observations (i), (ii) and (iii) we have

$$(x + a)^7 = {}^7C_0 x^7 + {}^7C_1 x^6 a + {}^7C_2 x^5 a^2 + {}^7C_3 x^4 a^3 + {}^7C_4 x^3 a^4 + {}^7C_5 x^2 a^5 + {}^7C_6 x a^6 + {}^7C_7 a^7$$

An expansion of a binomial to any positive integral index say n can now be visualised using these observations. We are now in a position to write the expansion of a binomial to any positive integral Index.

Binomial theorem for positive integral power [Statement only proof not for exam]

If n is a positive integer then,

$$(x + a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots\ldots\ldots\ldots + {}^nC_n a^n$$

Proof : We shall prove the theorem by the principle of mathematical induction.

Let P(n) be the statement

$$(x + a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots\ldots\ldots\ldots + {}^nC_n a^n \text{ for } n = 1$$

L.H.S. $= x + a$ R.H.S. $= x + a$ $\because {}^1C_0 = 1$ and ${}^1C_1 = 1$

Thus LHS = RHS

$\Rightarrow$ P(1) is true

Let us assume that P(n) is true for some positive integer k.

i.e. P(k) is true.

i.e. $(x + a)^k = {}^kC_0 x^k + {}^kC_1 x^{k-1} a + {}^kC_2 x^{k-2} a^2 + \ldots\ldots\ldots\ldots + {}^kC_k a^k$

We shall now show that P(k + 1) is true.

To this end we shall multiply both sides by $(x + a)$ we get

$(x + a)^{k+1} = (x + a)\ ({}^{k}C_0 x^k + {}^{k}C_1 x^{k-1} a + {}^{k}C_2 x^{k-2} a^2 + \ldots\ldots + {}^{k}C_k a^k)$

$= {}^{k}C_0 x^{k+1} + ({}^{k}C_1 + {}^{k}C_0)\ x^k\ a + ({}^{k}C_2 + {}^{k}C_1) x^{k-1}\ a^2 + ({}^{k}C_3 + {}^{k}C_2)\ x^{k-2}\ a^3 + \ldots\ldots$

$+ {}^{k}C_k\ a^{k+1}$

Now we have ${}^{k}C_0 = {}^{k+1}C_0$ and ${}^{k}C_k = {}^{k+1}C_{k+1} = 1$

Also we have,

$${}^{k}C_r + {}^{k}C_{r-1} = {}^{k+1}C_r$$

For r = 1, 2, 3, ... we get

$${}^{k}C_1 + {}^{k}C_0 = {}^{k+1}C_1$$

$${}^{k}C_2 + {}^{k}C_1 = {}^{k+1}C_2$$

$${}^{k}C_3 + {}^{k}C_2 = {}^{k+1}C_3 \text{ etc}$$

Thus we get have,

$(x + a)^{k+1} = {}^{k+1}C_0\ x^{k+1} + {}^{k+1}C_1\ x^k a + {}^{k+1}C_2\ x^{k-1} a^2 + {}^{k+1}C_3\ x^{k-2} + a^3 + \ldots\ldots\ldots\ldots + {}^{k+1}C_{k+1}\ a^{k+1}$

This shows that P(n) is true for n = k + 1. Thus P(k + 1) is true whenever P(k) is true. By principle of mathematical induction P(n) is true for all positive integers.

Hence the proof of Binominal theorem.

Observations

1. **There are *n* +1 terms in the expansion of $(x + a)^n$ i.e one more than the index.**
2. **The power of *x* in each term goes on decreasing by one and whereas the power of a goes on increasing by one.**
3. **The power of a in any term is same as lower suffix of C.**
4. **The sum of the indices of *a* and *b* is *n* in every term of the expansion.**

Example 1 : Expand $(3x + 2y)^4$

Solution :

Here $n = 4,\ x = 3x,\ a = 2y.$

$\therefore\ (3x + 2y)^4 = {}^{4}C_0 (3x)^4 + {}^{4}C_1 (3x)^3\ 2y + {}^{4}C_2 (3x)^2\ (2y)^2 + {}^{4}C_3 (3x)^1\ (2y)^3 + {}^{4}C_4 (3x)^0\ (2y)^4$

$= 1.\ 81\ x^4 + 4.\ 27\ x^3\ 2y + 6.\ 9x^2\ 4y^2 + 4.\ 3x\ 8y^3 + 1.1.16y^4$

$= 81\ x^4 + 216\ x^3 y + 216\ x^2 y^2 + 96\ xy^3 + 16\ y^4$

Example 2 : Expand $\left(x - \dfrac{1}{2x}\right)^5$

Solution :

Here $n = 5,\ x = x,\ a = -\dfrac{1}{2x}.$

$$\therefore \left(x-\frac{1}{2x}\right)^5 = {}^5C_0\, x^5 + {}^5C_1\, x^4\left(-\frac{1}{2x}\right) + {}^5C_2\, x^3\left(-\frac{1}{2x}\right)^2 + {}^5C_3\, x^2\left(-\frac{1}{2x}\right)^3 + {}^5C_4\, x\left(-\frac{1}{2x}\right)^4 + {}^5C_5\left(-\frac{1}{2x}\right)^5$$

$$\left(x-\frac{1}{2x}\right)^5 = x^5 - \frac{5x^3}{2} + \frac{5x}{2} - \frac{5}{4x} + \frac{5}{16x^3} - \frac{1}{32x^5}$$

Example 3 :

Expand $(1+x+x^2)^3$

Solution :

Here $n = 3$, $x = 1 + x$, $a = x^2$.

$$\therefore [(1+x)+x^2)]^3 = {}^3C_0(1+x)^3 + {}^3C_1(1+x)^2\,(x^2)^1 + {}^3C_2(1+x)^1\,(x^2)^2 + {}^3C_3(1+x)^0\,(x^2)^3$$

$$= (1+x)^3 + 3(1+x)^2\,x^2 + 3(1+x)\,x^4 + x^6$$

$$= 1 + 3x + 3x^2 + x^3 + 3(1 + 2x + x^2)\,x^2 + 3x^4 + 3x^5 + x^6$$

$$= 1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6.$$

Example 4:

Simplify $\left(3+\sqrt{5}\right)^6 + \left(3-\sqrt{5}\right)^6$ using binomial theorem.

Solution :

$$(3+\sqrt{5})^6 + (3-\sqrt{5})^6$$

$$= {}^6C_0\,3^6 + {}^6C_1\,3^5\sqrt{5} + {}^6C_2\,3^4(\sqrt{5})^2 + {}^6C_3\,3^3(\sqrt{5})^3 + {}^6C_4\,3^2(\sqrt{5})^4 + {}^6C_5\,3^1(\sqrt{5})^5 + {}^6C_6(\sqrt{5})^6$$

$$+ {}^6C_0\,3^6 - {}^6C_1\,3^5\sqrt{5} + {}^6C_2\,3^4(\sqrt{5})^2 - {}^6C_3\,3^3(\sqrt{5})^3 + {}^6C_4\,3^2(\sqrt{5})^4 - {}^6C_5\,3^1(\sqrt{5})^5 + {}^6C_6(\sqrt{5})^6$$

$$= 2[729 + 15.\,81.\,5 + 15.\,9.\,25 + 125]$$

$$= 2\,[729 + 6075 + 3375 + 125]$$

$$= 20608.$$

Example 5 : Expand $\left(3x+\frac{1}{x}\right)^5$

Solution :

Here $n = 5$, $x = 3x$, $a = \frac{1}{x}$

$$\left(3x+\frac{1}{x}\right)^5 = {}^5C_0\,(3x)^5+{}^5C_1(3x)^4\,\frac{1}{x}+{}^5C_2(3x)^3\,\frac{1}{x^2}+{}^5C_3(3x)^2\,\frac{1}{x^3}+{}^5C_4(3x)^1\,\frac{1}{x^4}+{}^5C_5(3x)^0\frac{1}{x^5}$$

$$= 243\,x^5 + 405\,x^3 + 270\,x + \frac{90}{x} + \frac{15}{x^3} + \frac{1}{x^5}$$

Example 6:

Expand $\left(x-\frac{1}{x^2}\right)^4$

Solution :

Here $n = 4,\ x = x\ ,\ a = -\frac{1}{x^2}$

$$\left(x-\frac{1}{x^2}\right)^4 = {}^4C_0\ x^4+{}^4C_1\ x^3\left(-\frac{1}{x^2}\right)+{}^4C_2\ x^2\left(-\frac{1}{x^2}\right)^2+{}^4C_3\ x\left(-\frac{1}{x^2}\right)^3+{}^4C_4\,x^0\left(-\frac{1}{x^2}\right)^4$$

$$= x^4 - 4x^3\,\frac{1}{x^2} + 6\,x^2\,\frac{1}{x^4} - 4\,x\,\frac{1}{x^6} + \frac{1}{x^8} \qquad \left(\because {}^4C_0 = 1,\ x^0 = 1\right)$$

$$= x^4 - 4x + \frac{6}{x^2} - \frac{4}{x^5} + \frac{1}{x^8}$$

Example 7:

Expand $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^6$

Solution :

$$\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^6 = {}^6C_0\left(\sqrt{x}\right)^6+{}^6C_1\left(\sqrt{x}\right)^5\left(-\frac{1}{\sqrt{x}}\right)+{}^6C_2\left(\sqrt{x}\right)^4\left(\frac{-1}{\sqrt{x}}\right)^2+{}^6C_3\left(\sqrt{x}\right)^3\left(-\frac{1}{\sqrt{x}}\right)^3$$

$$+\ {}^6C_4\left(\sqrt{x}\right)^2\left(-\frac{1}{\sqrt{x}}\right)^4+{}^6C_5\left(\sqrt{x}\right)^1\left(\frac{-1}{\sqrt{x}}\right)^5+{}^6C_6\left(\sqrt{x}\right)^0\left(-\frac{1}{\sqrt{x}}\right)^6$$

$$= \left(\sqrt{x}\right)^6 - {}^6C_1\left(\sqrt{x}\right)^5\frac{1}{\sqrt{x}}+{}^6C_2\left(\sqrt{x}\right)^4\frac{1}{\left(\sqrt{x}\right)^2}-{}^6C_3\left(\sqrt{x}\right)^3\frac{1}{\left(\sqrt{x}\right)^3}+{}^6C_4\left(\sqrt{x}\right)^2\frac{1}{\left(\sqrt{x}\right)^4}-$$

$${}^6C_5\sqrt{x}\,\frac{1}{\left(\sqrt{x}\right)^5}+\frac{1}{\left(\sqrt{x}\right)^6}$$

$$= x^3 - 6x^2 + 15x - 20 + \frac{15}{x} - \frac{6}{x^2} + \frac{1}{x^3}$$

Example 8: Expand $\left(\frac{1}{\sqrt{x}}+x^2\right)^7$

Solution : Here $n=7,\ x=\frac{1}{\sqrt{x}},\ a=x^2$

$$\left(\frac{1}{\sqrt{x}}+x^2\right)^7=\left(\frac{1}{\sqrt{x}}\right)^7+{}^7C_1\left(\frac{1}{\sqrt{x}}\right)^6x^2+{}^7C_2\left(\frac{1}{\sqrt{x}}\right)^5(x^2)^2+{}^7C_3\left(\frac{1}{\sqrt{x}}\right)^4(x^2)^3+{}^7C_4\left(\frac{1}{\sqrt{x}}\right)^3(x^2)^4$$

$$+{}^7C_5\left(\frac{1}{\sqrt{x}}\right)^2(x^2)^5+{}^7C_6\left(\frac{1}{\sqrt{x}}\right)(x^2)^6+{}^7C_7\left(\frac{1}{\sqrt{x}}\right)^0(x^2)^7$$

$$=\frac{1}{x^{7/2}}+\frac{7}{x}+21\,x\sqrt{x}+35x^4+35x^5\sqrt{x}+21x^9+7x^{11}\sqrt{x}+x^{14}$$

$$x^{-7/2}+7x^{-1}+21x^{3/2}+35x^4+35x^{11/2}+21x^9+7x^{23/2}+x^{14}$$

Example 9: Simplify $\left(2+\sqrt{3}\right)^5+\left(2-\sqrt{3}\right)^5$

Solution :

$$\left(2+\sqrt{3}\right)^5+\left(2-\sqrt{3}\right)^5={}^5C_0\,2^5+{}^5C_1\,2^4\sqrt{3}+{}^5C_2\,2^3\left(\sqrt{3}\right)^2+{}^5C_3\,2^2\left(\sqrt{3}\right)^3+{}^5C_4\,2^1\left(\sqrt{3}\right)^4$$

$$+{}^5C_5\,2^0\left(\sqrt{3}\right)^5+{}^5C_0\,2^5-{}^5C_1\,2^4\sqrt{3}+{}^5C_2\,2^3\left(\sqrt{3}\right)^2-{}^5C_3\,2^2\left(\sqrt{3}\right)^3$$

$$+{}^5C_4\,2^1\left(\sqrt{3}\right)^4-{}^5C_5\,2^0\left(\sqrt{3}\right)^5$$

$$=2[1.32+10.8.3+5.2.9]$$
$$=22\,[\,32+240+90]$$
$$=724$$

Example 10: Simplify $\left(3+\sqrt{2}\right)^6-\left(3-\sqrt{2}\right)^6$

Solution :

$$\left(3+\sqrt{2}\right)^6-\left(3-\sqrt{2}\right)^6=\Big[{}^6C_0\,3^6+{}^6C_1\,3^5(\sqrt{2})^2+{}^6C_2\,3^4(\sqrt{2})^2+{}^6C_3\,3^3(\sqrt{2})^3+{}^6C_4\,3^2(\sqrt{2})^4$$

$$+{}^6C_5\,3^1(\sqrt{2})^5+{}^6C_6\,3^0(\sqrt{2})^6\Big]-\Big[{}^6C_0\,3^6-{}^6C_1\,3^5(\sqrt{2})+{}^6C_2\,3^4(\sqrt{2})^2$$

$$-{}^6C_3\,3^3(\sqrt{2})^3+{}^6C_4\,3^2(\sqrt{2})^4-{}^6C_5\,3^1(\sqrt{2})^5+{}^6C_6\,3^0(\sqrt{2})^6\Big]$$

$$=2\,[6.243.\sqrt{2}+20.27.2\sqrt{2}+6.3.4\sqrt{2}\,]$$
$$=2.3\sqrt{2}\,[2.243+20.9.2+6.4]$$
$$=6\sqrt{2}\,[486+360+24]\;=\;6\sqrt{2}\,[870]$$
$$=5220\sqrt{2}$$

EXERCISE 4.1

I. (3 marks problems) using binomial thereom expand the following.

1. $(1-2x)^5$ 2. $\left(\frac{2}{x}-\frac{x}{2}\right)^5$ 3. $(2x-3)^6$ 4. $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

5. $\left(x+\frac{1}{x}\right)^6$ 6. $\left(x^2+\frac{3}{x}\right)^4$

II. Simplify: (5 marks problems)

1. $\left(\sqrt{3}+1\right)^5-\left(\sqrt{3}-1\right)^5$ 2. $\left(1+\sqrt{5}\right)^5-\left(1-\sqrt{5}\right)^5$

3. $\left(3+\sqrt{2}\right)^4+\left(3-\sqrt{2}\right)^4$ 4. $\left(\sqrt{2}+1\right)^6-\left(\sqrt{2}-1\right)^6$

ANSWERS 4.1

I 1. $1-10x+40x^2-80x^3+80x^4-32x^5$

2. $\frac{32}{x^5}-\frac{40}{x^3}+\frac{20}{x}-5x+\frac{5}{8}x^3-\frac{x^5}{32}$

3. $64x^6-576x^5+2160x^4-4320x^3+4860x^2-2916x+729$

4. $\frac{x^5}{243}+\frac{5x^3}{81}+\frac{10}{27}x+\frac{10}{9x}+\frac{5}{3x^3}+\frac{1}{x^5}$

5. $x^6+6x^4+15x^2+20+\frac{15}{x^2}+\frac{6}{x^4}+\frac{1}{x^6}$

6. $x^8+12x^5+54x^2+\frac{108}{x}+\frac{81}{x^4}$

II 1. 152 2. $160\sqrt{5}$ 3. 386 4. $140\sqrt{2}$

4.3 General term in the expansion of $(x+a)^n$

In the binomial expansion for $(x+a)^n$, we observe that the first term is ${}^nC_0x^n$, the second term is ${}^nC_2x^{n-1}a$, the third term is ${}^nC_2x^{n-2}a^2$, and so on. Looking at the pattern of the successive terms we can say that the $(r+1)^{th}$ term is ${}^nC_rx^{n-r}a^r$. The $(r+1)^{th}$ term is also called the **general term** of the expansion $(x+a)^n$. It is denoted by T_{r+1}.

Thus $\boxed{T_{r+1} = {}^nC_r\ x^{n-r}\ a^r}$

Example 11:

Find the 7^{th} **term in the expansion of** $\left(x^3 - \frac{1}{x}\right)^9$

Solution : Here $n = 9$, $x = x^3$, $a = -\frac{1}{x}$

Now $r + 1 = 7$

$r = 6$

We have $T_{r+1} = {}^nC_r\, x^{n-r} a^r$

$$T_{6+1} = {}^9C_6 (x^3)^{9-6} \left(-\frac{1}{x}\right)^6$$

$$T_7 = {}^9C_6 (x^3)^3 \frac{1}{x^6}$$

$$T_7 = {}^9C_6\, x^9 \frac{1}{x^6}$$

$$T_7 = {}^9C_6\, x^3$$

Example 12:

Find the 8^{th} term in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{12}$

Solution : Here $n = 12$, $x = 2x^2$, $a = -\frac{3}{x}$

Now $r + 1 = 8$

$r = 7$

We have $T_{r+1} = {}^nC_r\, x^{n-r} a^r$

$$T_{7+1} = {}^{12}C_7 (2x^2)^{12-7} \left(-\frac{3}{x}\right)^7$$

$$T_8 = {}^{12}C_7 (2x^2)^5 \left(-\frac{3}{x}\right)^7$$

$$T_8 = -{}^{12}C_7\, 2^5\, 3^7 x^3$$

Example 13:

Find *a* if the 17^{th} and 18^{th} terms of the expansion $(2 + a)^{50}$ are equal.

Solution :

For 17^{th} term we have

Now $r + 1 = 17$

$r = 16,\ x = 2,\ a = a,\ n = 50$

$T_{r+1} = {}^{n}C_{r}\ x^{n-r}\ a^{r}$

$T_{16+1} = {}^{50}C_{16}\ 2^{50-16}\ a^{16}$

$T_{17} = {}^{50}C_{16}\ 2^{34}\ a^{16}$

Similarly $T_{18} = {}^{50}C_{17}\ 2^{33}\ a^{17}$

Given that $T_{17} = T_{18}$

${}^{50}C_{16}\ 2^{34}\ a^{16} = {}^{50}C_{17}\ 2^{33}\ a^{17}$

Therefore $\dfrac{{}^{50}C_{16}\ 2^{34}}{{}^{50}C_{17}\ 2^{33}} = \dfrac{a^{17}}{a^{16}}$

$$a = \frac{{}^{50}C_{16} \times 2}{{}^{50}C_{17}} = \frac{50!}{16!\,34!} \times \frac{17!\,33!}{50!} \times 2 = 1$$

Example 14:

The second, third and fourth terms in the binomial expansion $(x + a)^n$ are 240, 720 and 1080 repsectively. Find x, a and n.

Solution :

Given that $T_2 = 240$

We have $T_2 = {}^{n}C_{1}\ x^{n-1}\ a$

So, ${}^{n}C_{1}\ x^{n-1}\ a = 240$ ——— (1)

Similarly ${}^{n}C_{2}\ x^{n-2}\ a^2 = 720$ ——— (2)

and ${}^{n}C_{3}\ x^{n-3}\ a^3 = 1080$ ——— (3)

Dividing (2) by (1), we get

$$\frac{{}^{n}C_{2}\ x^{n-2}\ a^2}{{}^{n}C_{1}\ x^{n-1}\ a} = \frac{720}{240}$$

$$\frac{(n-1)!}{(n-2)!}\ \frac{a}{x} = 6$$

or $$\frac{a}{x} = \frac{6}{n-1}$$ ——— (4)

Dividing (3) by (2), we have

$$\frac{a}{x} = \frac{9}{2(n-2)}$$ ——— (5)

From (4) and (5),

$$\frac{6}{n-1} = \frac{9}{2(n-2)}$$

Therefore $n = 5$

Hence, from(1), $5x^4a = 240$ and from (4), $\frac{a}{x} = \frac{3}{2}$

Solving theses equations for a and x,
We get $x = 2$ and $a = 3$

Example 15:

Find the 12th term in the expansion of $\left(3x + \frac{2}{\sqrt{x}}\right)^{18}$

Solution : Here $n = 18$, $x = 3x$, $a = \frac{2}{\sqrt{x}}$

$T_{r+1} = {}^nC_r\, x^{n-r} a^r$

$T_{11+1} = {}^{18}C_{11}\,(3x)^{18-11} \left(\frac{2}{\sqrt{x}}\right)^{11}$

$T_{12} = {}^{18}C_{11}\; 3^7 x^7 \frac{2^{11}}{x^{11/2}}$

$T_{12} = {}^{18}C_{11}\; 3^7\; 2^{11} x^{3/2}$

Example 16:

Find the 5th term in the expansion of $\left(2 - \frac{1}{x}\right)^8$

Solution : Here $n = 8$, $x = 2$, $a = -\frac{1}{x}$

$T_{r+1} = {}^nC_r\, x^{n-r} a^r$

$T_{4+1} = {}^8C_4\, 2^{8-4} \left(-\frac{1}{x}\right)^4 \quad \therefore T_5 = {}^8C_4\; 2^4\; \frac{1}{x^4}$

$T_5 = {}^8C_4\; \frac{16}{x^4}$

Example 17:

Find the 7th term in the expansion of $\left(3 + \frac{2}{x}\right)^{10}$

Solution : Here $n = 10$, $x = 3$, $a = \frac{2}{x}$

$$T_{r+1} = {}^{n}C_{r}\, x^{n-r}\, a^{r}$$

$$= {}^{10}C_{6}\, 3^{10-6} \left(\frac{2}{x}\right)^{6}$$

$$= {}^{10}C_{6}\, 3^{4}\, \frac{2^{6}}{x^{6}}$$

4.4 Middle term/terms in the expansion of $(x + a)^n$

Regarding the middle term in the expansion $(x + a)^n$, we have

(i) **If *n* is even,** then the number of terms in the expansion will be $n + 1$. Since n is even $n + 1$ is odd. Therefore, the middle term is $\left(\frac{n+1+1}{2}\right)^{th}$

i.e. $\boxed{\left(\frac{n}{2}+1\right)^{th}}$ term.

(ii) **If *n* is odd,** then $n + 1$ is even, so there will be two middle terms in the expansion namely,

$\boxed{\left(\frac{n+1}{2}\right)^{th}}$ term and $\boxed{\left(\frac{n+1}{2}+1\right)^{th}}$ term.

Example 18:

Find the middle term in the expansion of $\left(\sqrt{x} - \frac{5}{x^2}\right)^8$

Solution : Here, $n = 8$ (even) there is only one middle term namely 5th term.

Here $n = 8$, $x = \sqrt{x}$, $a = -\frac{5}{x^2}$

$r + 1 = 5$, $r = 4$

We have $T_{r+1} = {}^{n}C_{r}\, x^{n-r}\, a^{r}$

$$T_{4+1} = {}^{8}C_{4} \left(\sqrt{x}\right)^{8-4} \left(-\frac{5}{x^2}\right)^{4}$$

$$T_{5} = {}^{8}C_{4} \left(\sqrt{x}\right)^{4} \frac{5^4}{x^8}$$

$$= {}^{8}C_{4}\, x^{2}\, \frac{625}{x^8}$$

$$= \frac{8!}{4!\,4!} \times 625 \times \frac{1}{x^6} = \frac{43750}{x^6}$$

Example 19:

Find the middle terms in the expansion of $\left(2x^2 - \frac{1}{\sqrt{x}}\right)^{11}$

Solution : Here $n = 11$ (odd) $\therefore$ there are 2 middle terms.

6^{th} and 7^{th} terms are the middle terms. So we have to find T_6 and T_7.

Here $n = 11$, $x = 2x^2$, $a = -\frac{1}{\sqrt{x}}$

We have $T_{r+1} = {}^nC_r \, x^{n-r} a^r$

$$T_{r+1} = {}^{11}C_r \, (2x^2)^{11-r} \left(-\frac{1}{\sqrt{x}}\right)^r$$

$$T_{r+1} = {}^{11}C_r \, 2^{11-r} \, x^{22-2r} \frac{(-1)^r}{x^{r/2}} \ldots\ldots\ldots (1)$$

(i) To get T_6, put $r = 5$ in (1)

$$T_{5+1} = {}^{11}C_5 \, 2^{11-5} \, x^{22-2(5)} \frac{(-1)^5}{x^{5/2}}$$

$$T_6 = {}^{11}C_5 \, 2^6 \, x^{12-\frac{5}{2}} (-1)$$

$$T_6 = -{}^{11}C_5 \, 2^6 \, x^{19/2}$$

(ii) To get T_7, put $r = 6$ in (1)

$$T_{6+1} = {}^{11}C_6 \, 2^{11-6} x^{22-2(6)} \frac{(-1)^6}{x^{6/2}}$$

$$T_7 = {}^{11}C_6 \, 2^5 \, x^7$$

Example 20:

Find the middle terms in the expansion of $\left(3x + \frac{x^2}{2}\right)^7$.

Solution :

Here $n = 7$ (odd) $\therefore$ there are 2 middle terms. 4^{th} and 5^{th} terms are the middle terms. So we have to find T_4 and T_5.

Here $n = 7$, $x = 3x$, $a = \frac{x^2}{2}$

We have $T_{r+1} = {}^{n}C_{r}\ x^{n-r}\ a^{r}$

$$T_{r+1} = {}^{7}C_{r}\ (3x)^{7-r} \left(\frac{x^2}{2}\right)^{r}$$

$$T_{r+1} = {}^{7}C_{r}\ 3^{7-r}\ x^{7-r}\ \frac{x^{2r}}{2^{r}} \quad \ldots\ldots\ldots (1)$$

(i) To get T_4, put $r = 3$ in (1)

$$T_{3+1} = {}^{7}C_{3}\ 3^{7-3}\ x^{7-3}\ \frac{x^{6}}{2^{3}}$$

$$T_{4} = {}^{7}C_{3}\ 3^{4}\ x^{4}\ \frac{x^{6}}{8}$$

$$T_{4} = {}^{7}C_{3}\ \frac{81}{8}\ x^{10}$$

$$= 35 \times \frac{81}{8} \times x^{10} = \frac{2835}{8}\ x^{10}$$

(ii) To get T_5, put $r = 4$ in (1)

$$T_{4+1} = {}^{7}C_{4}\ 3^{7-4}\ x^{7-4}\ \frac{x^{8}}{2^{4}}$$

$$T_{5} = {}^{7}C_{4}\ 3^{3}\ x^{3}\ \frac{x^{8}}{16}$$

$$T_{5} = {}^{7}C_{4}\ \frac{27}{16}\ x^{11}$$

$$= 35 \times \frac{27}{16}\ x^{11} = \frac{945}{16}\ x^{11}$$

Example 21:

Find the middle terms in the expansion of $\left(2x - \frac{1}{x}\right)^{17}$.

Solution :

Since there are 18 terms in the expansion, 9th and 10th terms are the middle terms. So we have to find T_9 and T_{10}.

Here $n = 17$, $x = 2x$, $a = -\frac{1}{x}$

We have $T_{r+1} = {}^{n}C_{r}\ x^{n-r}\ a^{r}$

$$T_{r+1} = {}^{17}C_r\,(2x)^{17-r}\left(-\frac{1}{x}\right)^r$$

$$T_{r+1} = {}^{17}C_r\,2^{17-r}\,x^{17-2r}\,(-1)^r \quad \ldots\ldots\ldots (1)$$

(i) To get T_9, put $r = 8$ in (1)

$$T_{8+1} = {}^{17}C_8\,2^{17-8}\,x^{17-2(8)}\,(-1)^8$$

$$T_9 = {}^{17}C_8\,2^9\,x\,(1)$$

$$T_9 = {}^{17}C_8\,2^9\,x$$

(ii) To get T_{10}, put $r = 9$ in (1)

$$T_{9+1} = {}^{17}C_9\,2^{17-9}\,x^{17-2(9)}\,(-1)^9$$

$$T_{10} = {}^{17}C_9\,2^8\,x^{-1}\,(-1)$$

$$T_{10} = -{}^{17}C_9\,\frac{2^8}{x}$$

Example 22:

Find the middle terms in the $\left(\frac{2x^2}{3}-\frac{3}{2x}\right)^{10}$.

Solution :

Here $n = 10$ (even) $\therefore$ there is only one middle term namely 6th term.

Here $n = 10$, $x = \frac{2x^2}{3}$, $a = -\frac{3}{2x}$

We have $T_{r+1} = {}^nC_r\,x^{n-r}\,a^r$

$$T_{5+1} = {}^{10}C_5\left(\frac{2x^2}{3}\right)^{10-5}\left(\frac{-3}{2x}\right)^5$$

$$T_6 = {}^{10}C_5\left(\frac{2x^2}{3}\right)^5\frac{(-3)^5}{2^5x^5}$$

$$T_6 = -{}^{10}C_5\,\frac{\cancel{2^5}\,x^{10}}{\cancel{3^5}}\,\frac{\cancel{3^5}}{\cancel{2^5}\,x^5}$$

$$T_6 = -{}^{10}C_5\,x^5$$

Example 23:

Find the middle term in the expansion of $\left(3x^2 - \frac{1}{x}\right)^{12}$.

Solution :

Since there are $12 + 1 = 13$ terms in the expansion, there is only one middle term namely 7^{th} terms.

Here $n = 12$, $x = 3x^2$, $a = -\frac{1}{x}$

We have $T_{r+1} = {}^nC_r\ x^{n-r}\ a^r$

$$T_{r+1} = {}^{12}C_6\ (3x^2)^{12-6}\left(-\frac{1}{x}\right)^6$$

$$T_7 = {}^{12}C_6\ 3^6\ x^{12}\ \frac{1}{x^6}$$

$$T_7 = {}^{12}C_6\ 3^6\ x^6$$

4.5 Coefficient of a particular power *'x'* in the Binomial Expansion.

Example 22:

Find the coefficient of x^6y^3 **in the expansion of** $(x + 2y)^9$.

Solution :

Suppose x^6y^3 occurs in the $(r + 1)^{th}$ term of the expansion $(x + 2y)^9$

Now $T_{r+1} = {}^9C_r\ x^{9-r}\ (2y)^r$

$$T_{r+1} = {}^9C_r\ 2^r\ x^{9-r}\ y^r$$

Comparing the indices of x as well as y in x^6y^3 and in T_{r+1}

We get $r = 3$

Thus the coefficient of x^6y^3 is

$${}^9C_3\,2^3 = \frac{9.8.7}{3.2}\ 2^3 = 672$$

Example 23:

Find the term containing x^{-4} **in the expansion of** $\left(\sqrt{x} - \frac{2}{x\sqrt{x}}\right)^{12}$.

Solution :

Suppose x^{-4} occurs in the $(r + 1)^{th}$ term of the expansion $\left(\sqrt{x} - \frac{2}{x\sqrt{x}}\right)^{12}$

Now $T_{r+1} = {}^{12}C_r \left(\sqrt{x}\right)^{12-r} \left(-\frac{2}{x\sqrt{x}}\right)^r$

$T_{r+1} = {}^{12}C_r \ (-2)^r \ x^{6-2r}$

Comparing the indices of x in x^{-4} and in T_{r+1}

We get $6 - 2r = -4$

$-2r = -10 \quad \boxed{r = 5}$

Thus the term containning x^{-4} is

$T_6 = {}^{12}C_5 \ (-2)^5$ i.e -25344

Example 24:

If the Coefficient of $(2r + 4)^{th}$ and $(r - 2)^{th}$ terms in the expansion of $(1 + x)^{18}$ are equal, then find 'r'.

Solution :

$T_{r+1} = {}^{18}C_r \ 1^{18-r} \ x^r = {}^{18}C_r \ x^r$

Therefore the coefficient of x^r i.e the Coefficient of $(r+1)^{th}$ term is ${}^{18}C_r$

Therefore the coefficient of $(2r+4)^{th}$ term is ${}^{18}C_{2r+3}$

The coefficient of $(r-2)^{th}$ term is ${}^{18}C_{r-3}$

Given that ${}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$18 = 2r + 3 + r - 3$

$\therefore \quad r = 6$

Example 25:

Find the Coefficient of $\frac{1}{x^{17}}$ in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$

Solution :

Here, $x = x^4$, $a = \frac{-1}{x^3}$, n = 15

$T_{r+1} = {}^nC_r \ x^{n-r} \ a^r$

$= {}^{15}C_r \ (x^4)^{15-r} \left(\frac{-1}{x^3}\right)^r$

$T_{r+1} = {}^{15}C_r \ x^{60-4r} \ \frac{(-1)^r}{x^{3r}}$

$T_{r+1} = {}^{15}C_r \ x^{60-7r} \ (-1)^r$

Comparing the indices of x in x^{-17} and T_{r+1}

We get $60 - 7r = -17$

$-7r = -77$

$r = 11$

$\therefore T_{11+1} = {}^{15}C_{11}\ x^{60-7(11)}\ (-1)^{11}$

$T_{12} = {}^{15}C_{11}\ x^{-17}\ (-1)$

$T_{12} = -{}^{15}C_{11}\ \frac{1}{x^{17}}$

$\therefore$ Coefficient of $\frac{1}{x^{17}}$ is $-{}^{15}C_{11}$.

Example 26:

Find the Coefficient of x^{-7} in $\left(\sqrt{x} - \frac{4}{x^3}\right)^{21}$

Solution :

Here $x = \sqrt{x}$, $a = \frac{-4}{x^3}$, $n = 21$

$T_{r+1} = {}^{n}C_{r}\ x^{n-r}\ a^{r}$

$= {}^{21}C_{r}\ \left(\sqrt{x}\right)^{21-r}\ \left(\frac{-4}{x^3}\right)^{r}$

$= {}^{21}C_{r}\ x^{\frac{21-r}{2}}\ \frac{(-4)^{r}}{x^{3r}}$

$T_{r+1} = {}^{21}C_{r}\ x^{\frac{21-r}{2}-3r}\ (-4)^{r}$

Comparing the indices of x in x^{-7} and T_{r+1}

We get $\frac{21-r}{2} - 3r = -7$

$\frac{21-7r}{2} = -7$

$21 - 7r = -14$

$-7r = -35$

$r = 5$

$\therefore T_{5+1} = {}^{21}C_{5}\ x^{-7}(-4)^{5}$

$= -{}^{21}C_{5}\ x^{-7}\ 4^{5}$

$= -{}^{21}C_{5} x^{-7}(1024)$

$\therefore$ Coefficient of x^{-7} is $-{}^{15}C_{11}(1024)$.

Example 27:

Find the Coefficient of x^8 in $\left(3x^2 - \frac{1}{2x}\right)^{10}$

Solution :

Here $x = 3x^2$, $a = \frac{-1}{2x}$, $n = 10$

$T_{r+1} = {}^nC_r \; x^{n-r} \; a^r$

$T_{r+1} = {}^{10}C_r \; (3x^2)^{10-r} \left(\frac{-1}{2x}\right)^r$

$= {}^{10}C_r \; 3^{10-r} \; x^{20-2r} \frac{(-1)^r}{2^r \, x^r}$

$T_{r+1} = {}^{10}C_r \; 3^{10-r} \; x^{20-3r} \frac{(-1)^r}{2^r}$

Comparing the indices of x in x^8 and T_{r+1}

We get $20 - 3r = 8$

$-3r = -12$

$r = 4$

$T_{4+1} = {}^{10}C_4 \; 3^{10-4} \; x^{20-3(4)} \; \frac{(-1)^4}{2^4}$

$T_5 = {}^{10}C_4 \; 3^6 \; x^8 \; \frac{1}{2^4}$

$\therefore$ Coefficient of x^8 is ${}^{10}C_4 \; 3^6 \; \frac{1}{2^4}$.

4.6 Term independent of x or the Constant term in $(x + a)^n$

A term which does not contain x, is called a term independent of x or a Constant term and in this case, the **power of x is zero.**

Example 28:

Find the term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6$

Solution :

Here $n = 6, x = \frac{3}{2}x^2, \; a = -\frac{1}{3x}$

We have $T_{r+1} = {}^{n}C_{r}\ x^{n-r}\ a^{r}$

$$T_{r+1} = {}^{6}C_{r}\left(\frac{3}{2}x^{2}\right)^{6-r}\left(-\frac{1}{3x}\right)^{r}$$

$$T_{r+1} = {}^{6}C_{r}\left(\frac{3}{2}\right)^{6-r}(x^{2})^{6-r}\ \frac{(-1)^{r}}{3^{r}x^{r}}$$

$$T_{r+1} = {}^{6}C_{r}\left(\frac{3}{2}\right)^{6-r}(x)^{12-3r}\ \frac{(-1)^{r}}{3^{r}} \quad \text{——————} (1)$$

The term will be independent of x if the index of x is zero.

i.e $12 - 3r = 0$

Thus $r = 4$

Hence 5th term is independent of x and is given by ${}^{6}C_{4}\left(\frac{3}{2}\right)^{6-4}\ \frac{(-1)^{4}}{3^{4}} = \frac{5}{12}$

Example 29: Find the Constant term in the expansion of $\left(2x^{2} - \frac{3}{x^{3}}\right)^{25}$

Solution :

Here $n = 25,\ x = 2x^{2},\ a = \frac{-3}{x^{3}}$

We have $T_{r+1} = {}^{n}C_{r}\ x^{n-r}\ a^{r}$

$$T_{r+1} = {}^{25}C_{r}\ (2x^{2})^{25-r}\left(\frac{-3}{x^{3}}\right)^{r}$$

$$T_{r+1} = {}^{25}C_{r}\ (2)^{25-r}\ x^{50-2r}(-3)^{r}\ x^{-3r}$$

$$T_{r+1} = {}^{25}C_{r}\ (2)^{25-r}\ x^{50-5r}(-3)^{r}$$

The term will be constant if the index of x is zero.

i.e $50 - 5r = 0$ Thus $r = 10$

Hence 11th term is the constant term and is given by ${}^{25}C_{10}\ 2^{15}\ 3^{10}$.

Example 30:

Find the term independent of x in the expansion of $\left(\sqrt[3]{x} + \frac{1}{2\sqrt[3]{x}}\right)^{18}$, $x > 0$.

Solution : Here $n = 18, x = \sqrt[3]{x}$, $a = \dfrac{1}{2\sqrt[3]{x}}$

We have $T_{r+1} = {}^{n}C_{r}\ x^{n-r}\ a^{r}$

$$T_{r+1} = {}^{18}C_{r}\ \left(\sqrt[3]{x}\right)^{18-r}\ \left(\frac{1}{2\sqrt[3]{x}}\right)^{r}$$

$$T_{r+1} = {}^{18}C_{r}\ x^{\frac{18-r}{3}}\ \frac{1}{2^{r}\ x^{r/3}}$$

$$T_{r+1} = {}^{18}C_{r}\ \frac{1}{2^{r}}\ x^{\frac{18-2r}{3}}$$

Since we have to find a term independent of x i.e term not having x. So take $\dfrac{18-2r}{3} = 0$

We get $r = 9$

The required term is $T_{10} = {}^{18}C_{9} \cdot \dfrac{1}{2^{9}}$

Example : 31

Show that there is no term independent of x in the expansion of $\left(2x^{2} + \dfrac{3}{x}\right)^{25}$.

Solution :

Here $n = 25, x = 2x^{2}, a = \dfrac{3}{x}$

We have $T_{r+1} = {}^{n}C_{r}\ x^{n-r}\ a^{r}$

$$T_{r+1} = {}^{25}C_{r}\ (2x^{2})^{25-r}\ \left(\frac{3}{x}\right)^{r}$$

$$T_{r+1} = {}^{25}C_{r}\ 2^{25-r}\ x^{50-2r}\ \frac{1}{x^{r}}$$

$$T_{r+1} = {}^{25}C_{r}\ 2^{25-r}\ 3^{r}\ x^{50-3r}$$

For the term independent of x, index of x must be zero i.e $50 - 3r = 0$

Thus $r = \dfrac{50}{3}$ which is not an integer.

Therefore there is no term independent of x .

Example 32:

Find the term independent of x in $\left(\frac{\sqrt{x}}{2}-\frac{2}{x^2}\right)^{10}$.

Solution :

Here $x=\frac{\sqrt{x}}{2}$, $a=\frac{-2}{x^2}$, $n=10$

$$T_{r+1}={}^{n}C_{r}\ x^{n-r}\ a^{r}$$

$$T_{r+1}={}^{10}C_{r}\left(\frac{\sqrt{x}}{2}\right)^{10-r}\left(\frac{-2}{x^2}\right)^{r}$$

$$T_{r+1}={}^{10}C_{r}\ \frac{x^{\frac{10-r}{2}}}{2^{10-r}}\ \frac{(-2)^{r}}{x^{2r}}$$

$$T_{r+1}={}^{10}C_{r}\ \frac{x^{\frac{10-r}{2}-2r}}{2^{10-r}}\ (-2)^{r}$$

$$T_{r+1}={}^{10}C_{r}\ \frac{x^{\frac{10-5r}{2}}}{2^{10-r}}\ (-2)^{r}$$

For term independent of x, index of x must be zero.

i.e $\frac{10-5r}{2}=0$

$10-5r=0 \qquad \therefore\ r=2$

Required term is $T_3={}^{10}C_{2}\ \frac{1}{2^{8}}(-2)^{2}$

$$T_3={}^{10}C_{2}\ \frac{1}{64}$$

$$\boxed{T_3=\frac{45}{64}}$$

4.7 Application problems on Binomial theorem

Example 33:

Using binomial theorem find the value of $(1.01)^5$ corrected upto four decimal places.

Solution :

$(1.01)^5 = (1 + 0.01)^5$

$= 1 + {}^5C_1(0.01) + {}^5C_2(0.01)^2 + {}^5C_3(0.01)^3 + {}^5C_4(0.01)^4 + {}^5C_5(0.01)^5$

$= 1 + 5(0.01) + 10(0.001) + 10(0.000001) +$ ignoring the further terms

$= 1 + 0.05 + 0.001 + 0.00001$

$= 1.0510$ Correct upto four decimals.

Example 34:

Using binomial theorem find the value of $(99)^5$.

Solution :

$(99)^5 = (100 - 1)^5$

$= (100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3 + {}^5C_4(100)(1)^4 - {}^5C_5(100)^0 1^5$

$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$

$= 9509900499$

Example 35:

Find the value of $(0.99)^5$upto 4 places of decimal using binomial theorem.

Solution :

$(0.99)^5 = (1 - 0.01)^5$

$= 1^5 - {}^5C_1 1^4(0.01) + {}^5C_2 1^3(0.01)^2 - {}^5C_3 1^2(0.01)^3 + {}^5C_4 1(0.01)^4 - {}^5C_5 1^0(0.01)^5$

$= 1 - 5(0.01) + 10(0.0001) - 10(0.000001) + 5(0.00000001) - 1(0.0000000001)$

$= 1 - 0.05 + 0.001 - 0.00001 + 0.00000005 - 0.0000000001$

$= 0.95099$

Example 36:

Find the value of $(1.2)^5$ using Binomial theorem upto 4 places of decimals.

Solution :

$(1.2)^5 = (1 + 0.2)^5$

$= 1^5 + {}^5C_1(0.2) + {}^5C_2(0.2)^2 + {}^5C_3(0.2)^3 + {}^5C_4(0.2)^4 + {}^5C_5(0.2)^5$

$= 1 + 5(02) + 10(0.04) + 10(0.008) + 5(0.0016) + 1(0.00032)$

$= 1 + 1 + 0.4 + 0.08 + 0.008 + 0.00032$

$= 2.48832$

EXERCISE 4.2

2 marks questions

1. Find

(i) the 5th term in $\left(\frac{4x}{5}-\frac{5}{2x}\right)^{8}$

(ii) the 8th term in $\left(\frac{a}{2}-\frac{3}{b}\right)^{10}$

(iii) the 6th term in $\left(\sqrt{x}-\sqrt{y}\right)^{17}$

(iv) the 7th term in $\left(3x^2-\frac{y}{3}\right)^{9}$

(v) the 10th term in $\left(\frac{a}{b}-\frac{2b}{a^2}\right)^{12}$

(vi) the 11th term in $\left(x+\frac{1}{\sqrt{x}}\right)^{14}$

3 marks Questions

2. Find the middle term in the expansion of

(i) $\left(x-\frac{1}{2y}\right)^{10}$ (ii) $\left(\frac{a}{x}+bx\right)^{12}$ (iii) $\left(\frac{2a}{3}-\frac{3}{3a}\right)^{6}$

(iv) $\left(3x-\frac{1}{6}x^3\right)^{8}$ (v) $\left(\frac{a}{3}+\frac{b}{3}\right)^{8}$

3. Find the middle terms in the expansion of

(i) $\left(3x-\frac{2}{x^2}\right)^{15}$ (ii) $\left(\frac{x}{2}+\frac{3}{x^2}\right)^{19}$ (iii) $\left(2x^2+\frac{1}{\sqrt{x}}\right)^{11}$

(iv) $\left(\sqrt{x}-\frac{4}{x^2}\right)^{11}$ (v) $\left(\sqrt{x}-\frac{3}{x^2}\right)^{13}$

5 marks Questions

4. Find the Coefficient of

(i) x^{11} in $\left(x+\frac{2}{x^2}\right)^{17}$ (ii) y^3 in $\left(7y^2-\frac{2}{y}\right)^{12}$

(iii) x^{11} in $\left(\sqrt{x}-\frac{2}{x}\right)^{17}$ (iv) x^{18} in $\left(x^2-\frac{6}{x}\right)^{15}$

(v) x^{-2} in $\left(x+\frac{1}{x^2}\right)^{17}$ (vi) x^5 in $\left(x+\frac{1}{x^2}\right)^{17}$

(vii) x^{18} in $\left(x^2+\frac{3a}{x}\right)^{15}$

5. Find the term independent of x in

(i) $\left(\frac{4x^2}{3}+\frac{3}{2x}\right)^9$ (ii) $\left(x^3-\frac{3}{x^2}\right)^{15}$ (iii) $\left(\sqrt{x}+\frac{1}{3x^2}\right)^{10}$

(iv) $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$ (v) $\left(3x-\frac{2}{x^2}\right)^{15}$ (vi) $\left(x^2-\frac{2}{x^3}\right)^5$

(vii) $\left(x-\frac{1}{x^2}\right)^{21}$ (vii) $\left(\sqrt{x}-\frac{2}{x^2}\right)^{20}$

4 Marks Question

6. Use binomial theorem to evaluate upto 4 decimals place

(i) $(1.02)^6$ (ii) $(98)^4$ (iii) $(1.0005)^4$ (iv) $(0.99)^4$

7. The first three terms in $(1+ax)^n$ where n is a positive integer are $1, 6x, 16x^2$. Find the values of a and n.

8. In the expansion of $(3+kx)^9$ the Coefficients of x^2 and x^3 are equal. Find k.

9. Find the ratio of the coefficient of x^4 in the two expansions $(1+x)^7$ and $(1+x)^{10}$?

ANSWERS 4.2

1. (i) 1120 (ii) $-32805\, a^3b^{-7}$ (iii) $-6188\, x^6y^{5/2}$ (iv) $\frac{28}{9}x^6y^6$

(v) $-{}^{12}C_3\, 2^9\, \frac{b^6}{a^{15}}$ (vi) $\frac{1001}{x}$

2. (i) $\frac{{}^{-10}C_5\, x^5}{32\, y^5}$ (ii) ${}^{12}C_6\, a^6b^6$ (iii) -20 (iv) $\frac{35}{8}x^{16}$

(v) ${}^8C_4\left(\frac{ab}{6}\right)^4$

3. (i) ${}^{-15}C_7\ 3^8\, 2^7\, \frac{1}{x^6}$; ${}^{15}C_7\, 3^7\, 2^8\, \frac{1}{x^9}$ (ii) ${}^{19}C_9\, \frac{3^9}{2^9\, x^8}$; ${}^{19}C_{10}\, \frac{3^{10}}{2^9\, x^{11}}$

(iii) ${}^{11}C_5\, 2^6\, x^{19/2}$; ${}^{11}C_6\, 2^5\, x^7$ (iv) $-{}^{11}C_5\, 4^5\, x^{-7}$; ${}^{11}C_6\, 4^6\, x^{-19/2}$

(v) ${}^{13}C_6\, 3^6\, x^{-9}\, \sqrt{x}$; $-{}^{13}C_7\, 3^7\, x^{-11}$

4. (i) 544 (ii) $-{}^{12}C_5\ 7^5 2^7$ (iii) $-{}^{17}C_4\ 2^{13}$ (iv) ${}^{15}C_4\ 6^4$

(v) 0 (vi) ${}^{17}C_4$ (vi) ${}^{15}C_4.18.a^4$

5. (i) ${}^{9}C_3\ 27$ (ii) ${}^{15}C_9\ (-3)^9$ (iii) 5 (iv) $\frac{7}{18}$

(v) ${}^{-15}C_5.2^5 3^{10}$ (vi) 40 (vii) ${}^{-21}C_7$ (viii) $16.{}^{20}C_4$

6. (i) 1.1261624 (ii) 92236816 (iii) 1.0020 (iv) 0.9605

7. $a = \frac{2}{3}, n = 9$

8. $k = \frac{9}{7}$

9. $\frac{1}{6}$

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Chapter

5

PARTIAL FRACTIONS

5.1 Introduction :

Any given rational fraction may be expressed in terms of different simple fractions which are called **Partial Fractions.**

Consider the expression $\frac{3}{x+1}+\frac{2}{x-2}$. This simplifies to $\frac{5x-4}{(x+1)(x-2)}$

i.e., $\frac{5x-4}{(x+1)(x-2)} = \frac{3}{x+1}+\frac{2}{x-2}$

This process of splitting a given rational fraction of the form $\frac{5x-4}{(x+1)(x-2)}$ into $\frac{3}{x+1}+\frac{2}{x-2}$ is called **'Resolving into Partial Fractions'.**

5.2 Rational Function (Polynomial Fractions):

An expression of the form $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n \ (a_0 \neq 0)$, where $a_0, a_1, a_2, ...a_n$ are constants are called polynomial in x of degrees 'n'.

Examples:

1) $3x^7 - 7x^5 + 13x^3 + 8$ is a polynomial in x of degree 7.

2) $5x^4 + 30x^3 - 2x^2 - x + 1$ is a polynomial in x of degree 4.

3) $x^2 + 5x + 6$ is a polynomial in x of degree 2.

4) $2x + 3$ is a polynomial in x of degree 1.

Note : All non zero constants are considered to be polynomials of degree zero

Definition:

If $f(x)$ and $g(x)$ are two polynomials in x, then $\frac{f(x)}{g(x)}$, $g(x) \neq 0$ is called Rational Fraction or a Rational Function.

Examples:

1) $\frac{8x^2-5x+2}{3x^3+4x-9}$ 2) $\frac{x+7}{x^2+6x+1}$ 3) $\frac{x^2+4x+2}{x^2+6x+9}$ 4) $\frac{x^3+5x^2-10}{(x+3)(3x+1)}$

5.3 Proper and Improper Rational Fractions:

A rational fraction $\frac{f(x)}{g(x)}$ is called a proper rational fraction if the degree of the numerator polynomial is less than the degree of the denominator polynomial. It is called an improper rational fraction if the degree of the numerator polynomial is greater than or equal to the degree of the denominator polynomial.

Examples :

1) $\frac{3x+5}{x^2+x-2}$ is a proper rational fraction. $\begin{pmatrix}\text{Degree of} & 3x+5=1\\ \text{Degree of} & x^2+x-2=2\end{pmatrix}$

2) $\frac{x^2}{(2x+1)(x-2)(x+3)}$ is a proper rational fractions. $\begin{pmatrix}\text{Degree of } x^2=2\\ \text{Degree of } (2x+1)(x-2)(x+3)=3\end{pmatrix}$

3) $\frac{x^2+9x+19}{x^2+7x+5}$ is an improper rational fraction. $\begin{pmatrix}\text{Degree of } x^2+9x+19=2\\ \text{Degree of } x^2+7x+5=2\end{pmatrix}$

4) $\frac{x^3+5x^2-10}{(x+3)(3x+1)}$ is an improper rational fraction. $\begin{pmatrix}\text{Degree of } x^3+5x^2-10=3\\ \text{Degree of } (x+3)(3x+1)=2\end{pmatrix}$

5.4 An improper rational fraction can be expressed as a sum of a polynomial and a proper rational fraction. This is illustrated by the following examples.

Example 1 :

Consider $\frac{x^2+1}{x^2-5x+6}$, an improper rational fraction.

$x^2-5x+6)\ x^2+1 \quad (1$

$\underline{x^2-5x+6}$

$\quad 5x-5$

Solution : $\therefore \quad \frac{x^2+1}{x^2-5x+6}=1+\frac{5x-5}{x^2-5x+6}$

= a polynomial + a proper rational fraction

Example 2 : Consider $\dfrac{x^3+7x^2+17x+11}{x^2+5x+5}$

Solution :

$$x^2+5x+5)\ x^3+7x^2+17x+11\ (x+2$$
$$(-)\ \underline{x^3+5x^2+5x}$$
$$2x^2+12x+11$$
$$(-)\ \underline{2x^2+10x+10}$$
$$2x+1$$

$$\therefore\ \frac{x^3+7x^2+17x+11}{x^2+5x+5}=(x+2)+\frac{2x+1}{x^2+5x+5}$$

= a polynomial + proper rational fraction.

Exercise 5.1

Express the following as a sum of a polynomial and a proper rational fraction:

1) $\dfrac{x^2+x+1}{x^2-1}$ 2) $\dfrac{3x^2-4x+7}{x+7}$ 3) $\dfrac{x^2-1}{x^2+1}$ 4) $\dfrac{5x^2}{x^2+4x+3}$

5) $\dfrac{x^4}{(x-1)(x^2+1)}$ 6) $\dfrac{4x^3-2x^2+3x+1}{2x^2+4x-1}$ 7) $\dfrac{4x^2-4x-1}{2x-1}$ 8) $\dfrac{x^3+7}{x^2-2x+1}$

Answers 5.1

1) $1+\dfrac{x+2}{x^2-1}$ 2) $(3x-25)+\dfrac{182}{x+7}$ 3) $1-\dfrac{2}{x^2+1}$ 4) $5-\dfrac{20x+15}{x^2+4x+3}$

5) $(x+1)+\dfrac{1}{(x-1)(x^2+1)}$ 6) $(2x-5)+\dfrac{25x-4}{2x^2+4x-1}$ 7) $(2x-1)-\dfrac{2}{2x-1}$

8) $(x+2)+\dfrac{3x+5}{x^2-2x+1}$

5.5 Partial Fractions :

Rules to resolve a proper rational fraction $\dfrac{f(x)}{g(x)}$ into partial fractions are discussed below in two different cases.

Case 1: Denominator consists of non-repeated linear factors:

Corresponding to each non repeated linear factor $(ax+b)$ of the denominator $g(x)$, consider a partial fraction $\frac{A}{ax+b}$, where A is a constant to be determined.

Example : $\frac{3x+9}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

Case 2: Denominator consists of repeated linear factors:

If the denominator $g(x)$ consists of a linear factor $(ax+b)$ repeated 'r' times ie., in the form $(ax+b)^r$, then consider the partial fractions as:

$$\frac{A_1}{ax+b}, \frac{A_2}{(ax+b)^2}, \frac{A_3}{(ax+b)^3} \ldots\ldots \frac{A_r}{(ax+b)^r}$$

where $A_1, A_2, A_3, \ldots\ldots A_r$ are constants to be determined.

Example : $\frac{2x+3}{(x+1)^2(x-5)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-5)}$

Remark:

For an improper rational fraction, we first express it as a sum of a polynomial and a proper fraction as discussed in sec. 5.4. and apply the appropriate rule to the so obtained proper fraction.

Example 3 :

Resolve $\frac{x+2}{(x+4)(x+3)}$ into partial fractions.

Solution :

The given fraction is a proper rational fraction with the denominator containing two non repeated linear factors.

Let $\frac{x+2}{(x+4)(x+3)} = \frac{A}{x+4} + \frac{B}{x+3}$ ..... (1)

where A and B are constants to be determined.

Multiply equation (1) by $(x+4)(x+3)$ to get

$x + 2 = A(x + 3) + B(x + 4)$ ..... (2)

put $x = -4$ in eq (2) to find A as:

$-2 = A(-1) + B(0) \quad \therefore \ A = 2$

put $x = -3$ in eq (2) to find B as:

$-1 = A(0) + B(1) \quad \therefore \ B = -1$

Substitute for A and B in eq (1)

$$\therefore \frac{x+2}{(x+4)(x+3)} = \frac{2}{x+4} - \frac{1}{x+3}$$

Example 4 : Resolve $\frac{x+8}{2x^2 - x - 1}$ into partial fractions.

Solution :

The given fraction is a proper rational fraction with the denominator polynomial can be factorised into linear factors as

$$2x^2 - x - 1 = (x+1)(2x+1)$$

$$\therefore \frac{x+8}{2x^2 - x - 1} = \frac{x+8}{(x-1)(2x+1)}$$

Now, let $\frac{x+8}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1}$ .....(1)

Where A and B are constants to be determined

Multiply eq.(1) by $(x-1)(2x+1)$

We get $x + 8 = A(2x+1) + B(x-1)$ .....(2)

put $x = 1$ to find A as $9 = A(3) + B(0) \quad \therefore A = 3$

Again put $x = -\frac{1}{2}$ to get the value of B.

$$\frac{15}{2} = A(0) + B\left(-\frac{3}{2}\right) \quad \therefore B = -5$$

substituting A = 3 and B = –5 in eq. (2) we get

$$\frac{x+8}{2x^2 - x - 1} = \frac{3}{x-1} - \frac{5}{2x+1}$$

Example 5 :

Resolve $\frac{x-1}{x(x+2)(x+4)}$ into partial fractions.

Solution:

The given fraction is a proper fraction whose denominator has three non repeated linear factors.

Let $\frac{x-1}{x(x+2)(x+4)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+4}$ .....(1)

Multiply eq. (1) by $x(x+2)(x+4)$ so as to get

$x-1 = A(x+2)(x+4) + Bx(x+4) + Cx(x+2)$ .....(2)

Where A, B, C are constants to be found out.

Put $x = 0$ in eq. (2) $\Rightarrow -1 = A(2)(4) + B(0) + C(0)$ $\therefore A = -\frac{1}{8}$

Put $x = -2$ in eq. (2) $\Rightarrow -3 = A(0) + B(-2)(2) + C(0)$ $\therefore B = \frac{3}{4}$

Put $x = -4$ in eq. (2) $\Rightarrow -5 = A(0) + B(0) + C(-4)(-2)$ $\therefore C = -\frac{5}{8}$

substitute values of A, B and C in eq. (1) we get

$$\frac{x-1}{x(x+2)(x+4)} = \frac{-1}{8x} + \frac{3}{4(x+2)} - \frac{5}{8(x+4)}$$

Example 6 :

Resolve $\frac{3x+5}{(x+2)^2(x-3)}$ into partial fractions.

Solution :

The given proper rational fraction has the linear factor $(x+2)$ repeated twice and $(x-3)$ appearing once.

Let $\frac{3x+5}{(x+2)^2(x-3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-3}$ .....(1)

Multiply both sides of eq.(1) by $(x+2)^2(x-3)$

$\therefore 3x+5 = A(x+2)(x-3) + B(x-3) + C(x+2)^2$ .....(2)

Put $x = -2$ in eq. (2) we get

$$-1 = A(0) + B(-5) + C(0) \quad \therefore B = \frac{1}{5}$$

put $x = 3$ in eq. (2) we get

$$14 = A(0) + B(0) + C(25) \quad \therefore C = \frac{14}{25}$$

Put $x = 0$ in eq(2) to get

$$5 = 6A - 3B + 4C$$

$$5 = -6A - 3\left(\frac{1}{5}\right) + 4\left(\frac{14}{25}\right)$$

$$6A = -\frac{84}{25} \qquad \therefore A = \frac{-14}{25}$$

Aliter : To find the value of A, we can also equate the coefficients of x^2 in eq.(2) as:

$$O = A + C$$

$$\Rightarrow A = -C \qquad \therefore A = \frac{-14}{25}$$

Substituting the values of A, B and C in eq. (1), we get

$$\frac{3x+5}{(x+2)^2(x-3)} = \frac{-14}{25(x+2)} + \frac{1}{5(x+2)^2} + \frac{14}{25(x-3)}$$

Example 7 :

Resolve $\dfrac{x^2-2}{x^2+x-12}$ into partial fractions.

Solution :

This is an improper rational fraction. We shall first express it as a sum of polynomial and a proper fraction.

$$\frac{x^2-2}{x^2+x-12} = 1 + \frac{10-x}{x^2+x-12}$$

Now consider the proper rational fraction.

$$\frac{10-x}{x^2+x-12}=\frac{10-x}{(x+4)(x-3)}=\frac{A}{x+4}+\frac{B}{x-3} \quad \text{.....(1)}$$

Multiplying eq.(1) by $(x+4)(x-3)$, we get

$10-x$ A$(x-3)$ + B $(x+4)$ .....(2)

Put $x=-4$ in eq. (2) to find as

$14 = A(-7) + B(0) \quad \therefore A = -2$

Put $x = 3$ in eq. (2) to find B as

$7 = A(0) + B(7) \quad \therefore B = 1$

$\therefore$ eq. (1) becomes $\frac{10-x}{x^2+x-12}=\frac{2}{x+4}+\frac{1}{x-3}$

$\therefore$ we get $\frac{x^2-2}{x^2+x-12}=1-\frac{2}{x+4}+\frac{1}{x-3}$

Example 8 :

Resolve $\frac{x^3-2}{x(x+1)^2}$ into partial fractions.

Solution :

The given fraction is an improper rational fraction. So we first reduce it as a sum of a polynomial and a proper rational fraction.

$$\frac{x^3-2}{x(x+1)^2}=\frac{x^3-2}{x^3+2x^2+x}$$

$x^3+2x^2+x)\ x^3-2\ (1$

$\underline{x^3+2x^2+x}$

$\underline{-2x^2-x-2}$

$$\therefore \quad \frac{x^3-2}{x(x+1)^2}=1-\frac{2x^2+x+2}{x(x+1)^2}$$

Let $\frac{2x^2+x+2}{x(x+1)^2}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$ .....(1)

Multiply both sides of eq(1) by $x(x+1)^2$,

we get $2x^2 + x + 2 = A(x+1)^2 + Bx(x+1) + Cx$ .....(2)

Put $x = 0$ in eq(2) to get

$2 = A(1) + B(0) + C(0)$ $\therefore A = 2$

Put $x = -1$ in eq(2) to get

$3 = A(0) + B(0) + C(-1)$ $\therefore C = -3$

Put $x = 1$ in eq(2) to get

$5 = A(4) + B(2) + C(1)$

$5 = 4A + 2B + C$

$5 = 4(2) + 2B + (-3)$ $\therefore B = 0$

Substituting the values of A, B and C, eq(1) becomes

$$\frac{2x^2 + x + 2}{x(x+1)^2} = \frac{2}{x} - \frac{3}{(x+1)^2}$$

Hence, $\frac{x^3 - 2}{x(x+1)^2} = 1 - \frac{2}{x} + \frac{3}{(x+1)^2}$

Example 9:

Resolve into $\frac{2x^2 + 16x + 29}{(x+3)^2(x+4)}$ partial fractions.

Solution :

$$\frac{2x^2 + 16x + 29}{(x+3)^2(x+4)} = \frac{A}{(x+3)} + \frac{B}{(x+3)^2} + \frac{C}{(x+4)}$$

$$\frac{2x^2 + 16x + 29}{(x+3)^2(x+4)} = \frac{A(x+3)(x+4) + B(x+4) + C(x+3)^2}{(x+3)^2(x+4)}$$

$$2x^2 + 16x + 29 = A(x+3)(x+4) + B(x+4) + C(x+3)^2$$

put $x = -3$

$2(9) + 16(-3) + 29 = B(1)$ $\therefore B = -1$

put $x = -4$

$2(16) + 16(-4) + 29 = C(1)$

$61 - 64 = C \qquad \therefore C = -3$

$2 = A + C$

$2 = A - 3 \qquad \therefore A = 5$

$$\therefore \quad \frac{2x^2+16x+29}{(x+3)^2(x+4)} = \frac{5}{x+3} - \frac{1}{(x+3)^2} - \frac{3}{x+4}$$

Example 10 : Resolve $\frac{1}{x(x^2-9)}$ into partial fractions:

Solution : $\frac{1}{x\left(x^2-9\right)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+3}$

$1 = A(x^2-9) + B(x)(x+3) + C(x)(x-3)$

put $x = 3$

$1 = (B)\,(3)\,(6)$

$$\boxed{B = \frac{1}{18}}$$

put $x = -3$

$1 = C\,(-3)\,(-6)$

$$\boxed{C = \frac{1}{18}}$$

put $x = 0$

$-9A = 1$

$$\boxed{A = \frac{-1}{9}}$$

$$\therefore \frac{1}{x\left(x^2-9\right)} = \frac{-1}{9x} + \frac{1}{18(x-3)} + \frac{1}{18(x+3)}$$

Example 11 :

Resolve into partial fractions $\dfrac{x}{(x+1)^2(x+2)}$

Solution : $\dfrac{x}{(x+1)^2(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{\left(x+1^2\right)} + \dfrac{C}{(x+2)}$

$x = A(x+1)(x+2) + B(x+2) + C(x+1)^2$

put $x = -1$

$-1 = B(1)$

$\boxed{B = -1}$

put $x = -2$

$-2 = C(1)^2$

$\boxed{C = -2}$

$2A + 2B + C = 0$

$2A - 2 - 2 = 0$

$2A = 4$

$\boxed{A = 2}$

$\therefore \quad \dfrac{x}{(x+1)^2(x+2)} = \dfrac{2}{x+1} - \dfrac{1}{(x-1)^2} - \dfrac{2}{(x+2)}$

Example 2 :

Resolve into partial fractions $\dfrac{x^2+1}{(x+1)(x-2)^2}$

Solution :

$$\frac{x^2+1}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$x^2 + 1 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$

put $x = 2$

$5 = C(3)$

$$\boxed{\frac{5}{3} = C}$$

put $x = -1$

$$2 = A(9)$$

$$\boxed{\frac{2}{9} = A}$$

$$4A - 2B + C = 1$$

$$4 \times \frac{2}{9} - 2B + \frac{5}{3} = 1$$

$$-2B = 1 - \frac{5}{3} - \frac{8}{9}$$

$$-2B = \frac{9 - 15 - 8}{9} = \frac{-14}{9}$$

$$\boxed{B = \frac{7}{9}}$$

$$\therefore \frac{x^2 + 1}{(x+1)(x-2)^2} = \frac{2}{9(x+1)} + \frac{7}{9(x-2)} + \frac{5}{3(x-2)^2}$$

EXERCISE 5.2

I. Resolve the following into partial fractions: (3 and 5 marks)

1) $\frac{x}{(x+1)(x-4)}$

2) $\frac{x+1}{(x-2)(x-3)}$

3) $\frac{7x-1}{(1-2x)(1-3x)}$

4) $\frac{3-7x^2}{(1-3x)(1+2x)(1+x)}$

5) $\frac{x^2}{(x+1)(x+2)(x+3)}$

6) $\frac{2x^2+10x-3}{(x+1)(x-3)(x+3)}$

7) $\frac{3x+1}{x^2-6x+8}$

8) $\frac{x^2-10x+13}{(x+1)(x^2-5x+6)}$

9) $\frac{3x+20}{x^2+4x}$

10) $\frac{x+3}{(x-1)(x^2-4)}$

11) $\frac{x+3}{x^3-x}$

12) $\frac{1+3x+2x^2}{(1-2x)(1-x^2)}$

II. Resolve the following into partial fractions: (5 Marks)

1) $\frac{4}{(x-3)(x+1)^2}$ 2) $\frac{9}{(x+1)(x+2)^2}$ 3) $\frac{3x+4}{(x+1)^2(x-1)}$

4) $\frac{3x+2}{(x-2)(x+3)^2}$ 5) $\frac{2x^2-4x+1}{(x-2)(x-3)^2}$ 6) $\frac{x^2}{(x+1)^2(x-5)}$

7) $\frac{4+7x}{(2+3x)(1+x)^2}$ 8) $\frac{1+2x}{(x+2)^2(x-1)}$ 9) $\frac{9x-27}{(x+1)(x-2)^2}$

10) $\frac{2x+5}{(x+2)(x-1)^2}$ 11) $\frac{x}{(1+2x)^2(1-3x)}$

III. Resolve into partial fractions: (5 Marks)

1) $\frac{2x^2-7x+1}{x^2-3x-4}$ 2) $\frac{4x^2-3x+5}{(2-x)(1+x)}$ 3) $\frac{x^3+7x^2+17x+11}{x^2+5x+6}$

4) $\frac{2x^2+3x+2}{x^2-x-2}$ 5) $\frac{2x^3+x^2-x-3}{x(x-1)(2x+3)}$

Answers 5.2

I. 1) $\frac{1}{5(x+1)}+\frac{4}{5(x-4)}$ 2) $\frac{-3}{x+2}+\frac{4}{x-3}$

2) $\frac{-5}{1-2x}+\frac{4}{1-3x}$ 4) $\frac{1}{1-3x}+\frac{1}{1+2x}+\frac{1}{1+x}$

5) $\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}$ 6) $\frac{11}{8(x+1)}+\frac{15}{8(x-3)}-\frac{5}{4(x+3)}$

7) $\frac{13}{2(x-4)}-\frac{7}{2(x-2)}$ 8) $\frac{2}{x+1}+\frac{1}{x-2}-\frac{2}{x-3}$

9) $\frac{5}{x}-\frac{2}{x+4}$ 10) $\frac{-4}{3(x-1)}+\frac{5}{4(x-2)}+\frac{1}{12(x+2)}$

11) $\frac{-3}{x}+\frac{2}{x-1}+\frac{1}{x+1}$ 12) $\frac{4}{1-2x}-\frac{3}{1-x}$

II. 1) $\frac{1}{4(x-3)}-\frac{1}{4(x+1)}-\frac{1}{(x+1)^2}$

2) $\frac{9}{x+1}-\frac{9}{x+2}-\frac{9}{(x+2)^2}$

3) $\frac{-7}{4(x+1)}-\frac{1}{2(x+1)^2}+\frac{7}{4(x-1)}$

4) $\frac{8}{25(x-2)}-\frac{8}{25(x+3)}+\frac{7}{5(x+3)^2}$

5) $\frac{1}{x-2}+\frac{1}{x-3}+\frac{7}{(x-3)^2}$

6) $\frac{11}{36(x+1)}-\frac{1}{6(x+1)^2}+\frac{25}{36(x-5)}$

7) $\frac{-6}{2+3x}+\frac{2}{1+x}+\frac{3}{(1+x)^2}$

8) $\frac{-1}{3(x+2)}+\frac{1}{(x+2)^2}+\frac{1}{3(x-1)}$

9) $\frac{-4}{x+1}+\frac{4}{x-2}-\frac{3}{(x-2)^2}$

10) $\frac{1}{9(x+2)}-\frac{1}{9(x-1)}+\frac{7}{3(x-1)^2}$

11) $\frac{2}{25(1+2x)}-\frac{1}{5(1+2x)^2}+\frac{3}{25(1-3x)}$

III, 1) $2+\frac{1}{x-4}-\frac{2}{x+1}$

2) $-4+\frac{5}{2-x}+\frac{4}{1+x}$

3) $(x+2)+\frac{4}{x+3}-\frac{3}{x+2}$

4) $2+\frac{16}{3(x-2)}-\frac{1}{3(x+1)}$

5) $1+\frac{1}{x}-\frac{1}{5(x-1)}-\frac{8}{5(2x+3)}$

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Chapter

6

MATHEMATICAL LOGIC

6.1 Introduction :

Logic is the subject which deals with principles of reasoning. It is a mathematical process to arrive at valid conclusions using certain valid assumptions called the laws of logic. The first treatise on logic were written by Aristotle. The subject was conceived and developed by George Boole and De'Morgan in later years. In this chapter, we shall understand some of the logical terminologies and concepts related to them.

6.2 Propositions and Truth Values:

An assertive statement which is either true of false but not both at the same time is called a proposition (or a statement).

Consider the following statements:

1) 5 is a prime number
2) oxygen is a gas
3) $23 \div 2 \leq 5$
4) $\sqrt{2}$ is an irrational number.
5) The word LOGIC has 5 vowels in it.

These statements can be said either true or false. The propositions (1), (2) and (4) are true whereas (3) and (5) are false propositions.

Sentences involving questions, exclamations and expressions of opinion are not considered as propositions.

For example,

1) Weather is pleasant today.
2) Wish you good luck!
3) How old is this building?
4) Hurray! We won the match.

The above statements are not propositions.

The truthfulness or falsity of a proposition is called the truth value of the proposition. The truth value of a true proposition is denoted as 'T' and that of a false proposition as 'F'. Thus, the truth values of the propositions (1), (2) and (4) is 'T' and (3) and (5) is 'F'.

6.3 Logical Connectives:

A simple proposition is a statement which cannot be split into two or more statements. For example,

1) Set of natural numbers is infinite.
2) New Delhi is in India.

are simple propositions.

A Compound proposition is made up of two or more simple propositions, called it's components.

Example:

1) If $x \in A, y \in B$ then $x \in A \cup B$.

Here '$x \in A, y \in B$' is a proposition and '$x \in A \cup B$' is another simple proposition. They are combined together using 'if then'.

2) Mathematics is easy or it is not raining today.

In this example, the two simple propositions 'Mathematics is easy' and 'it is raining today' are combined together using the words 'or' and 'not'.

These kinds of words that are used to link two or more simple propositions to form a compound proposition are called logical connectives.

The five logical connectives are :

1) 'and' denoted by $\wedge$ 2) 'or' denoted by $\vee$ 3) 'not' denoted by $\sim$

4) 'if... then...' denoted by $\rightarrow$ 5) 'if and only if' denoted by $\leftrightarrow$

These logical connectives are discussed in the following segment.

Conjunction: If 'p' and 'q' are two simple propositions, then the compound proposition 'p and q' ie., '$p \wedge q$' is called the conjunction of p and q.

The truth value of the compound proposition 'p and q' is true if both the components p and q are true, in all other cases it will be false. This is written in the form of a table called the **Truth Table**, containing all the possible combinations of the truth values of it's components.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: Let p : e is an vowel

q : $3 + 4 = 7$

Then $p \wedge q$: e is an vowel and $3 + 4 = 7$

'p' is true , 'q' is true. $\therefore$ '$p \wedge q$' is true.

Disjunction: Two simple propositions 'p' and 'q' connected together by the word 'or' is called disjunction of p and q, which is denoted as '$p \vee q$', read as 'p or q'.

The compound proposition '$p \vee q$' is false when both the components are false. In all other cases, it will be true. This is shown in the truth table below:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example: Let p : $\sqrt{2}$ is irrational

q : 9 is a prime number

'p' is a true proposition, 'q' is a false proposition. The compound proposition '$p \vee q$' hence is true.

Negation: The negation of a proposition 'p' is 'not p', denoted as '$\sim p$'

Example : p : calculus is difficult.

$\sim p$: calculus is not difficult.

The truth table for negation is as given below:

p	$\sim p$
T	F
F	T

The negation of a proposition is obtained by using the word 'not' at the appropriate place in the statement.

Conditional (Implication) : If 'p' and 'q' are two simple propositions, then the compound statement 'if p then q' is called the conditional of p and q. It is then denoted as $p \rightarrow q (p \Rightarrow q)$.

Here 'p' is called the antecedent and 'q' is the consequent.

'$p \rightarrow q$' is false if the antecedent 'p' is true and consequent 'q' is a false proposition. In all other cases, $p \rightarrow q$ will be true. The corresponding truth table is as shown in the table.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: Let $p : (a+b)^2 = a^2 + b^2 + 2ab$

$$q : (3+4)^2 = 9+16 = 25$$

Then $p \rightarrow q$: If $(a+b)^2 = a^2 + b^2 + 2ab$ then

$$(3+4)^2 = 25$$

Here $p \rightarrow q$ is false as p is true but q is false.

Suppose we consider $p:(a+b)^2 = a^2 + b^2$

$$q:(3+4)^2 = 25$$

Then $p \to q$: If $(a+b)^2 = a^2 + b^2$ then

$$(3+4)^2 = 25$$

Now, $p \to q$ is true even though p is false and q is also false.

Biconditional (Double Implication): If 'p' and 'q' are two simple propositions, then the compound proposition 'p if and only if q', denoted by $p \leftrightarrow q$ $(p \Leftrightarrow q)$ is called the biconditional or double implication.

the Biconditional $p \leftrightarrow q$ is true when both the components p and q are having the same truth values, otherwise it is false. This may be shown in the truth table given below.

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Example: Let p : 8 – 3 = 0

q : 3 × 5 = 2

Then $p \leftrightarrow q$: 8 – 3 = 0 if and only if

3 × 5 = 2

Here, p is false, q is also false.

Hence $p \leftrightarrow q$ is true.

Truth table for all the logical connectives may be written as below:

p	**q**	**p ∧ q**	**p ∨ q**	**~p**	**p → q**	**p ↔ q**
T	**T**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**T**	**F**	**F**	**F**
F	**T**	**F**	**T**	**T**	**T**	**F**
F	**F**	**F**	**F**	**T**	**T**	**T**

Worked Examples

Example 1:

Given p : 3 is an odd number.

q : Logic is easy.

Write the verbal form of the following compound propositions:

(i) $p \vee q$ (ii) $p \wedge \sim q$ (iii) $q \wedge p$ (iv) $p \rightarrow q$ (v) $p \leftrightarrow q$ (vi) $\sim p \vee q$

Solution:

(i) $p \vee q$: 3 is an odd number or logic is easy.

(ii) $p \wedge \sim q$: 3 is an odd number and logic is not easy.

(iii) $q \wedge p$: Logic is easy and 3 is an odd number.

(iv) $p \rightarrow q$: if 3 is an odd number then logic is easy.

(v) $p \leftrightarrow q$: 3 is an odd number if and only if logic is easy

(vi) $\sim p \vee q$: 3 is not an odd number or logic is easy.

Example 2:

Symbolise the following propositions: (i) 2 + 5 = 6 or all integers are rationals.

Solution:

(i) Let p : 2 + 5 = 6, q : all integers are rationals.

The given proposition is $p \vee q$.

(ii) If two triangles are similar then their corresponding sides are not proportional.

Solution:

(ii) Let p : Two triangles are similar.

q : Their corresponding sides are proportional.

Given proposition is $p \rightarrow \sim q$

(iii) **My friends are coming and we go to a picnic if and only if the weather is fine.**

Solution:

(iii) Let p : My friends are coming

q : We go to a picnic

r : The weather is fine

Then, the given proposition is $p \wedge q \leftrightarrow r$

(iv) He is neither quiet nor practical.

Solution:

(iv) The above proposition may be understood as 'He is not quiet and is not practical.

So, let p : He is quiet

q : He is practical

Then, we can write : $\sim p \wedge \sim q$

(v) If a quadrilateral is a square then it is a rectangle.

Solution:

Let p : Quadrilateral is a square

q : Quadrilateral is a rectangle (It is a rectangle)

We are given $p \rightarrow q$

(vi) If I study hard and get a merit seat then I am happy.

Solution:

Consider p : I study hard

q : I get a merit seat

r : I am happy

Hence the proposition is $(p \wedge q) \rightarrow r$

Example 3:

Assign a truth value to the following propositions: (i) $(5 \times 4 = 20) \wedge (6 + 7 = 13)$

Solution:

(i) Truth value of "$5 \times 4 = 20$ is 'T'

Truth value of "$6 + 7 = 13$" is 'T'

$\therefore$ Truth value of given proposition is 'T'

(ii) If $\sqrt{2}$ is irrational then $\sqrt{2}$ is a real number.

Solution:

(ii) Truth value of "$\sqrt{2}$ is irrational" is 'T'

Truth value of "$\sqrt{2}$ is real number" is 'T'

$\therefore$ Truth value of the given proposition is 'T'.

(iii) $(3 > 5) \vee (2 > 3)$

Solution:

(iii) Truth value of "$3 > 5$" is 'F'

Truth value of "$2 > 3$" is 'F'

$\therefore$ Truth value of "$(3 > 5) \vee (2 > 3)$" is 'F'

Example 4 :

If the truth values of the propositions p, q, r are T, T, F respectively, then find the truth values of the following compound propositions:

(i) $p \rightarrow (q \wedge r)$ (ii) $(p \vee r) \wedge q$ (iii) $\sim (p \vee q)$ (iv) $p \leftrightarrow q \vee \sim r$

Solution:

(i) $p \rightarrow (q \wedge r)$

p	q	r	$q \wedge r$	$p \rightarrow q \wedge r$
T	T	F	F	F

(ii) $(p \vee r) \wedge q$

p	q	r	$p \vee r$	$(p \vee r) \wedge q$
T	T	F	T	T

(iii) $\sim (p \vee q)$

p	q	$p \vee q$	$\sim (p \vee q)$
T	T	T	F

(iv) $p \leftrightarrow q \vee \sim r$

p	q	r	$\sim r$	$q \vee \sim r$	$p \leftrightarrow q \vee \sim r$
T	T	F	T	T	T

Example 5:

If $(\sim p \vee q) \wedge \sim r$ is a false proposition, then find the truth values of p, q and r.

Solution:

$(\sim p \vee q) \wedge \sim r$ is false

$\Rightarrow$ " $\sim p \vee q$ " is 'F' and " $\sim r$ " is 'F'

$\Rightarrow$ " $\sim p$ " is 'F', " q " is 'F' and " r " is 'T'

$\Rightarrow$ " p " is 'T' , " q " is 'F' and " r " is 'T'

Example 6 :

Construct the truth tables for the following :

(i) $(p \wedge q) \rightarrow (p \vee q)$ (ii) $(p \wedge \sim q) \leftrightarrow \sim p$ (iii) $(p \rightarrow q) \rightarrow \sim r$

Solution:

(i) $(p \wedge q) \rightarrow (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$p \wedge q \rightarrow p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

(ii) $(p \wedge \sim q) \leftrightarrow \sim p$

p	q	$\sim q$	$p \wedge \sim q$	$\sim p$	$p \wedge \sim q \leftrightarrow \sim q$
T	T	F	F	F	T
T	F	T	T	F	F
F	T	F	F	T	F
F	F	T	F	T	F

(iii) $(p \to q) \to \sim r$

p	q	r	$p \to q$	$\sim r$	$(p \to q) \to \sim r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	T	T	T

EXERCISE 6.1

One mark questions:

1) Write the verbal form of the following compound propositions given.

   p : x is a real number

   q : Mathematics is easy

   r : Today is a holiday

   (i) $p \vee q$ (ii) $p \wedge q$ (iii) $\sim p \wedge r$ (iv) $q \wedge \sim r$ (v) $(p \vee r) \to q$

   (vi) $(p \leftrightarrow q) \vee r$ (vii) $(p \vee r) \wedge \sim q$ (viii) $(p \wedge \sim q) \to \sim r$

2) Symbolise the following propositions:

   (i) $3x = 9$ and $x < 7$ (ii) $33 \div 11 \neq 3$ or $8 - 6 = 2$

   (iii) If two numbers are equal then their squares are not equal.

   (iv) If oxygen is a gas then gold is a compound.

   (v) $y + 4 \neq 4$ or e is not a vowel.

Two and Three marks questions:

1) If p,q and r are propositions with truth values F, T and F respectively, then find the truth values of the following compound propositions:

   (i) $(\sim p \to q) \vee r$ (ii) $(p \wedge \sim q) \to r$

   (iii) $p \to (q \to r)$ (iv) $\sim (p \to q) \vee \sim (p \leftrightarrow q)$

   (v) $(p \wedge q) \vee \sim r$ (vi) $\sim (p \vee r) \to \sim q$

2) (i) If the compound proposition $\sim(p \to q) \wedge (p \wedge r)$" is true, then find the truth values of p, q and r.

   (ii) If the compound proposition $p \to (q \vee r)$ is false, then find the truth values of p, q and r.

(iii) If the compound proposition $p \to (\sim q \vee r)$ is false, then find the truth values of p, q and r.

(iv) If the truth value of the proposition $(p \wedge q) \to (r \vee \sim s)$ is false, then find the truth values of p, q, r and s.

3) Construct the truth table for the following:

(i) $\sim p \wedge \sim q$ (ii) $(p \vee q) \wedge \sim p$ (iii) $\sim (p \to q)$ (iv) $p \wedge \sim q$ (v) $\sim p \vee q$

(vi) $(p \vee q) \leftrightarrow (q \wedge p)$ (vii) $\sim (p \vee q)$ (viii) $\sim p \leftrightarrow (p \wedge \sim q)$

4) If $p : \sqrt[3]{3}$ is an integer

q : 2 is an odd number

r : 5 is a prime number

then what is the truth value of the proposition $(p \leftrightarrow q) \wedge r$

Five mark questions:

1) Construct the truth table for the following propositions:

(i) $p \vee (q \wedge \sim r)$ (ii) $(\sim p \wedge q) \wedge \sim r$

(iii) $(p \wedge q) \to (r \vee \sim q)$ (iv) $(p \wedge q) \vee r$

(v) $(p \to q) \to \sim r$ (vi) $(p \to r) \wedge (p \to q)$

(vii) $p \to (q \wedge r)$ (viii) $(p \to \sim q) \wedge r$

Answers 6.1

One mark questions:

1) (i) x is a real number or Mathematics is easy.

(ii) x is a real number and Mathematics is easy.

(iii) x is not a real number and today is a holiday.

(iv) Mathematics is easy and today is not a holiday.

(v) If x is a real number or today is a holiday then Mathematics is easy.

(vi) x is a real number if and only if Mathematics is easy or today is a holiday.

(vii) x is a real number or today is a holiday and Mathematics is not easy.

(viii) If x is a real number and Mathematics is not easy then today is not a holiday.

2) (i) $p \wedge q$ (ii) $\sim p \vee q$

(iii) $p \to \sim q$ (iv) $p \to q$

(v) $\sim p \vee \sim q$

Two and Three marks questions:

1) (i) T (ii) T (iii) T (iv) F (v) T (vi) F

2) (i) p is T, q is F, r is F (ii) p is T, q is F, r is F

(iii) p is T, q is T, r is F (iv) p is T, q is T, r is F, s is T

3) (i)

p	q	~ p	~ q	~ p ∧ ~ q
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(ii)

p	q	p ∨ q	~ p	(p ∨ q) ∧ ~ q
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

(iii)

p	q	p → q	~ (p → q)
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

(iv)

p	q	~ q	p ∧ ~ q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

(v)

p	q	~ p	~ p ∨ q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(vi)

p	q	p ∨ q	q ∧ p	(p ∨ q) ↔ (q ∧ p)
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

(vii)

p	q	$p \vee q$	$\sim (p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

(viii)

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \leftrightarrow (p \wedge \sim q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	F	F
F	F	T	T	F	F

4) True

Five mark questions:

1) (i)

p	q	r	$\sim r$	$q \wedge \sim r$	$p \vee (q \wedge \sim r)$
T	T	T	F	F	T
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	T	F	T
F	T	T	F	F	F
F	T	F	T	T	T
F	F	T	F	F	F
F	F	F	T	F	F

(ii)

p	q	r	$\sim p$	$\sim r$	$\sim p \wedge q$	$(\sim p \wedge q) \wedge \sim r$
T	T	T	F	F	F	F
T	T	F	F	T	F	F
T	F	T	F	F	F	F
T	F	F	F	T	F	F
F	T	F	T	T	T	T
F	F	T	T	F	F	F
F	F	F	T	T	F	F
F	T	T	T	F	F	F

(iii)

p	q	r	~ q	ⓧ p ∧ q	ⓨ r ∨ ~ q	ⓧ → ⓨ
T	T	T	F	T	T	T
T	T	F	F	T	F	F
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	T	F	F	F	F	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

(iv)

p	q	r	p ∧ q	(p ∧ q) ∨ r
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

(v)

p	q	r	~ r	p → q	(p → q) → ~ r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	T	F	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	T	F
F	F	F	T	T	T

(vi)

p	q	r	ⓧ $p \to r$	ⓨ $p \to q$	ⓧ $\wedge$ ⓨ
T	T	T	T	T	T
T	T	F	F	T	F
T	F	T	T	F	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

(vii)

p	q	r	$q \wedge r$	$p \to (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

(viii)

p	q	r	$\sim q$	$p \to \sim q$	$(p \to \sim q) \wedge r$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	T	T
F	T	F	F	T	F
F	F	T	T	T	T
F	F	F	T	T	F

6.4 Tautology and Contradiction

Definition 1 : A compound proposition is said to be a Tautology if it is true for all possible combinations of the truth values of it's components.

Definition 2 : A compound proposition is said to be a contradiction if it is false for all possible comibinations of the truth values of it's components.

Worked Examples

Example 7 :

Show that the proposition $(p \to q) \leftrightarrow (\sim p \vee q)$ is a Tautology.

Solution:

p	q	(A) $p \to q$	$\sim p$	(B) $\sim p \vee q$	A $\leftrightarrow$ B
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

The last column indicates that the given proposition is a Tautology.

Example 8:

Show that the proposition $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction.

Solution:

p	q	(A) $p \wedge q$	$p \vee q$	(B) $\sim (p \vee q)$	A $\wedge$ B
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

The last column indicates that the given proposition is a contradiction.

Example 9 :

Verify whether the proposition $(p \wedge \sim q) \wedge (\sim p \vee q)$ is a contradiction or not.

Solution:

p	q	$\sim q$	(A) $p \wedge \sim q$	$\sim p$	(B) $\sim p \vee q$	A $\wedge$ B
T	T	F	F	F	T	F
T	F	T	T	F	F	F
F	T	F	F	T	T	F
F	F	T	F	T	T	F

Since the last column is all 'F', the given proposition is a contradiction.

Example 10: Prove that $[p \vee (p \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \vee r)$ is a tautology.

Solution:

p	q	r	$q \wedge r$	(A) $p \vee (q \wedge r)$	(x) $p \vee q$	(y) $p \vee r$	(B) $x \wedge y$	$A \leftrightarrow B$
T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F	T
F	F	T	F	F	F	T	F	T
F	F	F	F	F	F	F	F	T

The last column indicates that the given proposition is a Tautology.

Example 11: Find whether $p \rightarrow (\sim p \vee q)$ is a tautology or a contradiction.

Solution:

p	q	$\sim p$	$\sim p \vee q$	$p \rightarrow (\sim p \vee q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The last column suggests that the proposition is neither a Tautology nor a contradiction.

6.5 Logical Equivalence

Two compound propositions 'p' and 'q' involving the same components are said to be logically equivalent, if their truth values are the same for each different combinations of the truth values of their components. We then write '$p \equiv q$' (read as 'p is logically equivalent to q').

Some of the important logically equivalent propositions are as follows:

1) Idempotent Laws : (i) $p \vee p \equiv p$ (ii) $p \wedge p \equiv p$
2) Law of double negation : $\sim (\sim p) \equiv p$
3) Commutative Laws : (i) $p \vee q \equiv q \vee p$ (ii) $p \wedge q \equiv q \wedge p$
4) Associative Laws : (i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$ (ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
5) Distributive Laws : (i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
   (ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
6) De'Morgans Laws : (i) $\sim (p \wedge q) \equiv \sim p \vee \sim q$ (ii) $\sim (p \vee q) \equiv \sim p \wedge \sim q$
7) $\sim (p \rightarrow q) \equiv p \wedge \sim q$
8) $\sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

All the above results may be verified by constructing their respective truth tables.

WORKED EXAMPLES

Example 12 : Show that (i) $\sim (p \wedge q) \equiv \sim p \vee \sim q$ (ii) $\sim (p \vee q) \equiv \sim p \wedge \sim q$
(Verification of De'Morgans Laws)

Solution:

(i)

1	2	3	4	5	6	7
p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

4th and 7th columns are identical.

$\therefore \sim (p \wedge q) \equiv \sim p \vee \sim q$

Solution:

(ii)

1	2	3	4	5	6	7
p	q	$p \vee q$	$\sim (p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

4th and 7th columns are identical.

$\therefore \sim (p \vee q) \equiv \sim p \wedge \sim q$

Example 13 :

Prove : $\sim (p \rightarrow q) \equiv p \wedge \sim q$

Note : This result is useful in writing the negation of the given implication.

Solution:

1	2	3	4	5	6
p	q	$p \rightarrow q$	$\sim (p \rightarrow q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Columns 4 and 6 are identical. Hence proved.

Example 14:

Prove : $p \to (q \wedge r) \equiv (p \to q) \wedge (p \to r)$

Solution:

1	2	3	4	5	6	7	8
p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$p \to q$	$p \to r$	$(p \to q) \wedge (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Columns 5 and 8 are identical, proving the logical equivalence of the given propositions.

Example 15 :

Negate the following propositions:

(i) $\sim p \vee q$ (ii) $p \to (q \wedge r)$ (iii) $p \wedge \sim q$ (iv) $p \to \sim q$

Solution:

(i) $\sim(\sim p \vee q) \equiv \sim(\sim p) \wedge \sim q$ [using De'Morgans law]

$\equiv p \wedge \sim q$ [using $\sim(\sim p) = p$]

$\therefore \sim(\sim p \vee \sim q) \equiv p \wedge \sim q$

(ii) $\sim[p \to (q \wedge r)] \equiv p \wedge \sim(q \wedge r)$ [using $\sim(p \to q) \equiv p \wedge \sim q$]

$\equiv p \wedge (\sim q \vee \sim r)$ [using De'Morgan's laws]

$\therefore \sim[p \to (q \wedge r)] \equiv p \wedge (\sim q \vee \sim r)$

(iii) $\sim(p \wedge \sim q) \equiv \sim p \vee \sim(\sim q)$ [using De'Morgan's law]

$\equiv \sim p \vee q$ [using $\sim(\sim p) \equiv p$]

$\therefore \sim(p \wedge \sim q) \equiv \sim p \vee q$

(iv) $\sim(p \to \sim q) \equiv p \wedge \sim(\sim q)$ [using $\sim(p \to q) \equiv p \wedge \sim q$]

$\equiv p \wedge q$ [using $\sim(\sim p) \equiv p$]

$\therefore \sim(p \to \sim q) \equiv p \wedge q$

Example 16:

Negate the proposition:

"cow is big and it is black".

Solution:

Let p : cow is big

q : cow is black

Given proposition is "$p \wedge q$"

we know that: $\sim (p \wedge q) \equiv \sim p \vee \sim q$

Therefore, negation of the given proposition is "cow is not big or it is not black".

Example 17:

Negate the proposition "If the number is real then it is either rational or irrational"

Solution:

Let p : The number is real.

q : The number is rational.

r : The number is irrational.

Then, given proposition is $p \rightarrow (q \vee r)$

$\sim [p \rightarrow (q \vee r)] \equiv p \wedge (\sim q \wedge \sim r)$

Thus, the negation of the given proposition is "A number is real and it is not rational and not irrational."

This may also be stated as "A number is real but it is neither rational nor irrational".

Example 18:

Negate : "If he is rich then he is happy".

Solution:

Let p : He is rich, q : He is happy.

we are given : $p \rightarrow q$

$\sim (p \rightarrow q) \equiv p \wedge \sim q$

we therefore get the negation as: "He is rich and he is not happy".

Example 19:

Negate : "6 is an even number or $\sqrt{5}$ is not rational".

Solution:

Let p : 6 is an even number.

q : $\sqrt{5}$ is rational.

The given proposition is $p \vee \sim q$

Then, $\sim (p \vee \sim q) \equiv \sim p \wedge q$

$\therefore$ the negation of the given statement is "6 is not an even number and $\sqrt{5}$ is rational".

Example 20 :

Negate : "A triangle is equiangular if and only if the corresponding sides are equal".

Solution:

Consider p : A triangle is equiangular.

q : It's corresponding sides are equal.

we are given $p \leftrightarrow q$

we know that $\sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$. Hence, the negation of the proposition becomes "a triangle is equiangular and the corresponding sides are not equal or the triangle is not equiangular and the corresponding sides are equal".

6.6 Converse, Inverse and Contrapositive of a Conditional:

Given a conditional $p \rightarrow q$,

(i) the conditional $q \rightarrow p$ is called the converse of $p \rightarrow q$

(ii) the conditional $\sim p \rightarrow \sim q$ is called the inverse of $p \rightarrow q$

(iii) the conditional is $\sim q \rightarrow \sim p$ is called the contrapositive of $p \rightarrow q$

It may easily be verified that $q \rightarrow p \equiv \sim p \rightarrow \sim q$ and $p \rightarrow q \equiv \sim q \rightarrow \sim p$

Worked Examples

Example 21 :

Write the converse, inverse and contrapositive of "If I work hard then I get the grade.

Solution:

p : I work hard

q : I get the grade

$p \rightarrow q$ is the given proposition.

converse : $q \rightarrow p$

ie., If I get the grade then I work hard.

inverse : $\sim p \rightarrow \sim q$

ie., If I do not work hard then I do not get the grade.

contrapositive : $\sim q \rightarrow \sim p$

ie., If I do not get the grade then I do not work hard.

Example 22 :

Write the converse, inverse and contrapositive of the implication "If $x \in (A \cup B)$ then $x \in A$ or $x \in B$"

Solution:

Let p : $x \in A \cup B$

q : $x \in A$

r : $x \in B$

$p \rightarrow (q \vee r)$ is the given proposition.

converse : $(q \vee r) \rightarrow p$

ie., If $x \in A$ or $x \in B$ then $x \in (A \cup B)$

inverse : $\sim p \rightarrow \sim (q \vee r) \equiv \sim p \rightarrow (\sim q \wedge \sim r)$

ie., If $x \notin (A \cup B)$ then $x \notin A$ and $x \notin B$

contrapositive : $\sim (q \vee r) \rightarrow \sim p \equiv (\sim q \wedge \sim r) \rightarrow \sim p$

ie., If $x \notin A$ and $x \notin B$ then $x \notin (A \cup B)$.

Example 23 :

Write the converse, inverse and contrapositive of "If *x* is less than 1 then it is a prime number".

Solution:

p : *x* is less than 1

q : *x* is a prime number

The given proposition is $p \rightarrow q$

converse : $q \rightarrow p$

ie., If *x* is a prime number then it is less than 1.

inverse : $\sim p \rightarrow \sim q$

ie., If *x* is not less than 1 then it is not a prime number.

contrapositive : $\sim q \rightarrow \sim p$

ie., If *x* is not a prime number then it is not less than 1.

EXERCISE 6.2

One mark questions:

1) Negate the following propositions:
   (i) $p \vee \sim q$ (ii) $\sim p \rightarrow q$ (iii) $\sim p \wedge \sim q$
   (iv) $p \wedge \sim q$ (v) $\sim p \rightarrow \sim q$
2) Negate the following:
   (i) 4 is an even integer or 7 is a prime number.
   (ii) He likes to run and he does not like to sit.
   (iii) He likes Mathematics and he does not like Logic.
   (iv) If 6 is a divisor of 120 then 486 is not divisible by 6.
   (v) If 2 triangles are similar then their areas are equal.
   (vi) It is cold or it is raining.

Two marks questions:

1) Negate :
   (i) $p \rightarrow (q \wedge r)$ (ii) $q \vee [\sim (p \wedge r)]$
   (iii) $(p \rightarrow q) \wedge (q \rightarrow p)$ (iv) $p \rightarrow (q \wedge \sim r)$
2) Negate the following:
   (i) If an integer is greater than 3 and less than 5 then it is a multiple of 5.
   (ii) If 'x' is divisible by 'y' then it is divisible by 'a' and 'b'.
   (iii) Weather is fine and my friends are not coming or we do not go to a movie.
   (iv) If a triangle is equilateral then it's sides are equal and angles are equal.
   (v) 14 is a divisor of 48 and 28 is not divisible by 82.
3) Determine whether the following propositions is a Tautology or a contradiction or neither.
   (i) $(p \wedge q) \wedge \sim p$ (ii) $[\sim p \wedge (p \vee q)]$ (iii) $(p \wedge q) \rightarrow (p \vee q)$
   (iv) $(p \wedge q) \rightarrow p$ (v) $\sim p \wedge \sim q$

Three marks questions:

1) Write the converse, inverse and contrapositive of the implications given below:
   (i) If $x(x-2) = 0$ then $x = 2$
   (ii) If $x \in A \cap B$ then $x \in A$ and $x \in B$
   (iii) If the questions are easy then students score better marks.
   (iv) If I get a seat then I will watch a cinema and have fun.

(v) If oxygen is a gas then accountancy is easy or the child is brave.

(vi) If $x^2 = y^2$ then $x = y$

(vii) If 2 straight lines are parallel then they do not intersect.

Five marks questions:

1) Check whether the following propositions is a Tautology or a contradiction:

(i) $(p \wedge \sim q) \rightarrow (p \wedge q)$ (ii) $[\sim p \wedge (p \vee q)] \rightarrow q$

(iii) $(p \rightarrow q) \leftrightarrow (\sim p \rightarrow \sim q)$ (iv) $[\sim (p \rightarrow \sim q)] \vee (\sim p \leftrightarrow q)$

(v) $(\sim p \vee q) \leftrightarrow (p \vee \sim q)$

2) Show that $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ is a Tautology.

3) Show that $\sim (p \vee q) \rightarrow (\sim p \wedge \sim q)$ is a Tautology.

4) Prove : $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a Tautology.

5) Prove that $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.

6) Show that $(\sim p \wedge q) \wedge (q \wedge r) \wedge (\sim q)$ is a contradiction.

7) Examine whether the following are logically equivalent:

(i) $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ (ii) $p \rightarrow (q \rightarrow r)$ and $(p \rightarrow q) \rightarrow r$

(iii) $(p \wedge \sim q) \vee q$ and $p \vee q$ (iv) $p \leftrightarrow q$ and $(\sim p \vee q) \wedge (\sim q \vee p)$

(v) $p \wedge q$ and $\sim (p \rightarrow \sim q)$ (vi) $\sim (p \leftrightarrow q)$ and $(p \wedge \sim q) \vee (q \wedge \sim p)$

(vii) $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$

ANSWERS 6.2

One mark questions:

1) (i) $\sim p \wedge q$ (ii) $\sim p \wedge \sim q$ (iii) $p \vee q$

(iv) $\sim p \vee q$ (v) $\sim p \wedge q$

2) (i) 4 is not an even integer and 7 is not a prime number.

(ii) He does not like to run or he likes to sit.

(iii) He does not like Mathematics or he likes Logic.

(iv) 6 is a divisor of 120 and 486 is divisible by 6.

(v) 2 triangles are similar and their areas are not equal.

(vi) It is not cold and it is not raining. (It is neither cold nor raining)

Two marks questions:

1) (i) $p \wedge (\sim q \vee \sim r)$ (ii) $\sim q \wedge (p \wedge r)$

(iii) $(p \wedge \sim q) \vee (q \wedge \sim p)$ (iv) $p \wedge (\sim q \vee r)$

2) (i) An integer is greater than 3 and less than 5 but is not a multiple of 5.

(ii) 'x' is divisible by 'y' and it is not divisible by 'a' and 'b'.

(iii) Weather i not fine or my friends are coming and we go to a movie.

(iv) A triangle is equilateral and it's sides are not equal or angles are not equal.

(A triangle is equilateral and neither it's sides nor angles are equal).

(v) 14 is not a divisor of 48 or 28 is divisible by 82.

3) (i) contradiction (ii) neither (iii) tautology

(iv) tautology (v) neither

Three marks questions:

1) (i) converse : If $x = 2$ then $x(x-2) = 0$

inverse : If $x(x-2) \neq 0$ then $x \neq 2$

contrapositive: If $x \neq 2$ then $x(x-2) \neq 0$

(ii) converse : If $x \in A$ and $x \in B$ then $x \in A \cap B$

inverse : If $x \notin A \cap B$ then $x \notin A$ or $x \notin B$

contrapositive: If $x \notin A$ or $x \notin B$ then $x \notin A \cap B$

(iii) converse : If the students score better marks then questions are easy.

inverse : If the questions are not easy then students do not score better marks.

contrapositive: If the students do not score better marks then questions are not easy.

(iv) converse : If I watch a cinema and have fun then I get a seat.

inverse : If I do not get a seat then I will not watch a cinema or not have fun.

contrapositive: If I will not watch a cinema or not have fun then I do not get a seat.

(v) converse : If accountancy is easy or the child is brave then oxygen is a gas.

inverse : If oxygen is not a gas then accountancy is not easy and the child is not brave.

contrapositive: If accountancy is not easy and the child is not brave then oxygen is not a gas.

(vi) converse : If $x = y$ then $x^2 = y^2$

inverse : If $x^2 \neq y^2$ then $x \neq y$

contrapositive: If $x \neq y$ then $x^2 \neq y^2$

(vii) converse : If 2 straight lines do not intersect then they are parallel.

inverse : If 2 straight lines are not parallel then they intersect.

contrapositive: If 2 straight lines intersect then they are not parallel.

Five marks questions:

1) (i) Neither (ii) Tautology (iii) neither

(iv) neither (v) neither

7) (i) Logically equivalent (ii) Not Logically equivalent

(iii) Logically equivalent (iv) Logically equivalent

(v) Logically equivalent (vi) Logically equivalent

(vii) Logically equivalent

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UNIT II - COMMERCIAL ARITHMETIC

Chapter	Title	No. of Teaching hrs.
7.	RATIOS AND PROPORTIONS	10 hrs
8.	BILL DISCOUNTING	06 hrs
9.	STOCKS AND SHARES	04 hrs
10.	LEARNING CURVE	04 hrs
11.	LINEAR PROGRAMMING PROBLEMS	06 hrs
12.	SALES TAX AND VALUE ADDED TAX	04 hrs
	TOTAL TEACHING HOURS	**34 hrs**

Chapter

7

RATIO AND PROPORTIONS

7.1 Introduction:

Ratio, and proportions are extensively used in many branches of Science like Physics, Chemistry and Mathematics etc.

7.2 Ratio

Ratio is a relationship between two quantities of the same kind with respect to their magnitude and denotes how many times one of the quantities is contained in the other.

Expression of a ratio:

If there are two quantities a and b, the relationship between them can be expressed as a fraction $\frac{a}{b}$ where in the relationship between 'a' and 'b' is expressed as 'how many times 'a' is to 'b' = $a : b$

Antecedent and Consequent

If $\frac{p}{q}$ is a ratio then p is called as **antecedent** and q is called as the **consequent.**

Example 1 :

If $\frac{2}{3}$ is a ratio. Find the antecedent and consequent.

Solution :

If $\frac{2}{3}$ is a ratio

2 = antecedent

3 = consequent

Example 2 :

x gets a salary of ₹15000, y gets a salary of ₹5000. Find the ratio of their salaries.

Solution :

$$\frac{x}{y} = \frac{15000}{5000} = \frac{3}{1} = 3:1$$

Rules Related to Ratios:

I. Inverse Ratio (Reciprocal Ratio)

If $\frac{p}{q}$ is a ratio then $\frac{q}{p}$ is called as the inverse ratio of $\frac{p}{q}$.

Example 3 : Find the inverse ratio of 2:3

Solution :

Inverse Ratio of 2:3 = 3 :2

Example 4 :

A house consumes 20 kgs of rice and 5 kgs of wheat. Compare the consumption of rice and wheat in the form of the ratio.

Solution :

$$\frac{\text{Rice}}{\text{Wheat}} = \frac{20}{5} = \frac{4}{1} = 4:1$$

Example 5 :

Mr. x completes a job in 2 hours and Mr. y completes the same job in 40 minutes. Find the ratio in time taken.

Solution :

x has taken 2 hrs = 120 minutes

y has taken 40 minutes

$$\frac{x}{y} = \frac{120}{40} = \frac{3}{1} = 3:1$$

II. Ratio of greater inequality:

A ratio of greater inequality is one in which the value of antecedent is greater than the value of consequent. Thus $\frac{p}{q}$ would be a ratio of greater inequality if $p > q$.

Ex : 3:2, 5:4.

III. Ratio of lesser inequality:

A ratio of lesser inequality is one in which the value of antecedent is lesser than the value of the consequent. Thus $\frac{p}{q}$ would be a ratio of lesser inequality if $p < q$ **Ex :** 2:3, 4:5

IV. Compound ratio:

A compound ratio is one, which is obtained by multiplying the antecedent term with the antecedent term, and the subsequent consequent term with the consequent terms of the given ratios.

Example 6 :

Find the compound ratio of 3:5 and 4:7

Solution :

$$\frac{3}{5} \times \frac{4}{7} = \frac{12}{35} = 12:35$$

Example 7 :

Find the compound ratio of the ratios 1 : 2, 2 : 3 and 3 : 4

Solution :

$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4} \text{ or } 1:4$$

Example 8 :

Determine the compound ratio of

(i) 1:2, 2:3, 3:4 and 4:5

(ii) 3:5, 2:7, 1:3 and 4:9

(iii) 6 : $\frac{1}{3}$, $\frac{1}{3}$: 3, 5 : 3 and 7 : 8

Solution :

(i) $\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{1}{5}$ or 1:5

(ii) $\frac{3}{5}\times\frac{2}{7}\times\frac{1}{3}\times\frac{4}{9}=\frac{8}{315}$ or 8:315

(iii) $\frac{6}{\frac{1}{3}}\times\frac{\frac{1}{3}}{3}\times\frac{5}{3}\times\frac{7}{8}$

$= \frac{18}{1}\times\frac{1}{9}\times\frac{5}{3}\times\frac{7}{8}$

$= \frac{35}{12}$ or 35 : 12

V. Duplicate ratio:

A duplicate ratio is one which is obtained by taking the square of the respective terms of the given ratio. i.e., duplicate ratio of 3 : 4 is $3^2 : 4^2$ or 9 : 16

VI. Subduplicate ratio:

A Subduplicate ratio is the one which is obtained by taking the square roots of the respective terms of a given ratio. Thus if the given ratio is 4 : 9, then the sub-duplicate ratio is $\sqrt{4} : \sqrt{9}$ or 2 : 3

VII. Triplicate ratio:

A triplicate ratio is the one which is obtained by taking the cubes of the respective terms of a ratio. Thus if the given ratio is 1 : 2 then the triplicate ratio would be $1^3 : 2^3$ or 1 : 8

VIII. Subtriplicate ratio:

A Subtriplicate ratio is the one which is obtained by taking the cube roots of the respective terms in the given ratio. Thus if the given ratio is 8 : 27 then its sub-triplicate ratio would be $\sqrt[3]{8} : \sqrt[3]{27} = 2 : 3$

IX. Continued ratio:

A continued ratio is the one which is obtained by taking the relation between the magnitude of three or more quantities of the same kind. Thus the ratio of x : y : z and 3 : 4 : 5 are examples of a continued ratio.

For example the monthly incomes of A, B and C are ₹300, ₹500 and ₹800 respectively. The ratio of incomes is 300 : 500 : 800 or 3 : 5 : 8 which is a matter of continued ratio.

Example 9 :

Find the duplicate and triplicate ratios of the following:

(i) 2 : 3 (ii) 5 : 3 (iii) 9 : 4

Solution:

	Given ratio	Duplicate	Triplicate
(i)	2 : 3	4 : 9	8 : 27
(ii)	5 : 3	25 : 9	125 : 27
(iii)	9 : 4	81 : 16	729 : 64

Example 10 :

Find the sub-duplicate and sub-triplicate ratio of 1 : 64

Solution:

Given ratio	Sub-duplicate	sub-triplicate
1 : 64	1 : 8	1 : 4

Example 11 :

Find the ratio between two numbers such as their sum is 50 and their difference is 8.

Solution:

Let the two numbers be x & y

$x + y = 50$ (1)

$x - y = 8$ (2)

Adding eq^n (1) & (2)

$2x = 58$

$\therefore\ x = \frac{58}{2} = 29$

substituting in eq^n (1) we have

$29 + y = 50$

$y = 50 - 29$

$y = 21$

$x : y = 29 : 21$

Example 12 :

A ratio in the lowest terms is 3 : 7. If the difference between the quantities is 24. Find the quantities.

Solution:

Let the two quantities be x & y

$\frac{x}{y} = \frac{3a}{7a} \Rightarrow x = 3a, y = 7a$

$7a - 3a = 4a = 24$

$4a = 24$

$a = 6$

$\therefore x = 3 \times 6 = 18 \qquad y = 7a = 7 \times 6 = 42$

$\therefore$ the 2 quantities are 18 and 42.

Example 13 :

Two numbers are in the ratio 3 : 5 If 5 is added to each, they are in the ratio 22:35. Find the numbers.

Solution:

Let the two numbers be x and y

$\frac{x}{y} = \frac{3}{5}$

$x = 3a \qquad y = 5a$

if 5 is added to each, the new ratio is

$$\frac{3a+5}{5a+5}=\frac{22}{35}$$

$105a + 175 = 110a + 110$

$65 = 5a$

$$a = \frac{65}{5}$$

$a = 13$

$\therefore x = 3a = 3 \times 13 = 39$

$y = 5a = 5 \times 13 = 65$

$\therefore$ the two numbrs are 39 and 65.

Example 14 :

What must be added to each term in the ratio 5 : 6, so that it becomes 8 : 9?

Solution:

Let the number x be added to each term in the ratio 5 : 6

$$\therefore \quad \frac{5+x}{6+x}=\frac{8}{9}$$

$45 + 9x = 48 + 8x$

$x = 48 - 45$

$x = 3$

$\therefore$ Number 3 must be added to each term in the ratio 5 : 6 so that it becomes 8 : 9

Example 15 :

Monthly incomes of A and B are in the ratio 2 : 3 and their monthly expenditure are in the ratio 3 : 5. If each save ₹100 per month. Find the monthly incomes of A and B.

Solution:

Let the monthly incomes be $2x$ and $3x$ respectively, since their savings are ₹ 100 each their expenditure would be $(2x - 100)$ and $(3x - 100)$

$$\frac{2x-100}{3x-100}=\frac{3}{5}$$

$5(2x - 100) = 3(3x - 100)$

$10x - 500 = 9x - 300$

$x =$ ₹ 200

Hence the monthly income of A = 200 × 2 = ₹400

the monthly income of B = 200 × 3 = ₹600

Example 16 :

If $a : b = 2 : 3$

$x : y = 4 : 5$

Find $(5ax + 3by) = (10ax + 4by)$

Solution :

$$\frac{a}{b} = \frac{2}{3} \Rightarrow a = \frac{2b}{3} \text{ and } \frac{x}{y} = \frac{4}{5} \quad \therefore x = \frac{4y}{5}$$

$$\frac{5ax + 3by}{10ax + 4by} = \frac{\left(5 \times \frac{2b}{3} \times \frac{4y}{5}\right) + 3by}{\left(10 \times \frac{2b}{3} \times \frac{4y}{5}\right) + 4by}$$

$$= \frac{\frac{8by}{3} + 3by}{\frac{16by}{3} + 4by}$$

$$= \frac{\frac{17by}{3}}{\frac{28by}{3}} = \frac{17}{28} \text{ or } 17 : 28$$

Example 17 :

Find the value of x if $32 : x = 75 : 50$

Solution : $\frac{32}{x} = \frac{75}{50}$

$$x = \frac{32 \times 50}{75} = \frac{64}{3} = 21.33$$

Example 18 :

If $a : b = 2 : 3$ and $b : c = 5 : 7$ and $c : d = 3 : 1$ find $a : d$

Solution :

$a : b = 2 : 3 \qquad b : c = 5 : 7 \qquad c : d = 3 : 1$

$$\frac{a}{d} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{2}{3} \times \frac{5}{7} \times \frac{3}{1} = \frac{10}{7} \quad \therefore a : d = 10 : 7$$

Example 19 :

The angles of a triangle in the ratio 2 : 3 : 4. Find the angles.

Solution :

Let the angles be $2x$, $3x$ and $4x$

$2x + 3x + 4x = 180$

$9x = 180$

$x = 20^o$

$\therefore$ the 1st angle $= 2 \times 20^o = 40^o$

2nd angle $= 3 \times 20^o = 60^o$

3rd angle $= 4 \times 20^o = 80^o$

Example 20 :

An article is sold at 20% gain on the cost price. Find the ratio of the selling price and cost price.

Solution : Let the cost price of the article $= x$

$$\text{Profit in Rs} = 20\% \text{ of } x = \frac{20}{100} \quad x = \frac{x}{5}$$

Selling price = cost price + profit

$$\text{Selling price} = x + \frac{x}{5} = \frac{6x}{5}$$

$$\text{SP : CP} = \frac{6x}{5} : x$$

Ratio of selling price to cost price

$$= \frac{\left(\frac{6x}{5}\right)}{x} = \frac{6}{5} = 6:5$$

Example 21 :

Divide ₹ 6000 in the ratio 3 : 4 : 5

Solution :

Let the three amounts are $3x$, $4x$ and $5x$

$3x + 4x + 5x = 6000$

$12x = 6000$

$$\therefore \quad x = \frac{6000}{12} = 500$$

The three amounts are

$3 \times 500 = ₹1500$

$4 \times 500 = ₹2000$

$5 \times 500 = ₹2500$

Example 22 :

If $x : y = 2 : 3$ find $\dfrac{2x^2 + 5y^2}{x^2 + y^2}$

Solution :

$$\frac{x}{y} = \frac{2}{3}$$

$$\frac{x^2}{y^2} = \frac{4}{9} \qquad \therefore \; x^2 = \frac{4}{9} y^2$$

$$\therefore \quad \frac{2x^2 + 5y^2}{x^2 + y^2} = \frac{\left(2 \times \frac{4}{9} y^2\right) + 5y^2}{\left(\frac{4}{9} y^2\right) + y^2}$$

$$= \frac{\frac{8}{9} y^2 + 5y^2}{\frac{4}{9} y^2 + y^2}$$

$$= \frac{53y^2}{13y^2} = \frac{53}{13}$$

$$\therefore \quad \frac{2x^2 + 5y^2}{x^2 + y^2} = \frac{53}{13} \text{ or } 53 : 13$$

Example 23 :

Distribute 632 amongst A, B and C in such a way that 'B' will have 20% more than 'A' and 'C' has 20% less than 'B'.

Solution :

Express the money received by B and C as ratio

i.e., B receives $= \dfrac{120}{100}$ A , If A = 100 then B = 120

$\therefore \frac{A}{B} = \frac{100}{120} = \frac{5}{6}$

C receives $= \frac{80}{100}$ B If B = 100, then C = 80

$\frac{B}{c} = \frac{100}{80} = \frac{5}{4}$

A : B = 5 : 6, B : C = 5 : 4

A : B, B : C

5 : 6, 5 : 4

Multiply by 5, Multiply by 6

25 : 30, 30 : 24

$\therefore$ A : B : C = 25 : 30 : 24

$25x + 30x + 24x = 632$

$79x = 632$

$x = \frac{632}{79}$ $\therefore x = 8$

$\therefore$ A receives $25 \times 8 =$ ₹ 200

A receives $30 \times 8 =$ ₹ 240

C receives $24 \times 8 =$ ₹ 192

Example 24 :

Rajeev planned his journey to Mumbai as follows. He will travel $\frac{5}{9}$ th of the total distance by an aeroplane $\frac{3}{4}$ th of the remaining by Train and the remaining distance of 200 km by a car. What is the total distance to Mumbai?

Solution :

Let the total distance be x

Aeroplane travel $\frac{5}{9}\ x\ km$

Remaining distance $= \left(x - \frac{5}{9}x\right)\ km$

$= \frac{4}{9}\ x\ km$

He covers $\left(\frac{3}{4}\times\frac{4}{9}\ x\right)\ km$ by train $= \frac{1}{3}\ x\ km$ by train

He covers 200 km by car

Total Distance travelled by (Aeroplane + train + car)

$\therefore$ Total distance $x = \frac{5}{9}x + \frac{x}{3} + 200$

$x - 200 = \frac{5x + 3x}{9}$

$9x - 1800 = 8x$

$x = 1800$ km

$\therefore$ The total distance to Mumbai 1800 km.

Example 25 :

Divide 1,647 into three parts such that $\frac{3}{7}$ th of the first, $\frac{2}{3}$ rd of the second and $\frac{4}{5}$ th of the third are equal.

Solution :

Let the 3 parts be A, B and C

$\frac{3}{7}\ A = \frac{2}{3}\ B$

$\frac{A}{B} = \frac{2}{3} \times \frac{7}{3}$

$\frac{A}{B} = \frac{14}{9}$

$\therefore$ A : B = 14 : 9

and $\frac{2}{3}\ B = \frac{4}{5}\ C$

$\frac{B}{C} = \frac{4}{5}\times\frac{3}{2} = \frac{12}{10} = \frac{6}{5} = 6:5$

A : B = 14 : 9 multiple by 2 $\therefore$ A : B = 28 : 18

B : C = 6 : 5 multiple by 3 $\therefore$ B : C = 18 : 15

$\therefore$ A : B : C = 28 : 18 : 15

$\therefore\ 28x + 18x + 15x = 1647$

$61x = 1647$

$$x = \frac{1647}{61}$$

$\therefore\ x = 27$

$\therefore$ A receives $= 28 \times 27 =$ ₹ 756

B receives $= 18 \times 27 =$ ₹ 486

C receives $= 15 \times 27 =$ ₹ 405

Example 26 :

x, y and z play cricket. The runs scored by x and y are in the ratio of 3 : 2. y's runs to z's runs are in the ratio 3 : 2. Together they all score 342 runs. How many runs did each score?

Solution :

$x : y = 3 : 2$

$y : z = 3 : 2$

$x : y : z = 9 : 6 : 4$

$\therefore\ 9x + 6x + 4x = 342$

$19x = 342$

$$x = \frac{342}{19}$$

$x = 18$

$\therefore\ x$ scores $= 9 \times 18 = 162$ runs

y scores $= 6 \times 18 = 108$ runs

z scores $= 4 \times 18 = 72$ runs

Example 27 :

Three numbers are in the ratio 2 : 3 : 4. If the sum of their squares is 1856. Find the numbers.

Solution :

Let the 3 numbers be $2x$, $3x$ and $4x$

$(2x)^2 + (3x)^2 + (4x)^2 = 1856 \qquad \therefore\ 29x^2 = 1856$

$$x^2 = \frac{1856}{29}$$

$x^2 = 64 \qquad \therefore\ x = 8$

$\therefore$ the 3 numbers are

1st number $2 \times 8 = 16$

2nd number $3 \times 8 = 24$

3rd number $4 \times 8 = 32$

Example 28 :

₹ 5625 is divided among A, B and C so that A receives one half as much as B and C together receive and B receives one fourth of what A and C together receive. Find the share of A, B and C

Solution:

Given $A + B + C = 5625$ ..... (1)

and $A = \frac{1}{2}(B + C)$

$2A = B + C$ ..... (2)

$B = \frac{1}{4}(A+C)$

$4B = A + C$ ..... (3)

from (1) & (2) we have

But $A + B + C = 5625$

$A + 2A = 5625$

$3A = 5625$

$\therefore \quad A = \frac{5625}{3}$

$A = ₹1875$

$\therefore$ A'S share is ₹1875

from (1) & (3) we have

$B + 4B = 5625$

$5B = 5625$

$B = \frac{5625}{5} \qquad \therefore B = ₹1125$

$\therefore$ A'S share is ₹ 1875

B's share is ₹ 1125

C's share is ₹ 2625

Example 29 :

The monthly incomes of A and B are in the ratio 9 : 7 and those of B and C are is the ratio 3 : 2. If 10% of A's income and 15% of C's incomes differ by Rs. 18. Find the incomes of A, B and C

Solution :

Let the income of A and B be $9x$ and $7x$

$B : C = 3 : 2 \Rightarrow \frac{B}{C} = \frac{3}{2} \quad \therefore C = \frac{2B}{3}$

$\therefore$ C's income $= \frac{2}{3} \times 7x = \frac{14x}{3}$

10% of A's income $= \frac{10}{100} \times 9x = 0.9x$

15% of C's income $= \frac{15}{100} \times \frac{14x}{3} = 0.7x$

10% of A's income - 15% C income = ₹18

$0.9x - 0.7x = 18$

$0.2x = 18$

$x = \frac{18}{2}$

$x = \frac{180}{2}$

$x =$ ₹90

Hence A's income $= 9 \times 90 =$ ₹ 810

B's income $= 7 \times 90 =$ ₹ 630

c's income is $= \frac{14}{3} \times 90 =$ ₹ 420

Example 30 :

The ratio of prices of two house was 16 : 23 two years later when the price of the first had risen by 10% and that of second by ₹ 477, the ratio of their prices becomes 11 : 20. Find the original prices of the two houses.

Solution :

Let the prices of the two houses be $16x$ and $23x$

$$\therefore \quad \frac{16x+1.6x}{23x+47}=\frac{11}{20} \qquad \left(\therefore\ 10\%\ \text{of}\ 16x=\frac{10}{100}\times 16x=1.6x\right)$$

$$320x + 32x = 253x + 5247$$

$$352x = 253x + 5247$$

$$99x = 5247$$

$$x = \frac{5247}{99}$$

$$x = 53$$

The original price of 1st house = $16 \times 53 = ₹\ 848$

The original price of 2nd house = $23 \times 53 = ₹\ 1219$

Exercise 7.1

One mark questions:

1. If 3 : 5 is a ratio, find the antecedent and consequent.
2. x gets a salary of ₹ 20000, y gets a salary of ₹ 5000. Find the ratio of their salaries.
3. Find the inverse ratio of 4 : 5
4. A house consumes 30 kgs of wheat and 4 kg of sugar compare the consumption of wheat and sugar in the form of ratio.
5. Mr. x completes a job is 3 hours and Mr y completes the same job is 45 minutes represent their time in ratio.
6. Find the compound ratio of 3 : 4 and 4 : 7
7. Find the compound ratio of 1 : 2, 2 : 3 and 3 : 5
8. Find the duplicate ratio of 5 : 4
9. Find the triplicate ratio of 3 : 5
10. Find the subduplicate ratio of 9 : 49
11. Find the subtriplicate ratio of 125 : 64
12. Find the value of x of 5 : 20 = 3 : x

2 marks questions

13. Find the ratio between two numbers such that their sum is 40 and their difference is 8.
14. A ratio is the lowest term is 3 : 8. If the difference between the quantities is 25. Find the quantities.
15. Two numbers are is the ratio 3 : 5. If 5 is added to each, they are is the ratio. 2 : 3 find the numbers.

16. What must be added to each term in the ratio 2 : 3 so that it becomes 5 : 6.
17. What must be added to each term in the ratio 4 : 5 so that is becomes 7 : 8.
18. What must be subtracted from each term in the ratio 7 : 4 so that it becomes 5 :2.
19. What must be subtracted from each term in the ratio 8 : 7 so that it becomes 4 : 3.
20. If $a : b = 2 : 3$, $b : c = 3 : 5$ and $c : d = 5 : 7$ find $a : d$.
21. If $a : b = 2 : 3$ and $b : c = 6 : 13$ Find $a : b : c$.
22. If $a : b = 3 : 4$, $b : c = 8 : 15$ Find $a : b : c$.

3 marks questions

23. Divide 1800 is the ratio 3 : 4 : 5
24. If x : y = 3 : 4 find $\frac{2x^2 + 3y^2}{x^2 + y^2}$
25. If a : b = 2 : 3 and x : y = 4 : 7

    Find $\frac{5ax + 4by}{8ax + 3by}$
26. The angles of a triangle are in the ratio 3 : 4 : 5. Find the angles.
27. An article is sold at 40% gain an the cost price. Find the ratio of the selling price and cost price.
28. If the monthly incomes of A and B are is the ratio 3 : 4 and their expenditures are is the ratio 1 : 2. If each saves ₹ 1000 find the monthly incomes.
29. If te monthly incomes of A and B are in the ratio 3 : 4 and their expenditure are in the ratio 1 : 2. If each saves ₹ 2000. Find their monthly incomes.
30. Two numbers are is the ratio 6 : 7. If the difference of their squares to 117. Find the numbers.
31. Two numbers are is the ratio 3 : 4. If the sum of their squares is 900 find the two numbers.
32. If $x : y = 3 : 4$ and $y : z = 7 : 9$ Find x : y : z.

5 marks questions

33. Divide ₹1890 is three parts such that three times of the first, five times of the second and six times the third are equal.
34. Divide ₹3262 among x, y and z such that if ₹ 35, ₹15 and ₹12 are deducted from their respective shares, the remainder are in the ratio 3 : 5 : 8.
35. If $x : y = 2 : 3$ Find the value of $\frac{2x^3 + 3y^3}{x^3 + y^3}$

36. Divide 5880 is to three parts, such that 'B' receive twice as 'A' and C receives $\frac{5}{6}^{th}$ of what B receives.

37. If $\frac{2x^2+3y^2}{x^3+y^2}=\frac{2}{41}$ Fin x : y.

38. Divide 6000 into three parts is the ratio $\frac{1}{2}:\frac{1}{3}:\frac{1}{6}$

39. Divide 17,640 a many P, Q, R and S such that Q gets $\frac{2}{5}^{th}$ of P, R gets $\frac{5}{8}^{th}$ of Q and S gets $\frac{2}{13}^{th}$ of the sum of Q and R.

40. The Rajdhani express takes 18 hours to reach Delhi from Bhubaneshwar while Nilachala Express takes 24 hours for the same of Delhi is 2880 kms from Bhubaneshwar find the ratio between the average speeds of the two trains.

Answers 7.1

1 mark question:

1) Antecedent = 3 , Consequent = 5 2) 4 : 1 3) 5 : 4 4) 15 : 2
5) 5 : 4 6) 3 : 7 7) 1 : 5 8) 25 : 16 9) 27 : 125 10) 3 : 7
11) 5 : 4 12) x = 12

2 marks questions:

13. 3 : 2
14. Quantities are 15 and 40.
15. The numbers are 15 and 25.
16. 3 must be added to each term is the ratio 2 : 3 so that it becomes 5 : 6.
17. 3 must be added to each term is the ratio 4 : 5 so that it becomes 7 : 8.
18. 2 must be subtracted from each term is the ratio 7 : 4 so that it becomes 5 : 2.
19. 4 must be subtracted from each term in the ratio 8 : 7 so that it becomes 4 : 3.
20. $\frac{a}{d}=\frac{2}{3}\times\frac{3}{5}\times\frac{5}{7}=\frac{2}{7}$
21. $a : b : c = 4 : 6 : 13$
22. $a : b : c = 6 : 8 : 15$

3 marks questions:

23. 1st part = ₹ 450, 2nd part = ₹ 600, 3rd part = ₹ 750

24. $\frac{2x^2+3y^2}{x^2+y^2}=\frac{66}{25}$

25. $\frac{124}{127}$

26. 1st angle $3\times15°=45°$, 2nd angle $4\times15°=60°$, 3rd angle $5\times15°=75°$

27. 7 : 5

28. Monthly income of A = ₹ 1500, Monthly income of B = ₹ 2000

29. Monthly income of A = ₹ 3000, Monthly income of B = ₹ 4000

30. 18 and 21 are the 2 numbers

31. 18 and 24 are the 2 numbers

32. $x : y : z = 21 : 28 : 36$.

5 marks questions:

33. 1st part ₹ 900, 2nd part ₹ 540, 3rd part ₹ 450

34. x receives ₹ 635, y receives ₹ 1015, z receives ₹ 1612

35. $\frac{108}{35}$

36. A receives ₹ 1260, B receives ₹ 2520, C receives ₹ 2100

37. $x : y = 5 : 4$

38. A receives ₹ 3000, B receives ₹ 2000, C receives ₹ 1000

39. P gets ₹ 10,080, Q gets ₹ 4032, R gets ₹ 2520, S gets ₹ 1008

40. 4 : 3.

7.3 Proportion :

The term proportion may be definied as equality of two ratios as a : b = c : d which is usually expressed as $\frac{a}{b}=\frac{c}{d}$ or $ad = bc$ or $a : b :: c : d$

For example the ratio 2 : 3 is equal to the ratio 6 : 9 and hence it can be said that 2, 3, 6 and 9 are is proportion.

i.e. $2\times9=3\times6=18$

Rules related to proportion

(i) Mean proportion :

a, b and c are said to be in mean proportion if

$a : b :: b : c$

i.e $b^2 = ac$

ie $b = \sqrt{ac}$

For ex : 1, 3 and 9 are in mean proportion as 1 : 3 : : 3 : 9

$b = \sqrt{1 \times 9} = \sqrt{9} = 3$

(ii) Simple proportion :

When the number of related terms remains within four it is a case of simple proportion.

i.e. $a : b : : c : d$

For example : 2 : 3 : : 8 : 12

(iii) Compound proportion :

When the number of related terms exceed four, it is case of compound proportion

$a : b : : c : d : : e : f$

eg : 2 : 3 : : 8 : 12 : : 6 : 9

(iv) Continued proportion :

A continued proportion is the one is which there is a chain of ratios between the related terms.

i.e. $a : b = b : c = c : d$ and so on

For example 1 : 2 = 2 : 4 = 8 : 16 and so on

(v) Direct proportion :

A direct proportion means a positive co-relation between the related terms where by an increase in the value of one is followed by a proportionate increase in the value of another and a decrease in the value of one is followed by a proportionate decrease in the value of another.

Examples for

(i) The ratio between cost and quantity of articles.

(ii) The height and weight of a person.

(vi) Inverse proportion :

An inverse proportion means a negative co-relation between the two related terms where by an increase in the value of one is followed by a proportionate decrease in the value of another and vice versa.

Examples for

1. The ratio between the number of workers and amount of time
2. The ratio between supply and price

7.4 Properties proportion :

(i) Invertendo :

According to this property If $a : b = c : d$ then invertendo is $b : a = d : c$

Proof : $\frac{a}{b} = \frac{c}{d}$

$$1 \div \frac{a}{b} = 1 \div \frac{c}{d}$$

$$\therefore \boxed{\frac{b}{a} = \frac{d}{c}}$$

(ii) Alternendo :

According to this property If $a : b = c : d$ then $a : c = b : d$

Proof : $\frac{a}{b} = \frac{c}{d}$ Multiplying both the side by $\frac{b}{c}$

$$\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$$

$$\boxed{\frac{a}{c} = \frac{b}{d}}$$

(iii) Componendo:

According this property $a : b = c : d$

then $a + b = b :: c + d = d$ or $\frac{a+b}{b} = \frac{c+d}{d}$

Proof : $\frac{a}{b} = \frac{c}{d}$

Adding 1 to both the sides we have

$$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

$$\boxed{\frac{a+b}{b} = \frac{c+d}{d}}$$

(iv) Dividendo:

According to this property if $a : b :: c : d$

then $a - b = b :: c - d : d$ or $\frac{a-b}{b} = \frac{c-d}{d}$

Proof $\frac{a}{b} = \frac{c}{d}$

Subtracting 1 from both the sides

$\frac{a}{b} - 1 = \frac{c}{d} - 1$

$$\boxed{\frac{a-b}{b} = \frac{c-d}{d}}$$

(v) Componendo and Dividendo:

According to this property

If $a : b : : c : d$ then

$a + b = a - b : : c + d : c - d$ or $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Proof By componendo if a : b : : c : d

then $\frac{a+b}{b} = \frac{c+d}{d}$ ------ (1)

By dividendo if $\frac{a}{b} = \frac{c}{d}$

then $\frac{a-b}{b} = \frac{c-d}{d}$ ------ (2)

Dividing componendo by dividendo

We have $\boxed{\frac{a+b}{a-b} = \frac{c+d}{c-d}}$

Example 1 : Find the fourth proportional of 6, 14, 15

Solution : 6 : 14 : : 15 : x

$6x = 15 \times 14$

$x = \frac{15 \times 14}{6} = \frac{5 \times 14}{2}$

$x = 35$

Example 2 : Find the mean proportional of 9 and 16

Solution : 9 : x : : x : 16

$x^2 = 144$ $\therefore x = 12$

Example 3 :

Find the 3rd proportional of 6 and 24

Solution : $6 : 24 : : 24 : x$

$$6x = 24 \times 24$$

$$x = 4 \times 24$$

$$x = 96$$

Example 4 :

If $a : b = 4 : 5$. Find $\dfrac{3a+2b}{3a-2b}$

Solution :

Let $a = 4k,\ b = 5k$

$$\therefore \quad \frac{3(4k)+2(5k)}{3(4k)-2(5k)} = \frac{12k+10k}{12k-10k} = \frac{22k}{2k} = 11$$

Example 5 :

If a : 3 : 15 = 5 : b : 5. Find the values of a and b

Solution :

$$\frac{a}{5} = \frac{3}{b} = \frac{15}{5} = 3 \qquad \therefore \quad \frac{a}{5} = 3 \Rightarrow \therefore a = 15$$

$$\frac{3}{b} = 3 \Rightarrow b = 1$$

Example 6 :

If $a + b = a - b = 4 : 3$ Find the value of a and b

Solution :

$$\frac{a+b}{a-b} = \frac{4}{3}$$

Applying componendo and dividendo

We have, $\dfrac{a+b+a-b}{a+b-a+b} = \dfrac{4+3}{4-3}$

$$\frac{2a}{2b} = \frac{7}{1} \qquad \therefore \quad \frac{a}{b} = \frac{7}{1}$$

$\therefore$ $a = 7$ and $b = 1$

Example 7 :

If $\frac{a}{b}=\frac{c}{d}$ prove that $\frac{2a+7b}{2c+7d}=\frac{2a-7b}{2c-7d}$

Solution :

If $\frac{a}{b}=\frac{c}{d}$

Multiplying by $\frac{2}{7}$ on both the sides

We have $\frac{2a}{7b}=\frac{2c}{7d}$

Applying componendo and dividendo we have

$\frac{2a+7b}{2a-7b}=\frac{2c+7d}{2c-7d}$ $\quad\therefore\quad \frac{2a+7b}{2c+7d}=\frac{2a-7b}{2c-7d}$

Example 8 :

If $\frac{2a+2b-3c-3d}{2a-2b-3c+3d}=\frac{a+b-4c-4d}{a-b-4c+4d}$ the prove that *a : b : : c : d*

Solution :

By applying componendo and dividendo we have

$$\frac{(2a+2b-3c-3d)+(2a-2b-3c+3d)}{(2a+2b-3c-3d)-(2a-2b-3c+3d}=\frac{(a+b-4c-4d)+(a+b-4c+4d)}{(a+b-4c-4d)-(a-b-4c+4d)}$$

$$\frac{4a-6c}{4b-6d}=\frac{2a-8c}{2b-8d}$$

OR

$\Rightarrow \frac{2a-3c}{2b-3d}=\frac{a-4c}{b-4d}$ By applying alternendo

$\Rightarrow \frac{2a-3c}{a-4c}=\frac{2b-3d}{b-4d}$

Subtracting 2 from both the sides

We have $\frac{2a-3c}{a-4c}-2=\frac{2b-3d}{b-4d}-2$

$$\frac{\cancel{2a} - 3c - \cancel{2a} + 8c}{a - 4c} = \frac{\cancel{2b} - 3d - \cancel{2b} + 8d}{b - 4d}$$

$$\frac{\cancel{5}c}{a - 4c} = \frac{\cancel{5}d}{b - 4d}$$

$$\frac{c}{a - 4c} = \frac{d}{b - 4d}$$

by applying Inventendo

$$\frac{a - 4c}{c} = \frac{b - cd}{d}$$

adding 4 to both the sides

$$\frac{a - 4c}{c} + 4 = \frac{b - 4d}{d} \neq 4$$

$$\frac{a - \cancel{4c} + 4c}{c} = \frac{b - \cancel{4d} + \cancel{4d}}{d} \qquad \therefore \ \frac{a}{c} = \frac{b}{d}$$

By applying alternendo we have $\boxed{\frac{a}{b} = \frac{c}{d}}$

Example 9 :

Four number are in proportion. The sum of the extremes is 54 and the sum of the means is 36. If the ratio of their means is 2 : 1 Find the numbers.

Solution :

Let the numbers be *a, b, c, d*

$a : b : : c : d$

$a + d = 54$ .....(1)

$b + c = 36$ .....(2)

Given $\frac{b}{c} = \frac{2}{1}$ $\therefore$ $b = 2c$ in (2)

$3c = 36$ $\therefore$ $c = 12$ and $b = 24$

$\therefore$ $bc = 288 = ad$

$\therefore \ a = \frac{288}{d}$ .....(3)

(3) in (1), $\frac{288}{d} + d = 54$

$288 + d^2 = 54d$

$d^2 - 54d + 288 = 0$

$d^2 - 48d + 68d + 288 = 0$

$d(d-48) - 6(d-48) = 0$

$(d-6)(d-48) = 0$

$d = 6$ or 48

$a + d = 54$

If $d = 6$ $\quad a = 48$

If $d = 48$ $\quad a = 6$

$\therefore$ a, b, c, d are 6, 24, 12, 48
OR
a, b, c, d are 48, 24, 12, 6

Example 10 :

Evaluate $\frac{x+a}{x-a} + \frac{x+b}{x-b}$ **where** $x = \frac{2ab}{a+b}$

Solution :

$$\frac{x+a}{x-a} + \frac{x+b}{x-b}$$

By applying componendo and dividendo
We have

$$\frac{x+\cancel{a}+x-\cancel{a}}{\cancel{x}+a-\cancel{x}+a} + \frac{x+\cancel{b}+x-\cancel{b}}{\cancel{x}+b-\cancel{x}+b}$$

$$\frac{2x}{2a} + \frac{2x}{2b} \qquad \frac{x}{a} + \frac{x}{b}$$

Putting the value of $x = \frac{2ab}{a+b}$ we have

$$\frac{2ab}{a(a+b)} + \frac{2ab}{b(a+b)}$$

$$\frac{2ab^2 + 2a^2b}{ab(a+b)} = \frac{2ab\cancel{(b+a)}}{ab\cancel{(a+b)}} = 2$$

EXERCISE 7.2

One mark questions:

1. State which of the following are proportions.
   (i) $2 : 3 :: 6 : 9$ (ii) $1 : 3 :: 4 : 15$
   (iii) $3 : 5 :: 6 : 15$ (iv) $7 : 9 :: 14 : 18$
2. Find the fourth proportional to each of the following
   (i) 4, 5, 24 (ii) 6, 12, 15
   (iii) 1.5, 4.5, 3.5 (iv) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$
3. Find the mean proportional to each of the following
   (i) 1 and 9 (ii) 2 and 8
   (iii) $\frac{1}{16}$ and $\frac{1}{25}$ (iv) 0.8 and 1.8
4. Find the third proportional to each of the following
   (i) 4, 6 (ii) 3, 12
   (iii) 2.4, 3.6 (iv) $2\frac{2}{3}, 4$

3 marks questions

5. If $\frac{a}{b} = \frac{c}{d}$ then prove that $\frac{3a+5b}{3c+5d} = \frac{3a-5b}{3c-5d}$
6. If $\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}} = \frac{a}{b}$ find x
7. Find the numbers which added to the terms numerator and denominator of 25 : 37 make it 5 : 6.
8. The ages of a father and his for son in the ratio 6 : 1. After 14 years their age will be in the ratio 8 : 3 what are their present ages?

5 marks questions

9. Four numbers formed by adding 1, 5, 10 and 15 to a certain number are in proportion. Find the number?
10. A concrete mixture was made of cement, chips and sand. The ratio of the cement and chip is the same as that of chips and the sand. If 144 bags of cement and 225 bags of sand were used is the mixture. Find the number of bags of chips used in the mixture.

Answers 7.2

1 Mark answers

1. (i) and (iv) are proportions

2. (i) 30 (ii) 30 (iii) 10.5 (iv) $\frac{1}{6}$

3. (i) 3 (ii) 4 (iii) $\frac{1}{20}$ (i) 1.2

4. (i) 9 (ii) 48 (iii) 5.4 (iv) 6

3 Mark answers

6. $x = \frac{2ab}{a^2 + b^2}$

7. 35

8. Father's age = 42 years, son's age = 7

5 Mark answers

9. 36, 40, 45, 50 are the four numbers

10. 180 bags of chips were used in the mixture.

7.5 Time and work, Time and distance and mixtures

Note :

(i) Time to complete, a work varies directly as the amount of work and inversely as the number of workers employed.

(ii) Time taken to travel a certain distance varies directly as the distance but vaires inversely as the speed.

Example 1 :

If ₹120 maintain a family of 4 persons for 30 days. How long ₹300 maintain a family of 6 persons?

Solution :

Person	Money	Days
4	120	30
6	300	x

$120 : 300 = 30 = x$

Money and days varies directly $120 : 300 = 30 : x$

Person & days are ..... inversly $6 : 4 = 30 : x$

$$120 \times 6 \times x = 300 \times 4 \times 30 \Rightarrow \frac{120 \times 6}{300 \times 4} = \frac{30}{x}$$

$$x = \frac{300 \times \cancel{120}}{\cancel{120} \times 6} = 50 \text{ days}$$

Example 2 :

500 workers can finish a work is 8 days. How many workers will finish the same work in 5 days.

Solution :

Worker	Days
500	8
x	5

First ratio of workers is $500 : x$

2nd ratio of days $8 : 5$

Here the number of workers and number of days are inversely proportional

$500 : x : : 5 : 8$

$5x = 500 \times 8$

$x = 800$

$\therefore$ 800 workers will be required to finish the same work is 5 days.

Example 3 :

3 carpenters can earn ₹360 is 6 days working at 9 hours a day. How much will 8 carpenters can earn is 12 days working 6 hours a day?

Solution :

Carpentar	days	hrs.	earning
3	6	9	360
8	12	6	x

1st ratio are carpenters $3 : 8$, 2nd ratio are days $6 : 12$, 3rd ratio are hours $9 : 6$

Here all are direct proportion to $360 : x$

$3 \times 6 \times 9 \times x = 8 \times 12 \times 6 \times 360$

$$x = \frac{\cancel{96}^{32} \times \cancel{6} \times \cancel{360}^{40}}{\cancel{54}_{6} \times \cancel{3}}$$

$x =$ ₹1280

Hence 8 carpenters can earn ₹1280 in 12 days working at the rate of 6 hours per day

Example 4 :

A mixture contains milk and water in the ratio 5 : 1 on adding 5 litres of water, the ratio of milk and water becomes 5 : 2, Find the quantity of milk in the orignial mixture.

Solution :

Let the quantity of milk be $5x$ and water be x

$$\frac{5x}{x+5} = \frac{5}{2}$$

$10x = 5x + 25$

$5x = 25$

$x = 5$

$\therefore$ the quantity of milk $= 5 \times 5 = 25$ litres

Example 5 :

A jar contains two liquids A and B in the ratio 7 : 5 when 9 litres of the mixture is drawn and the jar is filled with the same quantity of B, the ratio of A and B becomes 7 : 9. Find the quantity of A in the jar initially.

Solution :

Let the quantities of the two liquids be $7x$ and $5x$

Quantity of A and B in 9 liter of mixture drawn in $\left(\frac{7}{12}\times 9\right)$ and $\left(\frac{5}{12}\times 9\right)$

$\therefore$ the new ratio of A and B

$$\frac{7x-\left(\frac{7}{12}\times 9\right)}{5x-\left(\frac{5}{12}\times 9\right)+9} = \frac{7}{9}$$

$$\therefore \quad \frac{84x-63}{60x-45+108} = \frac{7}{9}$$

$\therefore x = 3$ liters

$\therefore$ Quantity of A $= 7x = 21$ liters.

Example 6 :

Two taps fill a tank separately is 24 minutes and 40 minutes respectively and a drain pipe can drain off 30 litres per minutes. If all the three pipes are opened the tank fills in 60 minutes. What is the capacity of the tank?

Solution :

Tap 1 can fill $\frac{1}{24}$ th of the tank in 1 minute

Tap 2 can fill $\frac{1}{40}$ th of the tank in 1 mintue

Let the drain pipe empty the tank in 'x' minute

$\frac{1}{x}^{th}$ In one minute the 3 pipes fill will be

$$\left(\frac{1}{24}+\frac{1}{40}-\frac{1}{x}\right)=\frac{1}{60}$$

$$\frac{1}{24}+\frac{1}{40}-\frac{1}{60}=\frac{1}{x}$$

$$\frac{10+6-4}{240}=\frac{1}{x}$$

$$\frac{12}{240}=\frac{1}{x} \qquad \therefore x = 20$$

$\therefore$ the capacity of the tank = $30 \times 20 = 600$ liters.

Example 7 :

If ten persons can do a job in 30 days. In how many days can fifteen persons do the same job?

Solution :

Person	Days
10	30
15	x

Ratio of persons = 10 : 15

Ratio of days = $30 : x \Rightarrow x : 30$ (inverse)

It is inverse proportion because more men will do job in lesser time

$10 : 15 : : x : 30$

$300 = 15x$

$$x = \frac{300}{15}$$

$x = 20$ days

15 persons can do the same job is 20 days

Example 8 :

If 8 men and 16 boys can do a price of work in 6 days and 12 men and 24 boys can do the same work in 8 days. In how many days can 16 men and 20 boys do it.

Solution :

8 men + 16 boys can do the job in 6 days, Multiply by 6

48 men + 96 boys can do the Job in 1 day

Next 12 men and 24 boys can do the same job is 8 days

12 men + 24 boys can do it is 8 days $\times^{ly}$ 8

96 men + 192 boys can do the Same job is 1 day

Hence in doing one day's work it may be stated that

48 men + 96 boys = 96 men + 192 boys

1 man = 2 boys

$\therefore$ 16 men + 20 boys = 32 boys + 20 boys = 52 boys can do the job in how many days.

This is the case of inverse proportion

Let the unknown quantity be x days

Boys	Days
32	6
52	x

32 : 52 : : x : 6

$32 \times 6 = 52\,x$

$$x = \frac{192}{52}$$

x = 3.7 days.

Example 9 :

If two men and four women can do a work is 33 days and 3 men and 5 women can do the same work is 24 days. How long shall 5 men and 2 women take to do the same work?

Solution :

2 men and 4 women can do a work in 33 days

66 men and 132 women can do (by × 33), work in 1 day ..... .(1)

Similarly 3 men and 5 women can do a work is 24 days (× 24)

72 men and 120 women can do a work is 1 day .....(2)

From (1) and (2) we have

66 men + 132 women = 72 men + 120 women

$\therefore$ 1 man = 2 women

5 men + 2 women = 10 women + 2 women = 12 women .....(3)

Let the required number of days be x

2men + 4 women = 4 women + 4 women = 8 women .....(4)

8W : 12W = x : 33

$12x = 33 \times 8$

$3x = 33 \times 2$

x = 22 days

Example 10 :

5 men each working 9 hours a day can finish a work is 30 days. How many men are required to finish eight times the work is 25 days each working 8 hours a day?

Solution :

Men	Hours	days	work
5	9	30	1
x	8	25	8

Hours : 8 : 9 = 5 : x ($\therefore$ more men less hrs. Inverse proportion)

Days 25 : 30 = 5 men : x men ($\therefore$ more men less day Inverse proportion)

work, 1 : 8 = $5x$ ($\therefore$ more work more men direct proportion)

$$x = \frac{\cancel{5} \times 9 \times \cancel{30}^{6} \times 8}{\cancel{8} \times \cancel{25}_{\cancel{5}} \times 1} = 54 \text{ men}$$

$\therefore$ 54 men are required to finish the work

Example 11 :

If 10 men or 20 boys can do a piece of work is 30 days, how long will 30 boys and 5 men take to do the same work?

Solution :

10 men can do a job is 30 days

1 men can do the job is $30 \times 10 = 300$ days

20 men can do the job in = $\frac{300}{20} = 15$ days

$\left[\because 20 \text{ Boys work} = 10 \text{ mens work}, \quad \therefore 1 \text{ Boy's work} = \frac{10}{20} = \frac{1}{2} \text{men}\right]$

$\therefore$ 30 boys = $\frac{1}{2} \times 30 = 15$ men

(5 + 15 = 20 men)

Example 12 :

Two taps can separately fill a tank in 12 min and 15 minutes respectively. The tank when full can be emptied by a drain pipe is 20 minutes. When the tank was empty, all the three were opened simultaneously. In what time will the tank be filled up?

Solution :

1st tap can fill $\frac{1}{12}^{th}$ tank in 1 min.

2nd tap can fill $\frac{1}{15}$ tank in 1 min.

drain pipe drain out $\frac{1}{20}^{th}$ tank in 1 min.

In 1 min $= \left(\frac{1}{12}+\frac{1}{15}-\frac{1}{20}\right)$

$= \frac{15+12-9}{180}$

$= \frac{18}{180} = \frac{1^{th}}{10}$ of tank will get filled

$\therefore$ the tank will get filled in 10 minutes

Example 13 :

Walking 4 kmph a student reaches his college 5 minutes late and if he walks at 5 kmph, he reach is 2½ minites early. What is the distance from his house to the college?

Solution :

Let the distance from his house to the college = x km

Time taken to cover x kms at 4 km ph $= \frac{x}{4}$ hrs.

Time to taken to cover x kms at 5 kmph $= \frac{x}{5}$ hrs

From the given problem - Time taken to cover x km is he reach 5 min late

Time taken to cover x km y he reach 2½ early

$\therefore \quad \frac{x}{4}-\frac{5}{60}=\frac{x}{5}+\frac{2½}{60}$

$\frac{x}{4}-\frac{5}{60}=\frac{x}{5}+\frac{5}{120}$

$$\frac{x}{4}-\frac{x}{5}=\frac{5}{60}+\frac{5}{120}$$

$$\frac{x}{20}=\frac{10+5}{120}$$

$$\frac{x}{2\cancel{0}}=\frac{15}{12\cancel{0}}$$

$$x = \frac{15\times 2}{12} = \frac{30}{12}$$

x = 2.5 km

$\therefore$ the distance from his house to the college = 2.5 km

Example 14 :

Two men 'X' and 'Y' starts from a place 'A' walking at 5 kms and 6 kms per hour respectively. How many kilometer will they be apart at the end of 5 hours. If (a) they walk in opposite directions and (b) they walk in the same direction?

Solution :

(a) If they walk in opposite directions, their relative speed will be 5 + 6 = 11 km ph. In one hour, they will be apart 11 kms. In 5 hous they will be apart by 5 × 11 = 55 kms.

(b) If they walk in the same direction their relative speed = 6–5 = 1 kmph.
In one hour, they will be 1 km apart
In 5 hours, they will be 5 × 1 = 5 kms.
They will be 5 kms apart.

Example 15 :

In a fort, there was ration for 560 soliders that would last the soldiers for 70 days After 20 days, 60 soldiers left the fort. For how many days the remaining ration can support the remaining soldiers?

Solution :

The remaining soldiers = 560 – 60 = 500

The remaining days = 70 – 20 = 50 days

The remaining food would have been sufficient for the soldiers for 50 days.

Since the number of soldiers is reduced to 500, the remaining food would now support them for more than 50 days.

Soldiers	days
560	50
500	x

$\therefore$ It is an Inverse Ratio

$560 : 500 = x : 50$

$$x = \frac{560 \times 50}{500} \qquad x = 56 \text{ days}$$

Hence the food would last for 56 days for the remaining soldiers.

EXERCISE 7.3

3 Mark Questions:

Q1. If ₹ 150 maintains a family of 4 persons for 30 days. How long ₹600 maintain a family of 6 persons?

Q2. 300 workers can finish a work in 8 days. How many workers will finish the same work in 5 days.

Q3. 5 carpenters can earn ₹ 540 in 6 days working 9 hours a day. How much will 8 carpenters can earn in 12 days working 6 hours a day?

Q4. A mixture contains milk and water in the ratio 6 : 1 on adding 5 litres of water, the ratio of milk and water becomes 7 : 2, find the quantity of milk in the original mixture.

5 Mark Questions:

Q5. A jar contains two liquids X and Y in the ratio 7 : 5. When 6 litres of the mixture is drawn and the jar in filled with the same quantity of Y, the ratio of X and Y becomes 7 : 9. Find the quantity X in the jar initially.

Q6. Two taps fill a cistern separately in 20 minutes and 40 minutes respectively and a drain pipe can drain off 30 litres per minute. If all the three pipes are opened, the cistern fills in 72 minutes what is the capacity of the cistern?

Q7. If ten persons can do a job in 60 days. In how many days can twenty persons do the same job?

Q8. A can do a piece of work in 20 days, B in 30 days and C in 60 days. All of them began to work together. However A left the job after 6 days and B quit work 6 days before the completion of work. How many days did the work last?

Q9. 8 men and 16 women can finish a job in 6 days but 12 men & 24 women can finish it in 8 days. How many days will 26 men and 20 women take to finish the job?

Q10. 4 men or 12 boys can do a piece of work in 5 days by working 8 hours per day. In how many days 2 men & 4 boys can do the same piece of work working 12 hours a day.

Q.11 A railway train 100 metres long is running at the speed of 30 kmph. In what time will it pass (i) a man standing near the line (ii) a bridge 100 metres long?

Q.12 The driver of a car is travelling at a speed of 36 kmph and spots a bus 80 metres ahead of him. After 1 hour the bus is 120 metres behind the car. What is the speed of the bus?

Answers 7.3

3 Mark Answers

1) 80 days 2) 480 days 3) ₹ 1152 4) 42 liters

5 MarkAnswers

5) 14 litres

6) 490.9 liters

7) 20 persons can do the same job in 30 days.

8) 15.6 days

9) 267 days

10) 4 days

11) (a) 12 sec (b) 24 sec

12) Bus speed = 35.8 kmph.

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Chapter

8

BILL DISCOUNTING

8.1 Bill of Exchange:

According to section 5 of the Indian Negotiable Instrument Act, 1881, "A Bill of Exchange is an instrument in writing containing an unconditional order, signed by the maker, directing a certain person to pay a certain sum of money only to or to the order of a certain person or the bearer of the instrument". In essence, it is a negotiable instrument by which one person agrees to pay another a sum of money at a specified date.

Specimen of a Bill of Exchange:

Example 1 :

Stamp — Place : Bangalore

Date : 06-07-2013

Three months after date, pay to me or my order, the sum of Rs.25,000 (Rupees Twenty Five Thousand only) for the value received.

Rs. 25,000

Accepted

sd/-
Subhash

sd/-
Pranathi

Fig. 1

The negotiable instrument becomes receivable only after it is signed by the drawee or his agent on the face of the instrument as shown above or on it's back. The bill of exchange should be written on a stamped paper of the court so that it becomes a legal document.

A Bill of exchange can be

(i) realised by its holder on maturity or retirement of the same from the party liable on it.

(ii) discounted by its holder with his banker before the due date.

(iii) endorsed to a creditor by its holder before the due date.

(iv) renewed for a further period by a mutual agreement between the parties.

We now discuss the various terms associated with this procedure of Bill discounting.

With respect to the bill of exchange in Example (1) [Refer Fig. (1)]

1) Creditor (or Drawer) is Ms. Pranathi
2) Debtor (or Drawee) is Mr. Subhash
3) Drawing Date : Date on which the bill is drawn
   For instance, 06-07-2013 in Example (1)
4) Bill period : The period after which the bill is due
   In Example (1), bill period is 3 months.
5) Grace period : The bill of exchange becomes legally due on a date 3 days after the due date. These 3 days is called the grace period.
6) Legally due date : Legally due date of a bill is calculated as follows:

Legally due date = Date of drawing + Bill period + Grace period

Thus, in Example (1),

Legally due date = 06-07-2013 → Dt of drawing
(+) 0 - 3 - 0 → Bill period
(+) 3 - 0 - 0 → Grace period
= 09-10-2013

Note : Nominal due date = Date of drawing
+ Bill period

7) Bill Amount : It is the amount due on the legal due date of a Bill. Bill amount is also called face value or nominal value of the bill. In Example (1), ₹ 25,000 is the face value.

8.2 Discount Date:

In order to meet one's financial urgency, a bill may be cashed from a bank or a broker before the bill falls legally due. This date on which the banker discounts the bill is the date of discount.

8.3 Discount Period

It means the period of early payment on the bill or the unexpired period of the bill. It is calculated by counting the number of days from the date of discount till the legal due date. Discount period is otherwise called the unexpired period, denoted as 't'.

Let the Bill in Example (1) be discounted on 30-08-2013.

Then the discounted period will be from 30-08-2013 to 09-10-2013.

The unexpired period is calculated as shown:

Discount Date : 30-08-2013, which is in August

$\therefore$ Number of days remaining in August = 31 – 30

= 1 day ..... (1)

Total Number of days in September = 30 days ..... (2)

We have the legally due date as 09-10-2013, which is in October.

This means, the number of days considered in October = 9 days ..... (3)

$\therefore$ Total number of days = (1) + (2) + (3)

= 1 + 30 + 9 days

= 40 days

ie., discount period = 40 days.

8.4 Discount Rate:

It is the SI charged by the banker for discounting the bill. It is the rate at which the face value of the bill is discounted to its present value.

8.5 True Present Value (or Present Worth):

Present value of a bill is the amount that would have to be invested today so as to obtain the maturity value of the bill on the due date. It is calculated using the formula

$$\boxed{P = \frac{F}{1+tr}}$$

where t is the unexpired period in years,

r is the discount rate in percentages

and F is the face value of the bill in rupees.

With reference to Example (1), we have already computed the discount period (in section 8.3) as 40 days

ie., $t = 40 \text{ days} = \frac{40}{365}$ years

F = ₹ 25,000

Assuming $r = 15\% \equiv 0.15$

we get $P = \frac{25000}{1+\frac{40}{365}\times 0.15}$

$\therefore$ P = ₹ 24,596.61

8.6 Discount

Discount on a bill can be allowed either to the debtor or to a banker who advances money on the bill. It is classified into two types viz.,

(i) True Discount and (ii) Banker's Discount

(i) True Discount : It is basically the difference between face value and the present value of a bill. It is the SI calculated on the present value of a bill, which is given by

$$\boxed{TD = Ptr}$$

where t is the unexpired period in years

r is the discount rate in percentages.

In Example (1), we get

$$TD = 24{,}596.61 \times \frac{40}{365} \times 0.15$$

$\therefore$ TD = ₹ 404.36

(ii) Banker's Discount: Bankers discount is the simple interest calculated on the face value of the bill, for the discount period, at the discount rate. It is calculated using

$$\boxed{BD = Ftr}$$

where F is the face value of the bill

t is the discount period

r is the discount rate.

Clearly, Banker's discount is always greater than the True Discount.

In Example (1), F = ₹ 25000

t = 40 days

r = 15%

$$\therefore \quad BD = 25000 \times \frac{40}{365} \times 0.15$$

$\therefore$ BD = ₹ 411

8.7 Banker's Gain:

The difference between Banker's discount and True discount is the Banker's gain.

$$\boxed{BG = BD - TD}$$

Thus, is Example (1),

BG = 411 – 404.36

$\Rightarrow$ BG = ₹ 6.64

8.8 Discounted Value of the Bill:

The amount obtained after deducting the bankers discount from the face value of the bill is called the discounted cash value of the bill.

Hence, Discounted value, $\boxed{DV = F - BD}$

In Example (1), $F = ₹\ 25{,}000$

$BD = ₹\ 411$

$\therefore$ Discounted Value $= 25{,}000 - 411 = ₹\ 24{,}589$

Discounted value is also termed as the **Banker's present value.**

We list below a few important formulae related to bill discounting:

1) $F = \dfrac{BD \times TD}{BG}$

2) $BG = \begin{cases} \dfrac{F(tr)^2}{1+tr} \\ TD.tr \\ BD - TD \end{cases}$

3) $TD = \dfrac{Ftr}{1+tr}$

4) Discounted Value $= F(1 - tr)$

5) **Legally due date : Draw date + Bill period + 3 days grace**

6) $BD = Ftr$

Worked Examples

Example 2:

For ₹ 512.50 due 6 months at 15% p.a., find the true present worth and discounted cash value.

Solution:

Given $F = ₹\ 512.50$

$t = 6\ m = \dfrac{1}{2}\ yr$

$r = 15\% = 0.15$

$\therefore\ P = \dfrac{F}{1+tr} = \dfrac{512.50}{1+\frac{1}{2}\times 0.15}$

ie., $P = ₹\ 476.44$

Discounted value $= F(1 - tr) = 512.50\left(1 - \dfrac{1}{2}\times 0.15\right)$

i.e., Discounted value $= ₹\ 474.06$

Example 3:

Banker's gain on a bill due after 6 months at 4% p.a. is ₹ 24. Find TD, BD, bill amount and discounted value of the bill.

Solution:

Given BG = ₹ 24

$t = 6\ m = \frac{1}{2}\ y$

r = 4% = 0.04

We have BG = TD tr

$\Rightarrow \quad 24 = TD \times \frac{1}{2} \times 0.04$

$\therefore \quad$ **TD = ₹ 1200**

Bankers gain, BG = BD – TD

$\therefore$ BD = BG + TD = 24 + 1200

ie., **BD = ₹1224**

Further, BD = Ftr

$\therefore\ 1224 = F \times \frac{1}{2} \times 0.04$

$\Rightarrow \quad F = ₹\ 61,200$

Discounted value = F – BD = 61200 – 1224

$\therefore$ Discounted value = ₹ 59,976

Example 4:

A Banker discounts a bill for a certain amount having 73 days to run before it matures at 15% p.a. The discounted value of the bill is ₹970. What is the face value of the bill? Also find the banker's discount.

Solution:

Given: $t = 73 \text{ days} = \frac{73}{365}\ y$

i.e., $t = \frac{1}{5}\ yr$

r = 15% = 0.15

Discounted value = ₹970

Discounted value = F(1 – tr)

$$970 = F\left(1 - \frac{1}{5} \times 0.15\right)$$

$\therefore$ **F = ₹1000**

BD = F – Discounted value = 1000 – 970

$\therefore$ **BD = ₹30**

Example 5:

A bill drawn for 3 months was legally due on 06-07-2009. Find the date of drawing of the bill.

Solution:

Given Legally due date as 06-07-2007

Bill period = 3 months

We know that

Legally Due Date = Date of drawing

(+) Bill period

(+) 3 days grace

Now, working backwards, we get

Date of drawing = Legally due date

(–) Bill period

(–) 3 days grace period

$\therefore$ Date of drawing = 06-07-2009

(–) 0 – 3 – 0

(–) 3 – 0 – 0

= 03 - 04 - 2009

Example 6 :

A bill of ₹1,460 was drawn on 1st April for 6 months after date and was discounted at 5% p.a. for ₹ 1451. On what date was the bill discounted?

Sol. Legal due date = 01 – 04 → Date of drawing

(+) 0 – 6 → Bill period

\+ 3 – 0 → Grace period

= 04 – 10

ie., Legal due date is 4 October.

Given that F = ₹1460

Discounted value = ₹1451

$\therefore$ BD = 1460 – 1451

BD = ₹9

But, BD = F t r

$9 = 1460 \times t \times 0.05$

$\Rightarrow \quad t = \frac{9}{73}$ years

$t = \frac{9}{73} \times 365 \Rightarrow$ **t = 45 days**

ie., the bill was discounted 45 days before the legally due date. Hence, we calculate backwards to find the date of discount.

Legal due date	:	4 October
ie., In October	:	4 days
In September	:	30 days
In August	:	11 days
		45 days

which means the date of discount falls in the month of August.

Total number of days in August = 31

(–) 11

20

ie., date of discount is 20 August.

Example 7:

A bill of ₹5000 drawn on 10-4-1998 at 3 months was discounted on 1-5-1998 at 12% p.a. For what sum was the bill discounted and how much has the banker gained in this?

Solution:

Here F = ₹ 5000

Bill period = 3 months

$\therefore$ Legally due date = 10 - 4 - 98 (Date of drawing)

(+) 0 – 3 – 0 (Bill period)

\+ 3 – 0 – 0 (Grace period)

= 13 - 7 - 98

Date of discount = 1 - 5 - 98

This means, the number of days from 1-5-98 to 13-7 - 98 becomes the unexpired period.

This may be calculated as follows:

1-5-98 to 13-7-98

$\therefore$ In may = 30 days (31 – 1)

In June = (+) 30 days

In July = (+) 13 ($\therefore$ 13-7-98 is the due date legally)

Total = 73 days

ie., $t = 73 \text{ days} = \frac{73}{365}\ year$

$t = \frac{1}{5}\ year$

Given r = 12%

$\therefore$ $BD = Ftr = 5000 \times \frac{1}{5} \times 0.12$

BD = ₹120

Discounted value = F – BD = 5000 – 120

$\therefore$ Discounted value = ₹4,880

To find BG, we must first calculate P and TD.

$$P = \frac{F}{1+tr} = \frac{5000}{1+\frac{1}{5}\times 0.12}$$

ie., P = ₹ 4882.81

$TD = P\,t\,r = 4882.81 \times \frac{1}{5} \times 0.12$

TD = ₹ 117.18

$\therefore$ BG = BD – TD = 120 – 117.18

ie., BG = ₹ 2.82

Example 8:

The banker's gain on a bill is $\frac{1}{5}$ th of the banker's discount and the rate of interest is 20% p.a. Find the unexpired period of the bill.

Solution:

Given: $BG = \frac{1}{5} BD, \quad r = 20\%$

we know that $F = \frac{BD \times TD}{BG} = \frac{BD \times TD}{\frac{1}{5} BD}$

$\Rightarrow \quad F = 5\ TD$

ie., $F = 5\left(\frac{Ftr}{1 + tr}\right)$

$\Rightarrow \quad 1 + tr = 5\ tr$

ie., $1 + t \times 0.2 = 5t \times 0.2$

solving for t, we get

t = 1.25 years.

Example 9 :

A bill for ₹ 2920 drawn at 6 months was discounted on 10-4-97 for ₹ 2916. If the discounted rate is 5% p.a., on what date was the bill drawn?

Solution:

Given: $F = ₹ 2920$

Bill period $= 6m = \frac{1}{2} y$

$r = 5\%$

$BD = F - \text{discounted value} = 2920 - 2916$

$\therefore BD = ₹ 4$

Since $BD = Ftr$, we get

$4 = 2920 \times t \times 0.05$

$\Rightarrow \quad t = 10$ days

i.e., Legally due date is 10 days after 10-4-97

$\therefore$ Legally due date is 20-4-97

Thus, date of drawing = 20-4-97 (Legal due date)

(–) 0-6-0 (Bill Period)

(–) 3-0-0 (Grace period)

Date of drawing = 17-10-96

Example 10 :

The difference between BD and TD on a certain sum of money due in 6 months is ₹ 27. Find the amount of the bill if the rate of interest is 6% p.a.

Solution:

Given: BD – TD = 27

ie., BG = ₹ 27

Since BG = TD. t.r, we get

$$27 = TD \frac{6}{12} \times 0.06$$

$\Rightarrow$ TD = ₹ 900

Then BD = BG + TD

$\therefore$ BD = ₹ 927

Now, BD = F t r

$$\Rightarrow 927 = F \times \frac{6}{12} \times 0.06$$

$\therefore$ **F = ₹ 30,900**

Example 11 :

A banker pays ₹ 4520 on a bill of ₹ 5000, 146 days before the legally due date. Find the rate of discount charged by the banker.

Solution:

Given: F = ₹ 5000

Discounted value = ₹ 4520

$$t = 146 \text{ days} = \frac{146}{365} \textit{ year}$$

r = ?

BD = F – Discounted value = 5000 – 4520

$\therefore$ BD = ₹ 480

$$BD = Ftr \Rightarrow 480 = 5000 \times \frac{146}{365} \times r$$

$$\Rightarrow r = 0.24$$

$$\Rightarrow r = 0.24 \times 100$$

$\therefore$ $\boxed{r = 24\%}$

Example 12 :

The present worth of a bill due sometime hence is ₹ 1100 and TD on the bill is ₹110. Find BD and BG.

Sol. Given P = ₹ 1100

TD = ₹ 110

To find BD and BG

We know that TD = P t r

$\therefore$ 110 = 1100 × tr

$\Rightarrow$ tr = 0.1 .....(1)

consider BG = TD × tr

$\therefore$ BG = 110 × 0.1 [using (1)]

$\therefore$ **BG = ₹ 11**

BD = BG + TD

BD = ₹ 121

EXERCISE 8.1

One mark questions:

1) A bill was drawn on 14-3-2013 for 3 months. When does the bill fall legally due?
2) A bill drawn for 3 months was legally due on 18-8-2012. Find the date of drawing of the bill.
3) Find the present value of ₹ 750 due 4 months hence at 15% p.a.
4) Find the banker's discount on a bill of ₹ 415 due 9 months hence at 15% p.a.
5) Define Banker's Gain.
6) Define Banker's Discount.
7) Find the present value of ₹ 2320, due 2 years hence, at 8% per annum.
8) Find the BD on ₹ 1015, payable after 3 months at 6% p.a.
9) Find the true discount on ₹ 1380, due $1\frac{1}{2}$ years after, at 10% p.a.

Two marks questions:

1) TD on a bill was ₹ 100 and BG was ₹ 10. What is the face value of the bill?
2) A banker pays ₹ 2380 on a bill of ₹ 2500, 73 days before the legal due date. Find the rate of discount charged by the banker.
3) The Banker's discount and true discount on a sum of money due 3 months hence are ₹ 154.50 and ₹ 150 respectively. Find the sum of money and the rate of interest.
4) BD and BG on a certain bill due after sometime are ₹ 1250 and ₹ 50 respectively. Find the face value of the bill.

Five marks questions:

1) The difference between banker's discount and true discount on a bill due after 6 months at 4% interest p.a. is ₹ 20. Find the true discount, banker's discount and face value of the bill.
2) The BG on a certain bill due 6 months hence is ₹ 10, the rate of interest being 10% p.a. Find the face value of the bill and the true present value.
3) A banker discounts a bill for a certain amount having 32 days to run before it matures at 15% p.a. The discounted value of the bill is ₹ 995.90. What is the face value of the bill, banker's discount, true discount and banker's gain?
4) A bill for ₹ 14,600 drawn at 3 months after date was discounted on 11-11-99 for ₹ 14,320. If the discount rate is 20% p.a., on what date was the bill drawn?
5) A bill for ₹ 2920 was drawn on September 11 for 3 months ater date and was discounted at 16% p.a. for ₹ 2875.20. On what date was the bill discounted.
6) A bill for ₹ 3500 due for 3 months was drawn on 27 march 2012 and was discounted on 18 April 2012, at the rate of 7% p.a. Find the banker's discount and discounted value of the bill.
7) A bill for ₹ 12900 was drawn on 3 Feb 2004 at 6 months and discounted on 13 March 2004 at 8% p.a. For what sum was the bill discounted and how much did the banker gain in this transaction?
8) The banker's gain on a bill is $\frac{1}{9}$ th of the banker's discount, rate of interest being 10% p.a. Find the unexpired period of the bill.
9) A bill for ₹ 2725.25 was drawn on 03-6-2010 and made payable 3 months after due date. It was discounted on 15-6-2010 at 16% per annum. What is the discounted value of the bill and how much did the banker gain?

ANSWERS 8.1

One mark questions:

1) 17 - 6 - 2013
2) 15 - 5 - 2012
3) ₹ 714.28
4) ₹ 46.68
7) ₹ 2000
8) ₹ 15.22
9) ₹ 180

Two marks questions:

1) ₹ 1100
2) 24%
3) F = ₹ 5150, r = 12%
4) ₹ 30,000

Five marks questions:

1) TD = 1,000
   BD = 1,020
   F = 51,000
2) F = ₹ 4,200
   P = ₹ 4,000
3) F = ₹ 1009.17
   BD = ₹ 13.27
   TD = ₹ 13.09
   BG = ₹ 0.18
4) 13 September 99
5) 9 November
6) BD = ₹ 49
   Discounted value = ₹ 3451
7) Discounted value = ₹ 12,487.20
   Bankers gain = ₹ 12.8
8) 15 months (1.25 years)
9) Discounted value = ₹ 2624.96
   Banker's gain = ₹ 3.57

* * * * *

Chapter

9

STOCKS AND SHARES

9.1 Introduction :

Wouldn't you love to be a business owner without ever having to show up at work? Imagine if you could sit back, watch your company grow and collect the dividend cheques as the money rolls in. This situation might sound like a dream, but its closer to reality than you might think.

As you've probably guessed we are talking about owning stocks or shares. This fabulous category of financial instruments is, without a doubt, one of the greatest tools ever invented for building wealth. Over the last few decades the average person's interest in the stock market has grown exponentially. What was once a toy of the rich has now turned into the vehicle of choice of growing wealth. This demand coupled with advances in trading technology has opened up the markets so that nowadays nearly anybody can own stocks.

9.2 The definition of a stock

Stock is a share in the ownership of a company. A stock is represented by a stock certificate. This is a fancy piece of paper that is proof of your ownership. In today's computer age, you won't actually get to see this document because your broker keeps these records electronically, which is also known as holding shares "in street name". This is done to make the shares easier to trade. In the past when a person wanted to sell his or her shares, that person physically took the certificates down to the broker. Now, trading with a click of the mouse or a phone call makes life easier for everybody.

Whether you say shares, equity or stock it all means the same thing. In today's financial markets, the distinction between stocks and shares has been some what blurred. Generally, these words are used interchangeably. However, the difference between the two words comes from the context in which they are used. For example, "Stock" is a general term used to describe the ownership certificates of any company, in general and 'shares' refers to the ownership certificates of a particular company. So, if investors say they own stocks, they are generally referring to their overall ownership in one or more companies technically if someone says that they own shares the question then becomes shares in which company? Bottom line, stocks and shares are the same thing.

Being an Owner: Holding a company's stock means that you are one of the many owners (shareholders) of a company and as such, you have a claim to everything the company owns. But, being a share holder of a public company does not mean you have a say in the day-to-day running of the business. Instead one vote per share to elect the board of directors at annual meetings is the extent to which you have a say in the company. For instance being a Microsoft shareholder does not mean you can call up Bill Gates and tell him how you think the company should be run.

9.3 Nominal Interest or dividend

Profits are sometimes paid out in the form of dividends. The more shares you own, the **larger the portion of the profits you get.** However some companies pay out dividends but many others do not. There is no obligation to pay out dividends even for those firms that have traditionally given them. Without dividends an investor can make money on a stock only through its appreciation in the open market. On the downside any stock may go bankrupt, in which case your investment is worth nothing.

Limited Liability: As an owner of a stock you are not personally liable if the company is not able to pay its debts. Owning a stock means that, no matter what, the maximum value you can lose is the value of your investment. Even if a company of which you are a shareholder goes bankrupt, you can never lose your personal assets.

Although risk might sound all negative, there is also a bright side Taking on greater risk demands a greater return on your investment. This is the reason why stocks have historically out performed, other investments such as bonds or saving accounts. Over the long term, an investment in stocks has historically had an average return of around 10-12%.

Nominal and market value of a share:

The original value of a share is called its nominal value or face value and the price at which it is selling is called its market value.

a) When the market value of a share is higher than its nominal value, it is said to be at premium or above par

b) When the market value of a share is equal to its nominal value the share is said to be at par

c) When the market value of a share is less than its market value it is said to be at discount or below par.

Example

1. 8% stock at 125 means.
   A stock of face value 100 is being sold at ₹125 (Market Value) and the nominal interest obtained on this share is ₹ 8.
2. 12% stock at 20 premium means face value is 100, Market Value is 120 and dividend is ₹12.
3. 7% stock at 10 discount means face value is 100, market value is 90 and interest is ₹ 7
4. 10% stock at par means
   Face value is 100, market value is 100, interest is ₹10.
   Note: (1) Nominal interest or dividend is always declared on face value of share.
   (2) The face value of a share can be ₹ 2, ₹ 5, ₹ 10 etc. but if nothing is mentioned in the problem then it is understood that the face value of one share is ₹ 100.
   (3) MV means market value and FV means face value.

Example 1:

What income can be obtained from ₹ 8000 of 4% stock.

Solution:

4% stock means if the face value stock is ₹100 income or dividend obtained is ₹4.

Face Value	Income
100	4
8000	?

$\therefore$ Income obtained is $\frac{8000 \times 4}{100} = ₹\ 320$

Example 2:

How much stock can be purchased by investing ₹ 9600 in 3.5% stock at 120.

Solution:

Note here that ₹ 9600 is market value and we are required to find face value.

Face Value	MV
100	120
?	9600

$\therefore$ Amount of stock $= \frac{100 \times 9600}{120} = ₹8000$

Example 3:

How much does Maya realize by selling ₹ 30,000 stock at 20 discount.

Solution:

Note here that Maya is selling stock worth ₹ 30,000. So here 30,000 is face value. Also ₹ 20 discount means cost of ₹100 stock is ₹ 80.

FV	MV
100	80
30,000	?

$\therefore$ Money obtained $= \frac{30,000 \times 80}{100} = ₹\ 24,000$

Example 4 :

Ayush buys ₹ 10,000 stock at 96 and sells when its price rises to ₹102. Find his gain.

Solution :

While Buying

FV	MV
100	96
10,000	?

Cost of stock $= \frac{10,000 \times 96}{100} = ₹\ 9600$ (Money spent)

While Selling

FV	MV
100	102
10,000	?

Money obtained = $\frac{102 \times 10,000}{100}$ = 10,200

$\therefore$ Gain = 10,200 – 9,600 = ₹ 600

OR

Gain on ₹ 100 share = 102 – 96 = 6

$\therefore$ gain on ₹ 10,000 share =

FV	gain
100	6
10,000	?

$= \frac{6 \times 10,000}{100}$ = ₹ 600

Example 5:

What is the market value of 12% stock when an investment of ₹ 6900 produces an income of ₹ 720.

Solution:

Given for an investment of ₹ 6900 income is ₹ 720. We have to find the investment (market value) which can earn an income of ₹ 12.

MV	Income
6900	720
?	12

$\therefore$ Market value of share = $\frac{6900 \times 12}{720}$ = ₹ 115

Example 6:

Sukanya holds ₹ 8000 of 3% stock. She sells it at ₹ 110 and invests the proceeds in 5% stock. Thereby her income increases by ₹ 260. Find the market price of 5%stock.

Solution: Case 1 : 3% stock

FV	Income
100	3
8000	?

Income obtained from 3% stock = $\frac{8000 \times 3}{100}$ = ₹ 240

To calculate cash obtained by selling 3% stock

FV	MV
100	110
8000	?

$\therefore$ Cash obtained = $\frac{8000 \times 110}{100}$ = ₹ 8800

Case 2 : 5% stock

Money invested in 5% stock is 8800

Income obtained from 5% stock

= 240 + 260 = ₹ 500

To calculate Market value of ₹ 100 at 5% stock

MV	Income
8800	500
?	5

$\therefore$ Market value = $\frac{5 \times 8800}{500}$ = ₹ 88

Example 7:

Raksha sells her 9,600 4% stock at a premium of 6. How much 3.5% stock at 4 discount can she buy from the sale proceeds of the former stock. What will be the difference in her income.

Solution : Case 1 : 4% stock

FV	Income
100	4
9600	?

Income obtained = $\frac{4 \times 9600}{100}$ = ₹ 384

By selling this stock

FV	MV
100	106
9600	?

Money obtained by selling = $\frac{9600 \times 106}{100}$ = ₹ 10,176

Case 2 : 3.5% stock

Money invested in 3.5% stock = 10,176

FV	MV
100	96
?	10,176

stock purchased from sale proceeds is

$$= \frac{10,176 \times 100}{96} = 10,600$$

To calculate income

FV	MV	Income
100	96	3.5
10,600	10,176	?

[Here 1st & 3rd column or 2nd & 3rd column may be used for calculation]

$$\therefore \text{ Income from 3.5\% stock} = \frac{10,600 \times 3.5}{100} = 371$$

Difference in Income = 384 – 371

Hence income decreases by ₹13

Example 8:

A man sells ₹5000, 4½% stock at 144 and invests the proceeds partly in 3% stock at 90 and partly in 4% stock at 108. He thereby increases his income by ₹25. How much of the proceeds were invested in each stock.

Solution: Case 1: 4½ stock

FV	MV	Income
100	144	4.5
5000	?	?

To calculate Income consider first and 3rd column

$$\text{Income} = \frac{5000 \times 4.5}{100} = 225 \quad .....(1)$$

$$\text{Money obtained by selling at 144} = \frac{5000 \times 144}{100} = 7200$$

[consider first and second column]

Let the amount invested in 3% stock be x

Let the amount invested in 4% stock be $7200–x$

Income from 3% stock $I_1 = \frac{3x}{90}$

[Note income $= \frac{\text{dividend of one share} \times \text{money invested}}{\text{market value of 1 share}}$

Income from 4% stock $I_2 = \frac{4}{108}(7200 - x)$

given total income increases by ₹ 25

$\therefore I_1 + I_2 = 225 + 25$

$\frac{3x}{90} + \frac{4(7200 - x)}{108} = 250$

$\frac{x}{30} + \frac{7200 - x}{27} = 250$

$\frac{27x + 2,16,000 - 30x}{810} = 250$

$2,16,000 - 3x = 2,02,500$

$2,16,000 - 2,02,500 = 3x$

$\frac{13500}{3} = x$

$\therefore x = 4500$

Investment in 3% stock = ₹ 4,500

Investment in 4% stock = 7200 – 4500 = ₹ 2700

Example 9:

A person sells out ₹ 4000 of 6.25 Government of India stock at 112.5 and re-invests the proceeds in 8% railway debentures, thereby increasing his income by ₹50. At what price did he buy the debentures.

Solution: Case 1 : 6.25 Govt. of India stock

To calculate income

FV	Income
100	6.25
4000	?

Income $= \frac{4000 \times 6.25}{100} =$ ₹ 250

To calculate amount received by selling

FV	MV
100	112.5
4000	?

$$\text{Amount received} = \frac{4000 \times 112.5}{100} = ₹\ 4500$$

Case 2: 8% railway debentures

given Income increases by 50

$\therefore$ New Income = 250 + 50 = 300

To calculate market value of ₹ 100 debenture

MV	Income
?	8
4500	300

$$\text{Market value} = \frac{4500 \times 8}{300} = ₹\ 120$$

Example 10:

Rahul invests ₹ 1,74,000 partly in 12% stock at 110 and partly in 15% stock at 86. If the total income from both is 24,600, find the amount invested by Rahul in 12% stock.

Solution : Let Amount invested in 12% stock be x. Then amount invested in 15% stock is 1,74,000–x.

Income from 12% stock

MV	Income
110	12
x	?

$$I_1 = \frac{12x}{110}$$

Income from 15% stock

MV	Income
86	15
1,74,000x	?

$$I_2 = \frac{(1,74,000 - x)15}{86}$$

Given total income = 24,600

$I_1 + I_2 = 24{,}600$

$$\frac{12x}{110} + \frac{(174000 - x) \times 15}{86} = 24{,}600$$

$$\frac{12x}{110} + \frac{174000 \times 15}{86} - \frac{15x}{86} = 24{,}600$$

$$\frac{174000 \times 15}{86} - 24{,}600 = \frac{15x}{86} - \frac{12x}{110}$$

$$\frac{2610000 - 2115600}{86} = \frac{1650x - 1032x}{86 \times 110}$$

$$\frac{494400 \times 86 \times 110}{86} = 618x$$

$88000 = x$

$\therefore$ Rahul invest 88000 in 12% stock.

Note : The nominal value or face value of a share is generally ₹ 10 or ₹ 25 or ₹ 100. If nothing is mentioned in the problem then we assume the face value is ₹ 100.

Example 11:

Shreya holds 1200 shares of a company each of face value ₹ 10. She sells 400 shares at the rate of ₹ 12 per share and the remaining at par. She invests the proceeds in a share of ₹ 100 at ₹ 80. Find the number of shares purchased by Shreya.

Solution :

Amount received by selling 400 shares = 400 × 12 = 4800

Amount received by selling remaining 800 shares = 800 × 10 = 8000.

$\therefore$ Total amount received = 4800 + 8000 = 12800

Cost of one share = 80

$\therefore$ No. of shares purchased for ₹ 12800

$$= \frac{12800}{80} = 160 \text{ shares.}$$

9.4 Yield :

An investor before purchasing a stock also looks at the return or yield or actual interest obtained on particular stock. **The actual percentage income on investment is called yield.**

$$\text{Yield} = \frac{\text{Interest}}{\text{Market Value}} \times 100$$

The yield on 6% stock at ₹ 143 is yield $= \frac{6}{143} \times 100$

In brief a) Dividend — Annual income on ₹ 100 stock (face value)

b) Yield — Annual income on ₹ 100 investment (market value)

Example 12:

Mr. Sandeep invests ₹15000 cash partly in 3% stock at 75 and partly in 6% debentures at 125 in such a way as to get a return on 4.5% for his money. How much does he invest his money. How much does he invest in each.

Solution:

[Note here return of 4.5% on 15000 invested means yield is 4.5%. However here total income can be calculated without yield formula also]

Total Income $= \frac{4.5}{100} \times 15000 =$ ₹ 675

Let Amount Invested in 3% stock be x then amount invested in 6% debentures is $15000–x$

Income from 3% stock $I_1 = \frac{3}{75} \times x$

Income from 6% debenture $I_2 = \frac{6}{125} \times (15000 - x)$

$I_1 + I_2 = 675$

$$\frac{3}{75}x + \frac{6}{125}(15000 - x) = 675$$

$0.04x + 0.048\ (15000–x) = 675$

$0.04x + 720 – 0.048x = 675$

$720 – 675 = 0.048x – 0.04x$

$45 = 0.008x$

$x =$ ₹ 5625

$\therefore$ Amount invested in 3% stock = ₹ 5625

Amount invested in 6% debentures = ₹ 9375

Example 13:

What is the market value of 6% stock if it earns an interest of 4.5% after deducting the income tax of 4%

Solution:

Let the Market value of ₹ 100 stock be x

Then income = ₹ 6

$$\text{Tax deduction} = \frac{4}{100} \times 6 = ₹\ 0.24$$

Net Income = 6 – 0.24 = ₹ 5.76

Given yield after tax deduction = 4.5%

$$\text{Yield} = \frac{\text{Income}}{\text{MV}} \times 100$$

$$4.5 = \frac{5.76}{x} \times 100$$

$$x = \frac{5.76 \times 100}{4.5} = ₹\ 128$$

$\therefore$ Market value of ₹ 100 stock is ₹ 128

Example 14:

Which is better investment 7.5% stock at 125 or 5% stock at 80.

Solution:

Here we need to calculate the yield and see which stock gives higher returns.

$$\text{yield on 7.5\% stock} = \frac{7.5}{125} \times 100 = 6\%$$

$$\text{yield on 5\% stock} = \frac{5}{80} \times 100 = 6.25\%$$

$\therefore$ 5% stock at 80 is better.

Example 15:

What rate of interest is obtained by investing in 9% stock at 180.

Solution : Rate of interest on money invested is yield itself.

$$\therefore \text{ yield} = \frac{\text{interest}}{\text{MV}} \times 100 = \frac{9}{180} \times 100 = 5\%$$

EXERCISE 9.1

One mark questions:

1. Find the income obtained by investing ₹ 3600 in 5% stock at 90.
2. How much stock at ₹ 75 can be bought for ₹ 3375
3. 6% stock is being sold at 15 discount. How much money is required to buy ₹ 6000 stock
4. Define yield.
5. What is the yield obtained when ₹ 5000, 3% stock is purchased at ₹ 125

Two marks questions:

1. What income can be obtained from an investment of ₹ 10,725 in 6.5% stock at 143. What is the amount of stock purchased.
2. 10% stock is quoted at ₹ 120. Find the rate of interest.
   [Hint: here rate of interest means yield]
3. How much of 8% stock at 96 can be purchased for ₹ 4800? Also find the income obtained.
4. How much money will a man get by selling ₹ 6000 stock at 5% stock at 20 premium ?
5. Abhi purchased 850 shares of a company the nominal value being ₹ 10 each. Company declared an annual dividend of 12%. How much dividend did Abhi receive.

Three marks questions:

1. Rakshitha invests ₹ 15000 in a company paying 7% per annum, when a share of value ₹ 100 is selling for ₹ 150. What is her annual income and what percentage does she get on her money.
2. Which is a better investment
   a) 8% stock at 110 or 9% stock at 98
   b) 6½% stock at 84 or 4% stock at 102
   c) 7% stock at 115 or 5% stock at 88
   d) 4% stock at 118 or 6% stock at 124
3. A man invests equal sums of money in 4%, 5% and 6% stock, each stock being at par. If the total income of the man is ₹ 3,600. Find his total investment.
4. What is the market value of 9.5% stock when an investment of ₹ 12,400 produces an income of ₹ 1472.5
5. What is the quoted value of 12% stock if it earns an interest of 8% after deducting the income tax of 8%.
6. Venu invested ₹ 5,02,950 in Infosys when price of ₹ 5 shares was 3353. He sold shares worth ₹ 500 when the price went upto 3583 and the remaining he sold when the price was ₹ 3253. How much did Venu gain.

7. Ramesh has invested ₹ 4,300 partly in 4.5% stock at ₹ 72 and partly in 5% stock at ₹ 95. If the total income from both is ₹ 250, find the investment in both the types of stock
8. Ramesh holds ₹ 2,100 of 3% stock. He sells at ₹ 121 and invests the proceeds in 5% stock. Thereby his income increases by ₹ 14. Find the market price of 5% stock.
9. How much must be invested in 14.25% stock at 98 to produce the same income as would be obtained by investing ₹ 9,975 in 15% stock at 105.
10. A stock yields 5% to an investor. A fall of ₹ 5 in its price causes it to yield 5½%. What was its market value if the two income are equal.
11. Sanjana invests ₹ 3240 in a stock at 108 and sells when the price falls to 104. How much stock at 130 can Sanjana now buy.
12. A person invested 42000 partly in 5% stock at 125 and the remaining in 7.5% stock at 75. If income derived from the two stocks is the same. Find the respective investments in each stock. Also find the total income.
13. If a person wishes to obtain 18% yield from his investment at what price should he buy 13.5% stock.
14. a) A invests a sum of money in 5.5% stock at 90 and B an equal sum in 3.5% stock. If A's income is 10% more than B's find the price of 3.5% stock.
    b) How much money has to be invested in 11.5% stock at 73 to obtain an income of ₹ 150 after a tax deduction at source of 20%.
15. a) A person invests ₹ 15000 partly in 3% stock at 75 and partly in 6% stock at 125. If the income from both is ₹ 675 find his investment in the 2 types of stock.
    b) At what price a person has bought 25% shares with face value 100 which has given him 20% returns.
16. Rohan invested ₹ 55000 partly in 8% stock at 80 and partly in 12% stock at 150 in such a way as to get a return of 9% for his money. How much did Rohan invest in each.
17. Prathik sells out ₹ 6000 of 7.5% stock at 108 and reinvests the proceeds in 9% stock. If Prathik's income increases by 270. At what price did Prathik buy 9% stock.
18. Jane sells her ₹ 12500, 4.5% stock at 94. How much of 9% stock at 125 can Jane purchase from the sale proceeds of the former stock. what is the change in Jane's income.
19. A man sells ₹ 25000, 13.5% stock when the shares were selling at a premium of 20. He invests the proceeds partly in 15% stock at 75 and partly in 16% stock at 128. Find how much he has invested in each stock if his income increases by ₹ 1875.
20. Nihal has ₹ 5 Maruthi shares worth ₹ 1000. He sells 120 shares when the shares are selling at ₹ 1400 and the remaining shares when the price goes upto ₹ 1600. He invests the proceeds in ₹ 10 Maruthi shares selling at ₹ 1184 find the number of ₹ 10 Maruthi shares purchased by Nihal.

ANSWERS 9.1

One mark questions:

1) ₹ 200 2) ₹ 4500 stock 3) ₹ 5100 4) 2.4%

Two marks questions:

1) ₹ 487.5, ₹ 7500 stock 2) 8.33%
3) ₹ 5000 stock, ₹ 400 4) ₹ 7200
5) ₹ 1020

Three marks questions:

1) ₹ 700, 4.66%
2) a) 9% stock b) 6½% stock
   c) 7% stock d) 6% stock
3) ₹ 72000 4) ₹ 80 5) ₹ 138
6) ₹ 18000 7) 2400, 1900 8) ₹ 165
9) ₹ 9800 10) ₹ 55 11) ₹ 2400
12) ₹ 30,000 in 5% stock, ₹ 12,000 in 7.5% stock. Income is 1200 + 1200 = 2400
13) ₹ 75
14) a) ₹ 63 b) ₹ 1190.21
15) a) ₹5625, ₹9375 b) ₹ 125
16) ₹27,500 in 8%, ₹ 27,500 in 12% 17) ₹ 81
18) ₹ 9400 stock, ₹ 283.5 increase
19) ₹ 20,000, ₹ 10,000
20) ₹250 = No. of shares

9.5 Brokerage :

Stock exchange is the name of the market where the transfer of stock are done through brokers. The brokers charge a small fee as commission from both the buyers as well as from the sellers. This commission is called brokerage. Brokerage is always calculated on the market value.

a) While selling shares, since you have to pay brokerage the net amount received will be market value – Brokerage.

b) While buying shares, since you have to pay brokerage, the total amount to be paid is market value + brokerage.

Example 16 :

Prashanth sold 25 Biocon shares when the market price was ₹ 400 per share. He then bought 10 Reliance shares which were selling at ₹ 800 per share. Brokerage for each transaction was 0.25%. The balance amount he gave to his daughter Sonu for shopping. How much did Sonu receive.

Solution :

Amount received by selling

Beocon shares = $25 \times 400 = ₹\ 10{,}000$

$$\text{Brokerage} = \frac{0.25}{100} \times 10000 = ₹\ 25$$

Net Amount received = $10{,}000 - 25 = ₹\ 9975$

Amount paid for Reliance shares = $800 \times 10 = ₹\ 8000$

$$\text{Brokerage} = \frac{0.25}{100} \times 8000 = ₹\ 20$$

Net Amount paid = $8000 + 20 = ₹\ 8020$

Balance Amount = $9975 - 8020 = ₹\ 1955$

$\therefore$ Amount paid to Sonu for shopping is ₹ 1955

Example 17:

If a client buys shares worth ₹ 1,25,000 and sells shares worth ₹ 75,000 through a broker assuming that the buying side brokerage is 0.5 and the selling side brokerage is 0.25%. Find the total brokerage paid to the broker.

Solution:

Brokerage while buying

$$= \frac{0.25}{100} \times 1{,}25{,}000$$

$$= ₹\ 625$$

Brokerage while selling

$$= \frac{0.25}{100} \times 75{,}000$$

$$= 187.5$$

$\therefore$ Total brokerage = $625 + 187.5 = ₹\ 812.5$

EXERCISE 9.2

Three marks questions:

1. A man owns 50 SBI shares which are now selling at the rate of 1800. He needs 50,000 for his daughter's education. He decides to sell 25 SBI shares. The brokerage charged is 0.25%. How much more money does he need to arrange after selling the share.
2. Rakshith decides to invest in TCS shares which are selling at 2020 per share. How much money is required to purchase 10 shares if the brokerage is 0.5%
3. Ritu purchased 200 HDFC shares when the price was 625 and then sold all the shares when the price went upto 715. If the brokerage for each transaction was 1%. How much did Ritu gain.
4. If a client buys shares worth ₹ 90,000 and sells shares worth ₹ 1,10,000 through a stock broker calculate the brokerage payable to the stock broker if the brokerage rate is 0.5% each side.
5. Veena buys 100 shares of Karnataka bank at ₹ 101 per share. She pays ₹ 10,130.3 to her broker. What is the total brokerage she paid and calculate the percentage rate of brokerage.
6. Mr. Ravi sold ₹ 2,250 stock at 75 and bought stock at 88.5 with the proceeds. How much stock does he buy if the brokerage is 2% for selling and 1.5% for buying.

ANSWERS 9.2

1) ₹ 5112.5 2) ₹ 20301 3) ₹ 15320 4) ₹ 1000

5) 0.3% 6) ₹ 1841.02

* * * * *

Chapter

10

LEARNING CURVE

10.1 Introduction :

By studying learning curve, we acquire skill and knowledge of performing a particular task repeatedly, we perfect the task and also gain efficiency in doing the same. This is called as **learning process.** The theory of learning curve deals with labour efficiency.

10.2 Learning curve :

Definition : Learning curve is defined as the curvilinear relation between the decrease in average labour hours per unit with increase in the total output.

10.3 Learning curve ratio :

The slope of the learning curve is called as the learning curve ratio. The learning curve ratio is expressed as a percentage

$$\textbf{Learning curve ratio} = \frac{\textbf{Average Labour cost of first 2N units}}{\textbf{Average labour cost of 1N units}}$$

Example 1 :

The average labour cost of a production for 100 units is ₹75. If the learning curve ratio is 0.75. Find the average cost of production for 50 units.

Solution :

Average labour cost of 100 units = ₹ 75

Learning curve ratio = 0.75

Let the average cost of 50 units = ₹ x

$$\text{Learning curve ratio} = \frac{\text{Average cost of 2N units}}{\text{Average cost of N units}}$$

$$0.75 = \frac{75}{x}$$

$$x = \frac{75}{0.75}$$

$$x = \frac{7500}{75}$$

$$x = 100$$

$\therefore$ the average labour cost of 50 units = ₹100

Example 2 :

The average labour cost of 100 units is ₹ 50. The learning curve ratio is 0.5. What is the average labour cost of 200 units?

Solution :

Let the average labour cost of 200 units = ₹x

$$\text{Learning curve ratio} = \frac{\text{Average labour cost of 2N units}}{\text{Average labour cost N units}}$$

$$0.5 = \frac{x}{50}$$

$$0.5 \times 50 = x$$

$$x = 25$$

$\therefore$ The average labour cost of 200 units = ₹ 25

Example 3:

The average labour cost for the first 150 units of production by a factory is ₹ 50 when the output was doubled to 300 units, the average labour cost was ₹ 25. Find the learning curve ratio and the learning curve percent.

Solution :

Average labour cost of 150 units = ₹ 50

Average labour cost of 300 units = ₹ 25

$$\text{Learning curve ratio} = \frac{\text{Average cost of first 2N units}}{\text{Average cost of first N units}}$$

$$= \frac{₹25}{₹50} = 0.5$$

Learning curve percent = $0.5 \times 100 = 50\%$

10.4 Learning effect:

The effeciency in performance of a labourer increases by repeatedly doing the task. This is due to the learning process and the learning effect improves the production. **For example** 80% learning effect means that when the cumulative output is doubled the cumulative average labour hours per unit will be 80% of the previous level.

The following tables show the learning effect of 90%, 80%, 70%, 60% and 50% when the time required to produce 1 unit is 100 hrs.

Table 1 : 90% learning effect

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours	Average hours per additional
1	1	100	100	100
1	2	90% of 100 $\frac{90\times100}{100} = 90$	180	$\frac{180-100}{2-1} = 80$
2	4	90% of 90 $\frac{90\times90}{100} = 81$	324	$\frac{324-180}{4-2} = 72$
4	8	90% of 81 = 72.9	583.2	$\frac{583.2324}{8-4} = 64.8$
8	16	80% of 72.9 = 65.6	11049.76	$\frac{1049.76-583.2}{16-8} = 58.32$

Table 2 : 80% learning effect

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours	Average hours per additional
1	1	100	100	100.00
1	2	80% of 100 = 80	160	60.00
2	4	80% of 80 = 64	256	48.00
4	8	80% of 64 = 51.2	409.6	38.40
8	16	80% of 57.2 = 40.96	655.3	30.72

Table 3 : 70% learning effect

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours	Average hours per additional
1	1	100	100	100.00
1	2	70% of 100 = 70	140	40.00
2	4	70% of 70 = 49	196	28.00
4	8	70% of 49 = 34.30	274.4	19.60
8	16	70% of 34.30	384.16	13.72

Table 4 : 60% learning effect

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours	Average hours per additional
1	1	100	100	100
1	2	60% of 100 = 60	120	20
2	4	60% of 60 = 36	144	12
4	8	60% of 36 = 21.6	172.8	7.2
8	16	60% of 21.6 = 12.96	207.36	4.32

Table 5 : 50% learning effect

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours	Average hours per additional
1	1	100	100	100.00
1	2	50% of 100 = 50	100	00.00

Note : From this table it is clear that learning curve of 50% or less is mathematically not possible. Generally 80% learning effect is commonly used in day to day life.

10.5 Learning curve equation

The equation $\boxed{y = a \cdot x^b}$ represents learning curve equation. Here

y = Cumulative average time per unit

a = Time for first unit

x = Cumulative total number of units produced

$$b = \text{index of learning} = \frac{\log (\text{learning effect})}{\log 2}$$

Taking log on both the sides of learning equation we have;

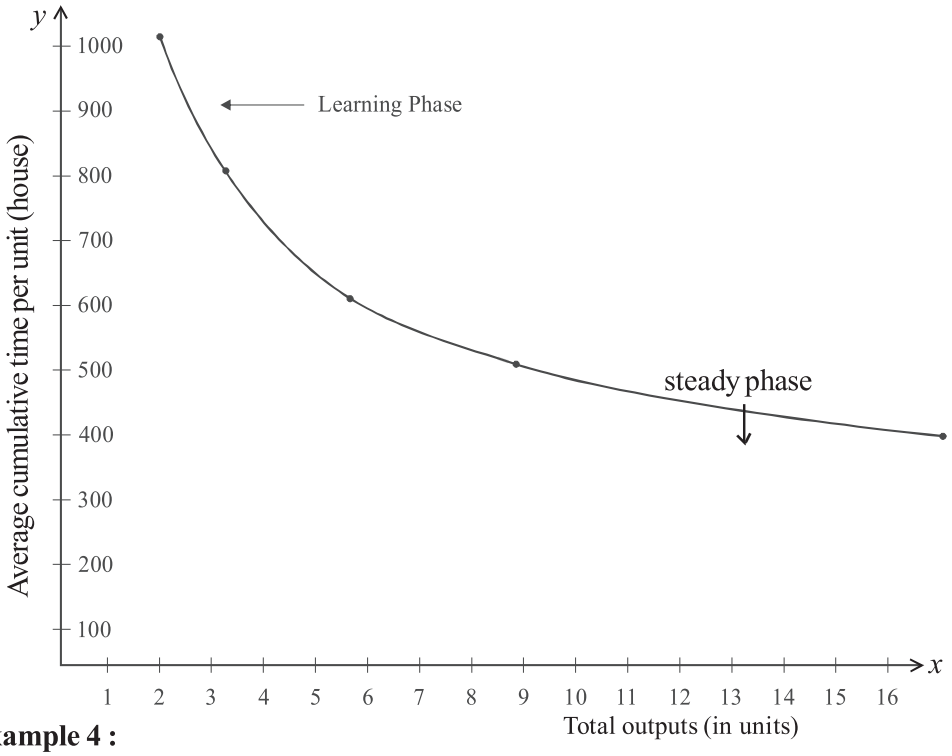
Which is a Linear Equation

$\log y = \log a + b \log x$

Table for the Graph

Total output in units	Cumulative and time per unit
1	1000
2	800
4	640
8	512
16	409.6

Graphical representation of the learning curve (80% learning effect)



Example 4 :

Give the formula for learning index.

Solution :

$$\text{Learning index} = \frac{\log (\text{learning effect})}{\log 2}$$

Example 5 :

The production manager of a company obtained the following equation for the learning effect.

$$\mathbf{y = 1356\ x^{-0.3219}}$$

This function is based on the company's experience for assembling the first 50 units of the product. Find the labour hours required to assemble 100 units.

Solution :

$y = a\,x^b$ Apply log on both sides

$\log y = \log a + b \log x$

$y =$ average cumulative time period

a = time taken for I lot / I unit

x = cumulative total no. of lots

b = Index of learning

where $a = 1356.4$, $x = 2$, $b = -0.3219$

$\log y = \log a + b \log x$

$\log y = \log 1356 + -0.3219 \log 2$

$\log y = 3.1324 - 0.3219\ (0.3010)$

$\log y = 3.1324 - 0.0969$

$\log y = 3.0355$

y = Anti log 3.0355 = 1085.18 hrs

Total time for 2 units = 2 × 1085.18 hrs = 2170.35 hrs.

Example 6 :

A Motor Company Ltd., has observed that a 90% learning effect applies to all labour related costs. Whenever a new product is taken up for production, the anticipated production to 320 units for the coming year. The production is done in lots of 10 units each. Each lot requires 1000 hours at ₹ 15/hour. Calculate the total labour hours and labour cost to manufacture 320 units.

Solution : 1 cost = 10 unit

Lots produced	Total output in Lots	Cumulative average time per unit (hours)	Total hours
1	1	1000	1000
1	2	90% of 1000 = 900	1800
2	4	90% of 900 = 810	3240
4	8	90% of 810 = 729	5832
8	16	90% of 729 = 656.1	10,497.6
16	32	90% of 656.1 = 590.49	18,895.68

The total hours is 18,895.68 hrs. and the labour cost at

₹ 15 per hr = 18,895.68 × 15

= ₹ 2,83,435.20

Example 7 :

An Engineering company has 80% learning effect and spends 500 hours for the prototype. Estimate the labour cost of producing 7 engines of new order if the labour cost is ₹ 40 per hour.

Solution :

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours
1	1	500	500
1	2	80% of 500 = 400	800
2	4	80% of 400 = 320	1280
4	8	80% 320 = 256	2048

Total hours of 8 engines = 2048
Total hours of 1 engine = –500
Total hours for 7 engines = 1548 hrs.

If the labour cost ₹ 40 per hour.

Total labour cost of producing 7 engines = 1548 × 40 = ₹ 61920

Example 8 :

The average labour cost for first 100 units produced by a firm is ₹ 25. When the output was doubled to 200 units, the average labour cost was ₹ 20. Determine learning curve ratio and learning curve percent.

Solution : Learning curve ratio = $\frac{\text{Average labour cost for 2 N units}}{\text{Average labour cost for N units}}$

$$= \frac{20}{25} = \frac{4}{5} = 0.8$$

Learning curve percent = 0.8 × 100 = 80%

Example 9 :

xyz company supplies water tankers to the Government. The first water tanker takes 20,000 labour hours. The government auditors suggest that there should be 90% learning effect rate. The management expects an order of 8 water tankers in the next year. what will be the labour cost the company will incur at the rate of ₹ 20 per hour ?

Solution :

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours
1	1	20,000	20,000
1	2	90% of 20,000 = 18,000	36,000
2	4	90% of 18,000 = 16,200	64,800
4	8	90% of 16,200 = 14,580	1,16,640

Time taken for 8 water tankers = 116640 hrs.

Total cost the company will incur at the 20 per hour

= 116640 × 20 = ₹ 23,32,800

Example 10 :

A company requires 100 hours to produce the first 10 units at ₹ 15 per hour. The learning curve effect is 80%. Find the total labour cost to produce a total of 160 units.

Solution : Assume 10 unit = 1 lot

Units produced	Total output in lots	Cumulative average time per lot (hours)	Total hours
1	1	100	100
1	2	80% of 100 = 80	160
2	4	80% of 80 = 64	256
4	8	80% of 64 = 51.2	409.6
8	16 (160 unit)	80% of 51.2 = 40.96	655.36

Time taken for 160 units = 655.36 hrs.

Total cost the company has to incur at the rate of 15 ₹ per hour

= 655.36 × 15 = ₹ 9830.40

EXERCISE 10.1

One mark questions:

1. Define learning curve.
2. Write the formula for learning index.
3. Find the index of learning for 70% learning effect.
4. Find the index of learning for 80% learning effect.
5. Write the learning curve equation.

Five mark questions:

6. An engineering company has 80% learning effect and spends 1000 hours to produce 1 lot of the product. Estimate the labour cost of producing 8 lots of the product if the labour cost is ₹ 40 per hour.
7. ABC company required 1000 hours to produce first 30 engines. If the learning effect is 90%. Find the total labour cost at ₹ 20 per hour to produce total of 120 engines?
8. The production manager of a company obtained the following equation for the learning effect.

$$y = 1400\ x^{-0.3}$$

This function is based on the company's experience for assembling the first 50 units of the product. The company was asked to bid a new order of 100 additional units and the labour cost for producing an additional 100 units at the rate of ₹ 20 per hour.

9. An aircraft manufacturer supplies aircraft engines to different airlines. They have just completed an initial order for 30 engines involving a total of 6000 direct labour hours at ₹ 20 per hour. They have been asked to bid for a prospective contract for a supply of 90 engines. It is expected that there will be 80% learning effect. Estimate the labour cost for the new order.
10. A first sample batch of 50 units of product A took 80 hours to make. The company now wishes to estimate the average time per unit will be if the total output of product A is 200 units and 80% learning rate applies.

Answers 10.1

1. Definition : Learning curve is the curvilinear relationship between the decrease in average labour hours per unit with increase in the total output

2. Learning index $= \dfrac{\text{Log (learning effect)}}{\log 2}$

3. Learning index $= \dfrac{\text{Log (70\%)}}{\log 2} = -0.5146$

4. Learning index $= \dfrac{\log (80\%)}{\log 2} = -0.3219$

5. Learning curve *eq* is $y = a\,x^b$
   y = Cumulative avg. time per unit
   a = Time for 1st unit
   x = Cumulative total number of units
   b = Index of learning

6.

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours
1	1	1000	1000
1	2	80% of 1000 = 800	1600
2	4	80% of 800 = 640	2560
4	8	80% of 640 = 512	4096

Total hours to produce 8 lots = 4096 hrs

Total labour cost at ₹40 per hr = 4096 × 40

= ₹ 1,63,840

7) Assume 1 lot = 30 engine

Units produced	Total output in units	Cumulative average time per unit (hours)	Total hours
1	1	1000	1000
1	2	90% of 1000 = 900	1800
2	4	90% of 900 = 810	3240

The total number of hours to produce 4 lot (120 engine) = 3240

The lab cost at 20 per hour = 3240 × 20 = ₹ 64,800

8) $y = 1400\ x^{-0.3}$

$\log y = \log 1400 - 0.3 \log 2$

$\log y = 3.14612 - (0.3)\ 0.4471$

$\log y = 3.14612 - 0.14313$

$\log y = 3.05582$

y = Anti log 3.00297

$y = 1007$

The total labour hours = 1007 × 3 = 3021 hours

The labour cost at ₹ 20 per hr. = 1621 × 20 = ₹ 32,420

Total labours hours for additional 100 unit = 3021 – 1400 = 1621

9) 1 lot = 30 engine

Lot produced	Total output in lot	Cumulative average time per lot (hours)	Total hours
1	1	6000	6000
1	2	80% of 6000 = 4800	9600
2	4	80% of 4800 = 3840	15360

The total hours to manufacture 4 lots = 15360

$$= \frac{6000}{9360}$$

Total lab cost for 9360 hrs at ₹ 20 per hour = 9360 × 20 = ₹1,87,200

10)

Lot produced	Total output in Lot	Cumulative average time per Lot (hours)	Total hours
1	1	80	80
1	2	80% of 80 = 64	128
2	4	80% of 64 = 51.2	204.8

Total Labourhours = 204.8 (Assume 50 unit = 1 lot)

* * * * *

Chapter

11

LINEAR PROGRAMMING PROBLEMS

11.1 Introduction :

In the field of Business and Industrial world, decision making is a very important task. Any organisation-big or small, has variety of resources like men, machine, money and material at their disposal. However the supply of these resources may be limited. In such situations, the management must find the best allocation of the available resources in order to maximize the profit or minimize the cost etc., utilize the resources to the best possible extent. In these contexts Linear Programming provides an optimal solution with the available information.

Linear Programming is a mathematical technique which deals with the optimization i.e. minimization or maximization of activities subject to the available resources.

For instance :

(i) A biscuit manufacturing factory may be interested in taking decision regarding the type of biscuits to be manufactured and the quantity produced so as to earn maximum profit.

(ii) A factory having some machines to manufacture cars would like to know the best way to utilize its machines so that maximum production is made possible in minimum time available.

(iii) A dietician may want to suggest food with certain basic vitamins and proteins. She may like to know the best way to prescribe food with optimum requirement of vitamins and proteins at the minimum cost.

These are few examples where Linear Programming provides an optimal solution. Linear programming takes optimal decisions from the available set of choices. Programming (or planning) refers to the process of determining a particular plan of action.

11.2 Definition:

Linear programming deals with the optimization of a linear function of variables subject to a set of linear constraints.

11.3 Formulation of Linear Programming problem

Consider the following steps for the construction of L.P.P. in a given situation.

Step 1 : Identify the unknown variables-also known as decision variables and denote them in terms of algebraic symbols.

Step 2 : Identify the objective of the problem and represent it with Z.

Step 3 : Identify all the restrictions-also known as constraints. Represent them as equations or inequations in terms of symbols.

Let us consider some examples in order to explain the formulation of L.P.P.

WORKED EXAMPLES

Example 1 :

Production allocation problem

A manufacturer Akhil produces two types of models M_1 and M_2. Each model M_1 requires 4 hours of grinding and 2 hours of polishing, whereas each model M_2 requires 2 hours of grinding and 6 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works for a maximum of 80 hours a week and each polisher works for a maximum of 180 hours a week. Profit on a M_1 model is ₹ 3 and on a M_2 model is ₹4. Whatever is produced in a week is sold in the market. Formulate L.P.P.

Solution :

Let 'x' be the number of M_1 models and 'y' be the number of M_2 models. The weekly profit of the manufacturer is given by

$Z = 3x + 4y$ which is the objective function. This is on the assumption that he has enough grinders and polishers capacity to produce these number of models. Now in order to produce these number of models, the total number of grinding hours needed per week is given by $4x + 2y$ and the total number of polishing hours needed per week given by $2x + 6y$. Since the manufacturer does not have more than 80 hours of grinding nor does he have 180 hours of polishing, we must have

$$4x + 2y \leq 80$$

$$2x + 6y \leq 180$$

Also it is not possible for the manufacturer to produce a negative number of models, it is obvious that we must also have $x \geq 0,\ y \geq 0$. Hence the manufacturer's allocation problem can be put in the following mathematical form

To maximize the profit

$$z = 3x + 4y$$

Subject to the constraints

$$4x + 2y \leq 80$$

$$2x + 6y \leq 180$$

$$x \geq 0,\ y \geq 0$$

The above mathematical expression is known as mathematical formulation. From the above in equations we find out the values of x and y. The values of x and y are substituted in the objective function z. The maximum value of the objective function is known as optimal value.

Example 2 :

A resourceful home decorater Ananthu manufactures two tyes of lamps say A and B. Both lamps go through two technicians, first a cutter and second a finisher. Lamp 'A' requires 2 hours of cutter's time and 1 hour of finisher's time. Lamp ' B' requires 1 hour of cutter's time and 2 hours of the finisher's time. The cutter has 104 hours and the finisher has 76 hours of available time. Profit on a lamp 'A' is ₹ 6 and profit on a lamp 'B' is ₹ 11. assuming that she can sell all that she produces, formulate L.P.P.

Solution :

Let the manufacturer makes x and y lamps of type A and B respectively. Then the total profit which is the objective function z is given by $z = 6x + 11y$. Total time of the cutter used in preparing 'x' lamps of type A and 'y' lamps of type B is $2x + y$. Since the cutter has only 104 hours, we have $2x + y \leq 104$. Similarly,

Total time of the finisher in preparing x lamps of type A and y lamps of type B is $x + 2y \leq 76$

Hence the decorater's problem is to find x and y

which maximizes $Z = 6x + 11y$ subject to the constraints

$$2x + y \leq 104$$

$$x + 2y \leq 76$$

$$x \geq 0, \quad y \geq 0$$

Example 3:

(Diet problem)

Pratheek wants to decide the constituents of diet which will fulfill his daily requirements of proteins, fats and carbohydrates at minimum cost. The combination is made among 2 food products A and B whose contents are indicated below:

Food	A	B	Minimum requirements
Proteins (mg)	5	2	800
Fats (mg)	6	5	200
Carbohydrates (mg)	4	3	700
Cost / Unit in ₹	70	50	

Formulate L.P.P

Solution:

Let the number of units of food A be x and that of food B be y respectively.

Therefore objective function is given by

Minimize $z = 70x + 50y$

Subject to constraints:

$$5x + 2y \geq 800$$
$$6x + 5y \geq 200$$
$$4x + 3y \geq 700$$

where $x, y \geq 0$

Example 4:

Arvind, a chemist provides his customers at the least cost, the minimum daily requirements of 2 vitamins A and B by using a mixture of 2 products M and N. The amount of each vitamin in 1 gm of each product, the cost per gram of each product and the minimum daily requirements are given below:

Products	No. of units of each vitamin in one gram of each product			
	Vitamin A	Vitamin B	Minimum requirement of each vitamin	Cost/gm of each product
M	6	2	12	20
N	2	2	8	16

Formulate L.P.P

Solution :

Let the amount of product M be x and the amount of product N be y. Looking at the table, we can formulate the L.P.P. problem as follows:

Objective function is to minimize $z = 20x + 16y$

Subject to the constraints $6x + 2y \geq 12$

$$2x + 2y \geq 8$$
$$x, y \geq 0$$

Example 5 :

A company owned by Swathi can produce two types of high quality shoes. Each shoe of the first kind requires thrice as much as time as the second kind. If all shoes are of the second kind only, the company can produce a total of 600 pairs a day. Only a maximum of 150 pairs of the first kind and 400 of the second kind can be sold a day. If the profit per pair of the first kind is ₹400 and per pair of the second kind is ₹150 then formulate L.P.P.

Solution :

Let the number of pair of shoes produced per day of the first kind requires 3 times as of the

second kind and the maximum possible production of the second kind is 600 pairs, we obtain the constraint : $3x + y \leq 600$

Next we have the sales constraint. Since only a maximum of 150 pairs of the first kind and 400 pairs of the second kind can be sold, we get $x \leq 150,\ y \leq 400$

Since neither production can be negative, we get $x \geq 0,\ y \geq 0$

Since profits on the two kinds per pair are respectively ₹400 and ₹150, the objective of the maximizing profit requires to maximize $z = 400x + 150y$.

Summarising we get the following L.P.P.

Maximize $z = 400x + 150y$ subject to the constraints

$$3x + y \leq 600$$

$$x \leq 150,\ y \leq 400$$

$$x \geq 0,\ y \geq 0$$

Example 6:

A company owned by Vaishnavi wants to advertise its products in two magazines I and II. Magazine I reaches 2000 potential customers and magazine II reaches 3000 potential customers. The two magazines charge ₹400 and ₹600 per page. The company can spend not more than ₹6000 per month on advertisement. The total reach for the income group under ₹20,000 p.a. should not exceed 4000 potential customers. The reach for this income group in the two magazine I and II is 400 and 200. How many days should the advertisement be placed in the two magazines to reach the maximum.

Formulate L.P.P.

Solution :

Let x and y be the number of days on which the advertisement is placed in the two magazines I and II respectively.

Then the required LPP will be:

Maximise $z = 2000x + 3000y$

Subject to the constraints:

$$400x + 600y \leq 6000$$

$$400x + 200y \leq 20000$$

$$x, y \geq 0$$

Example 7 :

A producer named Samarth has 30 and 17 units of labour and capital respectively which he can use to produce two types of goods A and B. To produce one unit of A,

2 units of labour and 3 units of capital are required. Similarly 3 units of labour and 1 unit of capital is required to produce 1 unit of B. If A and B are priced at ₹100 and ₹120 per unit respectively how should the producer use his resources to maximize the total revenue.

Form a LPP to maximize his revenue.

Solution:

The above information is given in the following table:

Goods	Labour	Capital	Cost (in ₹)
A	2	3	100
B	3	1	120
Available units	30	17	

Let x be the number of units of A and y be the number of units of B produced.

Mathematical formulation of given problem is as follows:

Maximise $z = 100x + 120y$

Subject to the constraints

$$2x + 3y \leq 30$$

$$3x + y \leq 17$$

$$x, y \geq 0$$

Example 8 :

A soft drink firm owned by Ektha has two bottling plants one located at P and the other located at Q. Each plant produces three different soft drinks A,B and C. The capacities of two plants in number of bottles per day are as follows:

Product	Plant P	Plant Q
A	**3000**	**1000**
B	**2000**	**6000**
C	**1000**	**1000**

A market survey indicates that during the month of April, there wil be a demand for 24000 bottles of A, 48000 bottles of B, 16000 bottles of C. The cost of running the two plants P and Q are respectively ₹6000 and ₹8000 per day.

Formulate L.P.P.

Solution :

Let us suppose that the plants P and Q should run for x days and y days respectively, then the total cost of running the two plants is ₹ $6000x + 8000y$. The following table helps us in formulating the problem.

Plant	No. of working days	Production			Cost of running
		A	B	C	
P	x	$3000x$	$2000x$	$1000x$	$6000x$
Q	y	$1000y$	$6000y$	$1000y$	$8000y$

Our problem is to find x and y, so that the cost $z = 6000x + 8000y$ is minimum subject to constraints

$$x \geq 0, \quad y \geq 0$$

$$3000x + 1000y \geq 24000$$

$$2000x + 6000y \geq 48000$$

$$1000x + 1000y \geq 16000$$

Example 9:

Yatharth runs a company which manufactures two products A and B. Product A is sold at ₹200 per unit and takes 30 minutes to make. Product B is sold at ₹300 per unit and takes 1 hour to make. There is a permanent order of 14 units of product A and 16 units of product B. The working week for the production of both A and B consists of 40 hours and the weekly turnover must not be less than ₹10,000. If the profit on each unit of product A is ₹20 and on product B is ₹30, then formulate L.P.P.

Solution :

Let x units of product A and y units of product B be manufactured.

The problem is to maximise $P = 20x + 30y$

Subject to constraints

$$x \geq 0, \quad y \geq 0$$

$$x \geq 14, \quad y \geq 16$$

$$200x + 300y \geq 10000$$

$$\frac{1}{2}x + 1y \leq 40$$

11.3.1 General Linear programming problem

A problem in linear programming is formulated mathematically by first identifying a set of decision variables say $x_1, x_2, x_3 \ldots\ldots x_n$ which are subjected to certain linear constraints written in the form of inequalities

$$a_{11}x_1 + a_{12} \times x_2 + ...+ a_{1n}x_n (\leq, \geq) b_1$$

$$a_{21}x_1 + a_{22} \times x_2 + ...+ a_{2n}x_n (\leq, \geq) b_2$$

$$a_{m1}x_1 + a_{m2} \times x_2 + ...+ a_{mn}x_n (\leq, \geq) b_m$$

where the coefficients aij and b_1, b_2, b_3 ..... b_m are constants and the non-negative constraints are given by $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 .....x_n \geq 0$

The aim of this mathematical formulation is to optimize a linear expression

$z = c_1x_1 + c_2x_2 + c_3x_3 + ...+ c_nx_n$ for a given constraints $c_1,\ c_2 .....c_n$

This is called as the **objective function**. Thus the optimisation (maximization or minimization) of the objective function is called the General Linear Programming problem.

11.3.2 Structure of Linear Programming Model:

The general structure of Linear Programming model consists of three basic components or elements.

1) **Objective function:** This is the function which is to be maximised or minimised in the problem. This function is always a linear function consisting of variables. It is denoted by '*Z*'.

2) **Decision variables :** These are the variables which affect the value of the objective function. These variables are denoted by $x_1, x_2, x_3 ....x_n$.

3) **Constraints :** These are the conditions existing with regard to the decision variables. These constraints will be in the form of linear inequalities.

Also there exists some other terminologies which are as follows:

(i) **Feasible region** : while solving the linear programming problem by graphical method, every constraint is satisfied by a part of the co-ordinate plane. The region of the cartesian plane which is the intersection of all these parts, thus representing the region which satisfies all the constraints is called the feasible region.

(ii) **Feasible solution :** A solution which is a set of real values which satisfies all the constraints is called Feasible solution.

(iii) **Optimal solution :** The feasible solution which achieves the objective on hand that is either maximizes or minimizes the objective function as the case may be is the optimal solution. It optimizes the objective. In the graphical method this solution occurs at one of the corners of the feasible region.

Exercise 11.1

Four mark questions:

1) Smaran being a manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts while it takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹2.50 per package on nuts and ₹ 1 per package on bolts. Form a linear programming problem to maximize his profit, if he operates each machine for at most 12 hours a day.

2) Vishal consumes two types of food, A and B everyday to obtain minimum 8 units of protein, 12 units carbohydrates and 9 units of fat which is his daily requirements. 1kg of food A contains 2, 6, 1 units of protein, carbohydrates and fat respectively. 1 kg of food B contains 1, 1 and 3 units of protein, carbohydrates and fat respectively. Food A cost ₹8 per kg and food B cost ₹ 5 per kg. Form an L.P.P. to find how many kilogram of each food should he buy daily to minimize his cost of food and still meet minimal nutritional requirements.

3) Archana, a dietician wishes to mix two types of foods F1 and F2 in such a way that the vitamin contents of the mixture contains at least 6 units of vitamin A and 8 units of vitamin B. Food F1 contains 2 units/kg of vitamin A and 3 units/kg of vitamin B while food F2 contains 3 units/kg of vitamin A and 4 units/kg of vitamin B. Food F1 costs ₹50 per kg and food F2 costs ₹75 per kg. Formulate the problem as L.P.P. to minimize the cost of the mixture.

4) A furniture maker Jatin has 6 units of wood and 28 hours of free time in which he will make decorative screens. Two models have sold well in the past, so he will restrict himself to those two. He estimates that model 1 requires 2 units of wood and 7 hours of time. Model 2 requires 1 unit of wood and 8 hours of time. The prices of the models are ₹120 and ₹ 80 respectively. Formulate LPP to determine how many screens of each model should the furniture maker assemble if the wishes to maximize his sales revenue.

5) A firm owned by Sheshnag manufactures two types of products A and B and sell them at a profit of ₹2 on type A and ₹3 on type B. Each product is processed on machines M1 and M2. Type A requires one minute of processing time on M1 and two minutes on M2. Type B requires one minute of time on M1 and one minute on M2. The machine M1 is available for not more than 6 hours 40 minutes while M2 is available for 10 hours during any working day. Formulate the L.P.P. in order to find how many products of each type should the firm produce each day so that profit is maximum.

6) Shreya company produces 2 types of shoes A and B. A is of superior type and B is of ordinary type. Each shoe of the first kind requires thrice as much time as the second kind. If all shoes are of the second kind only, the company can produce a total of 650 pairs a day. Only a maximum of 150 pairs of the first kind and 400 of the second kind can be sold per day. If the profit per pair of the first kind is ₹400 and per pair of the second kind is ₹150. Find the optimal product mix to be produced to maximize the profit by formulating on L.P.P. model.

7) Producer Rahul has 50 and 85 units of labour and capital respectively which he can use to produce two types of goods A and B. To produce one unit of A, 1 unit of labour and 2 units of capital are required. Similarly 3 units of labour and 2 units of capital is required to produce 1 unit of B. If A and B are priced at ₹100 and ₹150 per unit respectively, how should the producer use his resources to maximize the total revenue. Formulate the L.P.P. to maximize his total revenue?

8) Nikhil pesticide company must produce 200 kg mixture consisting of chemicals A and B daily. A cost ₹3 per kg and B cost ₹8 per kg. Maximum 80 kg of chemical A and atleast 60 kg. of chemical B should be used. Formulate L.P.P. model to minimize the cost.

Answers 11.1

1. Maximize, $z = 2.5x + y$
subject to the constraints;
$$x + 3y \le 12$$
$$3x + y \le 12$$
$$x, y \ge 0$$

2. Minimize, $z = 8x + 5y$
subject to the constraints;
$$2x + y \ge 8$$
$$6x + y \ge 12$$
$$x + 3y \ge 9$$
$$x, y \ge 0$$

3. Minimize, $z = 50x + 75y$
subject to constraints; $2x + 3y \ge 6$
$$3x + 4y \ge 8$$
$$x, y \ge 0$$

4. Maximize, $z = 120x + 80y$
subject to constraints; $2x + y \le 6$
$$7x + 8y \le 28$$
$$x, y \ge 0$$

5. Maximize, $z = 2x + 3y$
subject to constraints, $x + y \le 400$
$$2x + y \le 600$$
$$x, y \ge 0$$

6. Maximize, $z = 400x + 150y$
subject to constraints, $3x + y \le 650$
$$x \le 150$$
$$y \le 400$$
$$x, y \ge 0$$

7. Maximize, $z = 100x + 150y$

subject to constraints; $x + y \leq 50$
$2x + 2y \leq 85$
$x, y \geq 0$

8. Minimize, $z = 3x + 8y$

subject to constraints; $x + y \geq 200$
$x \leq 80$
$y \geq 60$
$x, y \geq 0$

11.4 Graphical solutions for Linear Programming problems:

The graphical method of solving a Linear Programming problem consists of the following steps.

Step 1: Represent the given problem in mathematical form i.e. formulate linear programming for the given problem.

Step 2: Represent the given constraints as equalities on x, y (or $x_1, x_2 \ldots$) i.e. consider each inequality constraint as equation. Plot each equation on the graph as each will geometrically represent a straight line.

Step 3: Mark the region. If the inequality constraint corresponding to the line is $\leq$, then the region below the line lying in the first quadrant (due to the non-negativity of the variables) is shaded. This region includes the origin. For the inequality constraint $\geq$ sign, the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region.

Step 4 : Measure the values of the two decision variables x and y graphically at each of the vertices of the feasible region. The values of x and y at one of the vertices represents optimal solution.

Step 5: Calculate the value of the objective function of the corner points on the feasible region and select the one which gives the optimal solution i.e. in the case of maximization problem, optimal point corresponds to the corner point at which the objective function has a maximum value and in the case of minimization, the corner point which gives the objective function the minimum value is the optimal solution.

Note:

1) Stretch the objective function line till the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through atleast one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through atleast one corner of the feasible region.

The following illustrations demonstrates how L.P.P. solutions can be obtained by the graphical method.

Example 1:

A company owned by shree group produces two products P and Q. Each P requires 4 hours of grinding and 2 hours of polishing and each Q requires 2 hours of grinding and 5 hours of polishing. The total available hours for grinding is 20 hours and for polishing is 24 hours. Profit per unit of P is ₹6 and that of Q is ₹8. Formulate L.P.P. and solve graphically.

Solution :

Let 'x' be the units of P manufactured and 'y' be the units of Q manufactured. Consider the table formed as per the given details.

Products	Grinding hours	Polishing hours	Profit per unit
P	4	2	6
Q	2	5	8
Availability of time	20	24	

Formulation of L.P.P. can be done as follows

Maximise, $z = 6x + 8y$.

Subject to constraints $4x + 2y \leq 20$

$2x + 5y \leq 24$

$x, y \geq 0$

The given L.P.P. can be solved graphically by following the steps as given below.

Step 1: The given constraints in the form of inequalities to be converted in the equations format and then draw the graph of the obtained lines equations.

Refering to the given details, we get the equations $4x + 2y = 20$ and $2x + 5y = 24$ from the constraints.

Now to draw the graph of the above lines first construct the table.

$4x + 2y = 20$

x	0	5
y	10	0

(0,10) (5, 0)

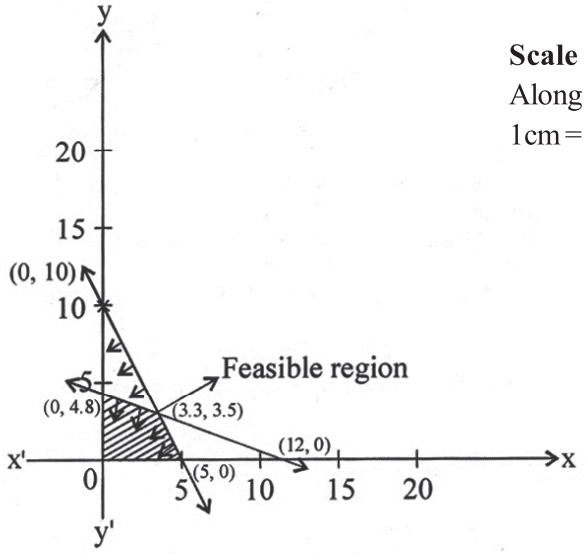
$2x + 5y = 24$

x	0	12
y	4.8	0

(0,48) (12, 0)

Using the obtained points we can draw the graph of the lines separately.

Step 2: Since both the given constraints are "less than on equal to" type the shading of the graph to be done in such a way that origin is to be included i.e. the part below the obtained lines to be shaded. Where over lapping of the shaded region occurs, we get feasible region step (1) and (2) are together shown in the graph which is drawn below.



Scale
Along x and y axis
1cm = 5 units

Note: Shading direction of the given lines is shown by the arrow marks. The region OABC is the feasible region where we find the overlapping of the shades (arrow marks) and this region gives the corner points O(0,0), A(0,4.8), B(3.3, 3.5) and C(5, 0)

Step 3: Optimal solution is obtained by substituing the values of x and y of the corner points in the objective function.

At O(0,0), $Z = 6(0) + 8(0) = 0$

At A(0, 4.8), $Z = 6(0) + 8(4.8) = 38.4$

At B(3.3, 3.5), $Z = 6(3.3) + 8(3.5) = 47.8$

At C(5, 0), $Z = 6(5) + 8(0) = 30$

Step 4: Thus the optimal solution of the graphical problem is obtained as

When $x = 3.3$ and $y = 3.5$ the maximum value of $Z = 47.8$

Example 2:

A diet for Sahana must contain at least 4000 units of vitamins, 50 units of minerals and 1400 calories. Two foods A and B available at a cost of ₹5 and ₹ 4 per unit respectively. If one unit of A contains 200 units of vitamin, 1 unit of mineral and 40 calories and one unit of food B contains 100 units of vitamins, 2 unit of minerals and 40 calories, find by graphical method what combination of foods be used to have least (minimum) cost.

Solution:

Variables	Food	Unit content of			Cost per unit (₹)
		Vitamins	Minerals	Calories	
x	A	200	1	40	5
y	B	100	2	40	4
Minimum requirement		400	50	1400	

x = number of units of food A, y = number of units of food B.

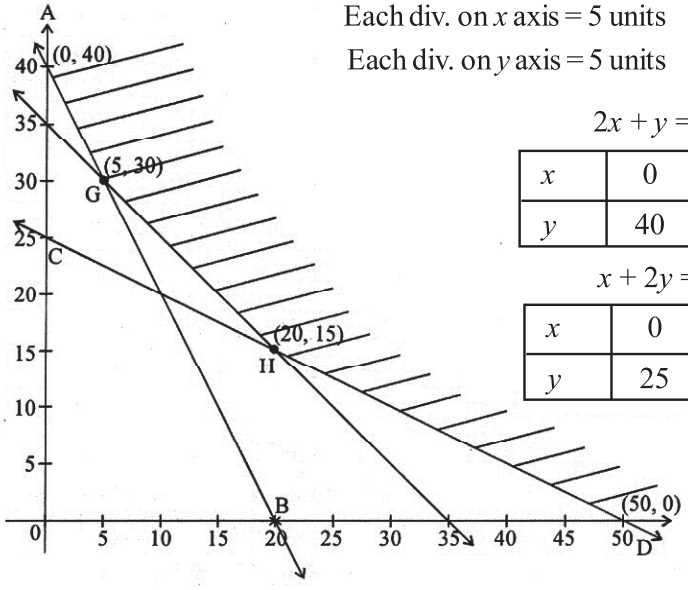
We shall formulate a Linear Programming model minimize $z = 5x + 4y$, subject to the constraints $200x + 100y \geq 4000; x + 2y \geq 50; 40x + 40y \geq 1400 \quad x, y \geq 0$

The above constraints can now be written as $2x + y \geq 40; x + 2y \geq 50; x + y \geq 35$ and $x \geq 0, y \geq 0$ indicates that the region is to lie in the first quadrant only.

As already stated in example 1 in equalities are to be graphed taking them as equalities. To draw the graph of $2x + y = 40$, put $x = 0$, we get i.e. A(0,40) as a point. Similarly put $y = 0$, we get $x = 20$ i.e. B(20,0) is another point joining the point A (0, 40) and B (20,0) by a line, we obtain the line graph $2x + y = 40$. Any point (x, y) lying one or above the line $2x + y = 40$ satisfies the constraint $2x + y \geq 40$. Similarly to draw the graph of $x + 2y = 50$, $y = 25$ ie C(0, 25) is a point and put $y = 0, x = 50$ ie D(50,0) is another point. Joining these two points by a line, we obtain the graph of $x + 2y = 50$. Finally we have $x + y = 35$, put $x = 0, y = 35$ and then put $y = 0, x = 35$. So E (0,35) and F(35, 0) are the points on the line $x + y = 35$. Now plot all the points as shown in the figure.

$x + y = 35$

x	0	35
y	35	0



Each div. on x axis = 5 units

Each div. on y axis = 5 units

$2x + y = 40$

x	0	20
y	40	0

$x + 2y = 50$

x	0	50
y	25	0

The set of points $\{(x, y) : x \geq 0, y \geq 0\}$ satisfying all the constraints constitute a feasible region which is shown as shaded region in the figure.

Solving $2x + y = 40, x + y = 35$ we get $x = 5$, $y = 30$ $\therefore$ G = (5, 30), Also by solving $x + 2y = 50$ and $x + y = 35$, we get $x = 20$, $y = 15$ $\therefore$ H = (20, 15).

We observe that the feasible region is unbounded on the upper side. But since the objective function $z = 5x + 4y$ is to be minimized, the optimum solution to the problem shall be located only at one of the extreme points A, G, H, D.

We shall evaluate the objective function $z = 5x + 4y$ at these extreme points.

At A (0, 40) z = $5x + 4y = 5(0) + 4(40) = 160$

At G (5, 30) z = $5x + 4y = 5(5) + 4(30) = 145$

At H (20, 15) z = $5x + 4y = 5(20) + 4(15) = 160$

At D(50, 0) z = $5x + 4y = 5(50) + 4(0) = 250$

Since the minimum value of the objective function z = 145 occurs at the extreme point G(5, 30) we conclude that the optimum solution to the given LPP is $x = 5$, $y = 30$ and minimum z = 145. Hence to have least cost, the diet should contain 5 units of food A and 30 units of food B.

Note : Cross check: Solving equations $2x + y = 40$, $x + 2y = 50$ and $x + y = 35$ by pair we get G = (5, 30) and H = (20, 15) Graphically also this must be correct.

Example 3:

Solve the following LPP graphically maximize $z = 6x + 8y$, subject to the constraints

$4x + 2y \leq 20$

$2x + 5y \leq 24$ and $x \geq 0, y \geq 0$

Mark the feasible region.

Solution :

Consider the first constraint $4x + 2y \leq 20$, taking it as equality we get $4x + 2y = 20$

put $x = 0$, we get $2y = 20 \rightarrow y = \frac{20}{2} = 10$

put $y = 0$, we get $4x = 20 \rightarrow x = \frac{20}{4} = 5$

$\therefore$ $4x + 2y = 20$ passes through the points A(0, 10) and B(5, 0)

Similarly for the 2nd constraint $2x + 5y = 24$

put $x = 0$, we get $5y = 24 \rightarrow y = \frac{24}{5} = 4.8$

put $y = 0$, we get $2x = 24 \rightarrow x = \frac{24}{2} = 12$

$\therefore$ $2x + 5y = 24$ passes through the points C(0, 48) and D(12,0)

Plot each equation on the graph. The shaded portion in the figure represent the feasible region which is bounded by the two axes and the two lines,

$4x + 2y = 20$ and $2x + 5y = 24$

The shaded area OCEB represents the set of all feasible solutions.

Each div. on x axis = 1 unit

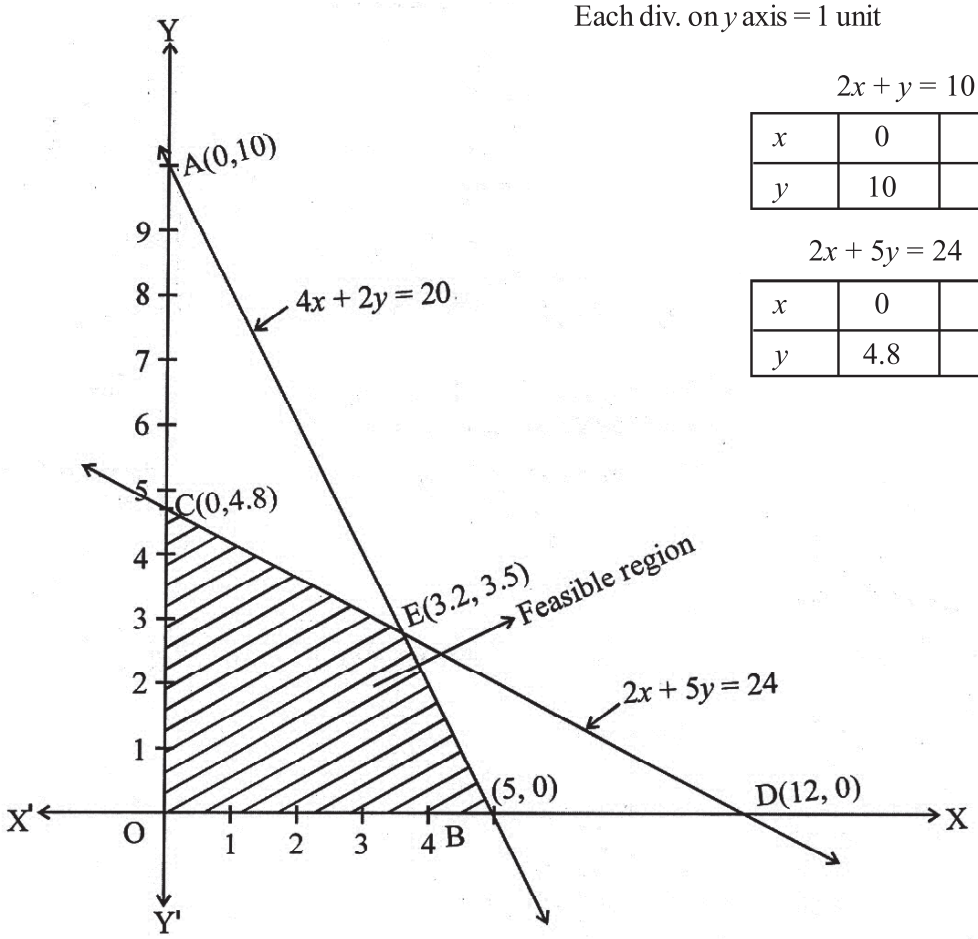
Each div. on y axis = 1 unit

$2x + y = 10$

x	0	5
y	10	0

$2x + 5y = 24$

x	0	12
y	4.8	0



Solving $4x + 2y = 20$ and

$$2x + 5y = 24$$

we get $y = 3.5$ and $x = 3.2$

thus E = (3.2, 3.5)

The four extreme points of the feasible region are 0(0,0), C(0,4.8), E(3.2, 3.5) and B(5,0)

Now we have to evaluate the objective function $z = 6x + 8y$ at these extreme points.

At O (0,0) ; z = 6(0) + 8(0) = 0

At C (0, 4.8) ; z = 6(0) + 8(4.8) = 38.4

At E (3.2, 3.5 ; z = 6(3.2) + 8(3.5) = 19.2 + 28 = 47.2

At B (5, 0) ; z = 6(5) + 8(0) = 30

Thus the optimum solution to the given problem occurs at E(3.2, 3.5) ie $x = 3.2$, $y = 3.5$ with the objective function value 47.2

$\therefore$ Optimal solution is obtained when $x = 3.2$ and $y = 3.5$

Example 4 :

Maximize $z = -x + 2y$ subject to the constraints:

$x \geq 3$, $x + y \geq 5$, $x + 2y \geq 6$, $y \geq 0$, $x \geq 0$

Solution :

Our problem is to maximize

$$z = -x + 2y$$

subject to the constraints

$$x \geq 3, x + y \geq 5, x + 2y \geq 6$$

$$x \geq 0,\ y \geq 0$$

First consider the lines $x = 3$, which passes through A(3,0) and is parallel to y axis. Then $x + y = 5$, which passes through the points B(5,0) and C(0,5) and the line $x + 2y = 6$ if drawn is found to be passing through D(6,0) and E(0,3).

Lines BC and DE meet at R(4,1); BC meets AT in Q(3,2) and DE meets AT in $P\left(3, \frac{3}{2}\right)$.

Feasible region is shown shaded in the figure. Note that the point D(6,0) lies in this region but O (0,0) and $P\left(3, \frac{3}{2}\right)$ do not lie in this region.

The corner points which are to be examined for optimum solution are D(6,0) R(4,1) and Q(3,2).

We find from the graph drawn below:

At D(6,0) $\Rightarrow$ z = –6 + 2 × 0 = –6

At R(4,1) $\Rightarrow$ z = –4 + 2 × 1 = –2

At Q(3,2) z = –3 + 2 × 2 = 1

Hence it appears that max z = 1 at (6, 0)

Since the feasible region is unbounded

we find the graph of the inequality $-x + 2y > 1$

As the half plane represented by this inequality has points common with the feasible region, therefore 1 is not the maximum value of z. Thus z has no maximum value.

$x + y = 5$

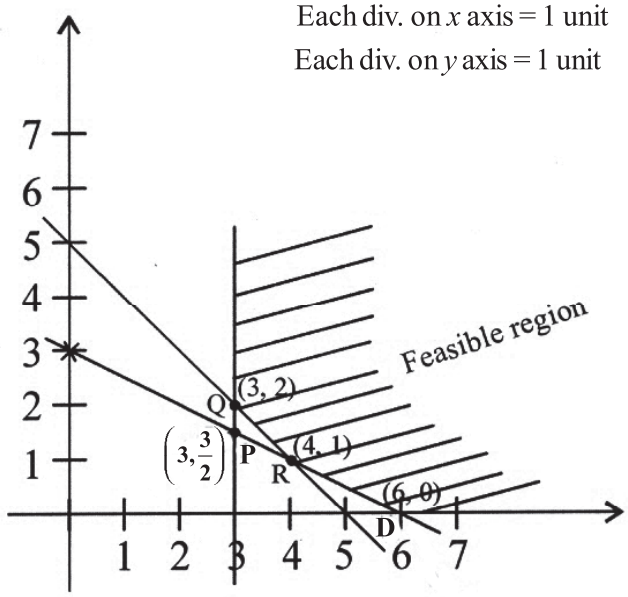
x	0	5
y	5	0

$x + 2y = 6$

x	0	6
y	3	0

Each div. on x axis = 1 unit

Each div. on y axis = 1 unit



Example 5 : Solve graphically

Maximize $z = x + y$, subject to the constraints; $2x - y + 1 \geq 0$, $x + y \leq 3$, $x \leq 2$ and $x, y \geq 0$.

Solution : $Z = 3$ $B(2,1)$ $C\left(\frac{3}{2}, \frac{3}{2}\right) 0(0,0)$

Consider the lines $2x - y + 1 = 0$, $x + y = 3$ and $x = 2$. Lines $2x - y + 1 = 0$ passes through $\left(\frac{-1}{2}, 0\right)$, (0, 1) and $\left(\frac{3}{2}, \frac{3}{2}\right)$ whereas the line $x + y = 3$ passes through (0,3), $\left(\frac{3}{2}, \frac{3}{2}\right)$, (2,1)

and (3,0). Line $x = 2$ is parallel to y axis and passes through (2,1). We find lines $2x - y + 1 = 0$ and $x + y = 3$ meet at $B\left(\frac{3}{2}, \frac{3}{2}\right)$ and the lines $x + y = 3$ and $x = 2$ meet at C(2,1) whereas lines $x = 2$ meets the x axis at (2,0) and the line $2x - y + 1 = 0$ meets the y axis at A(0,1) and 0(0,0). Thus the corner points A(0,1), $B\left(\frac{3}{2}, \frac{3}{2}\right)$, C(2,1), D(2,0) determines the feasible region as shaded below.

To find the optimal solution we find

at A(01) z = 1, at C(2,1) z = 3

at $B\left(\frac{3}{2}, \frac{3}{2}\right)$ z = 3, at D(2,0) z = 2

at 0(0,0) z = 0

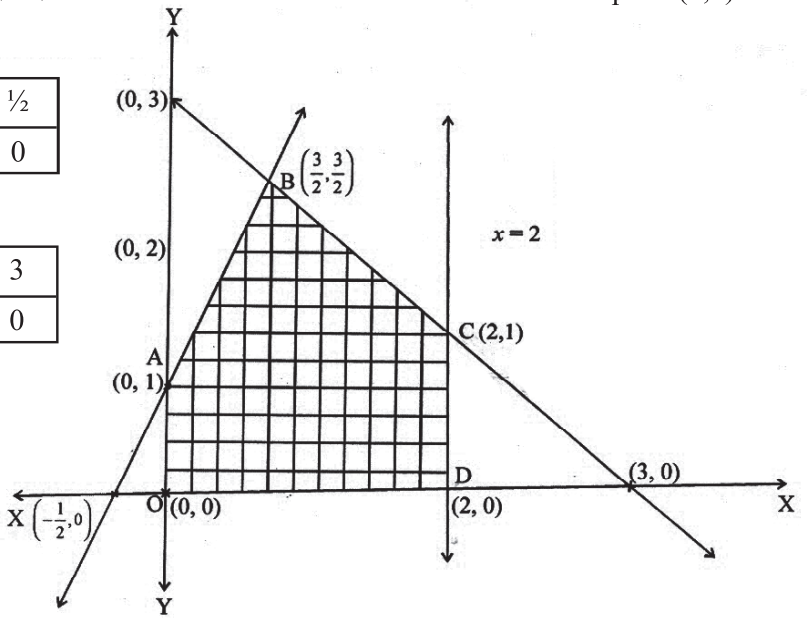
Z, the objective function is having the maximum value 3 at the point (2,1)

$2x - y - 1$

x	0	½
y	+1	0

$x + y = 3$

x	0	3
y	3	0



Example 6 :

Varsha electric appliance company produces two items, Refrigerator and ranges production takes place in two separate departments. Refrigerator are produced is department I and ranges in department II. The company's two product are produced and sold on a weekly basis. The weekly production cannot exceed 25 refrigerators and 35 ranges in department II, because of limited available facilities in these two departments. The company regularly employs a total of 60 workers in the two

departments. A refrigerator requires 2 man-week of labour, while a range requires 1 man week of labour. A refrigerator contributes a profit of ₹ 60 and a range contributes a profit of ₹ 40. How many units of refrigerators and ranges should the company produce to realise maximum profit?

Solution:

Formulation of the problem: Let x and y represent the number of refrigerators and ranges produced and sold every week.

Since a refrigerator contributes a profit of ₹ 60 and a range contributes a profit of ₹ 40, the objective function will be

$$\text{Maximise } z = 60x + 40y$$

The first constraint is of the available facilities in the two departments. By data $x \le 25,\ y \le 35$

The second constraint is of labour. Since a refrigerator requires 2 man week of labour and a range requires 1 man week of labour,

$2x + y \le 60$ is the constraint

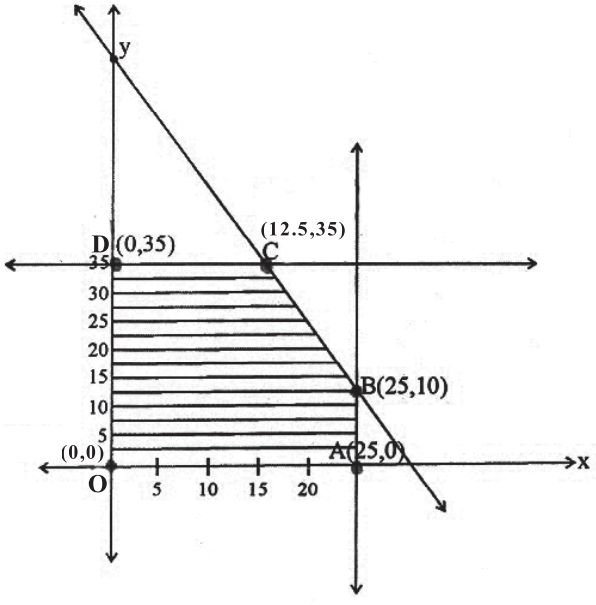
Also x and y cannot be negative, we have $x \ge 0,\ y \ge 0$.

Consolidating all the above data, the final LPP will be

Maximise $z = 60x + 40y$ subject to $x \le 25,\ y \le 35,\ 2x + y \le 60,\ x \ge 0,\ y \ge 0$

We first draw the constraints treating all of them as inequalities.

Since the origin satisfies all three of these inequalities, we consider the intersection o the areas in the I quadrant on that side of the half plane determined by these lines containing the origin as the feasible region. In the adjoining graph, this area is represented by OABCD.



$2x + y \le 60$

x	0	30
y	60	0

we evaluate the value of the objective function z at each of the corners of this region.

At 0(0,) z = 0

At A(25,0) z = 1500

At B(25,10) z = 1900

At C(12.5, 35) z = 2150

At D (0, 35) z = 1400

Clearly z is maximum at C(12.5, 35)

Since product of either product cannot be in fractional value, 12 refrigerators and 35 ranges should be produced to earn maximum profit.

Example 7:

A firm owned by Vaidhyanathan plans to purchase atleast 200 quintals of scrap containing high quality metal X and low quality metal Y. It decides that the scrap to be purchased must contain atleast 100 quintals of X and not more than 35 quintals of Y. The firm can purchase the scrap from two supplies A and B in unlimited quantities. The percentage of X and Y in terms of weight is the scraps supplied by A and B is given below.

Metals	Supplier A	Supplier B
X	25%	75%
Y	10%	20%

The price of A's scrap is ₹200 per quintal and that of B's is ₹400 per quintal. Formulate this problem as a L.P.P. and solve to determine the quantities to be bought from the two suppliers to minimize the total purchase cost.

Solution :

Let 'x' and 'y' be the number of quintals of scrap bought by the suppliers A and B respectively. Since the firm is to minimise the cost of purchase and each quintal form supplier A cost ₹ 2000 and that from B costs ₹400, the objective is to minimise $C = 200x + 400y$

Since it must buy atleast 200 quintals

$$x + y \geq 200$$

Also, the minimum of metal X in the scrap bought is to be 100 quintals. Since scrap from supplier A contains 25% and that from B contains 75% of metal X, we must have

$$0.25x + 0.75y \geq 100$$

Similarly, the scrap must not contain more than 35 quintals of metal y. Since the scrap from supplier A contains 10% of y and that from B contains 20% of y, we get

$$0.10x + 0.20y \leq 35$$

Also since the purchases cannot be negative $x, y \geq 0$

Consolidating, we arrive at the following L.P.P. minimise $C = 200x + 400y$ subject to

$$x + y \geq 200$$

$$0.25x + 0.75y \geq 100$$

$$.10x + 0.20y \leq 35$$

$$x, y \geq 0$$

Drawing the graph of each of the inequality constraints treating them as equalities we find the corner points as G(100, 100) H(250,50) and I(50, 150). Graph is shown below with the proper markings of the points.

We then evaluate the objective function C at these points.

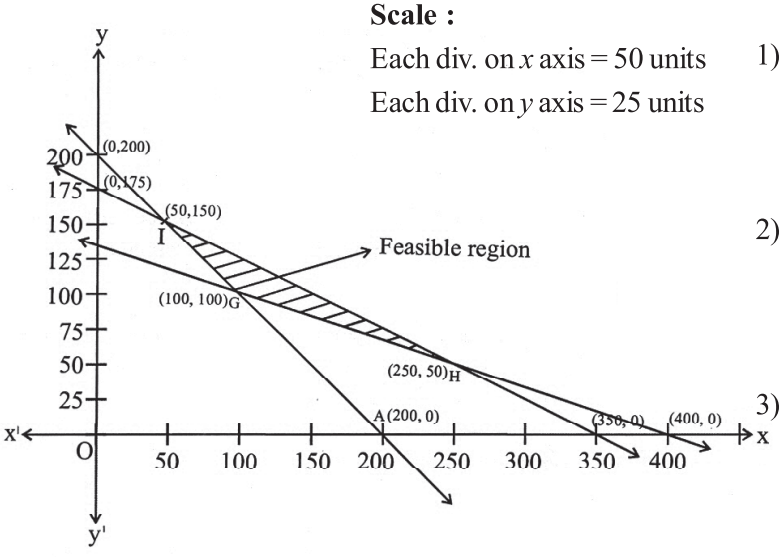
At G (100, 100) C = 200(100) + 400(100) = 60,000

At H (250, 50) C = 200(250) + 400(50) = 70,000

At I (50, 150) C = 200(50) + 400(150)

= 70,000

Since cost is minimum at G(100, 100) that is the optimal point and hence it is recommended that 100 quintals of scrap should be bought by each of the suppliers A and B to minimise the cost of buying.



Scale :

Each div. on x axis = 50 units

Each div. on y axis = 25 units

Construction of lines

1) $x + y = 200$

x	0	200
y	200	0

2) $0.25x + 0.75y = 100$

x	400	0
y	0	133

3) $.10x + 20y = 35$

x	350	0
y	0	175

Example 8 :

A co-operative society of farmers presided by Lalith has 50 hectare of land to grow two crops X and Y. The profits from crop X and Y per hectare are estimated as ₹10,500 and ₹9000 respectively. To control weeds a liquid herbicide has to be used for crops X and Y at the rate of 20 litres and 10 litres per hectare. Further not more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects, drainage from this land. How much land should be allocated to each crop so as to maximise the total profit of the society? Also find the maximum profit.

Solution :

Let 'x' hectare of land be allocated to crop X and Y hectare of land to crop Y.

Obviously $x \geq 0,\ y \geq 0$

profit per hectare on crop x = ₹10,500

profit per hectare on crop y = ₹ 9000

Therefore total profit = $10500x + 9000y$

Formulation of the problem is as follows

Maximise $z = 10500x + 9000y$

Subject to the constraints

$x + y \leq 50$(1)

$20x + 10y \leq 800$ which can be otherwise

written as $2x + y \leq 80$ (2)

$x \geq 0,\ y \geq 0$ (3)

Let us draw the graph of the system of inequalities (1) to (3). The feasible region OABC is shown in the figure.

Now $z = 10500x + 9000y$

At O(0,0), z = 0

At A(40, 0), z = 420000

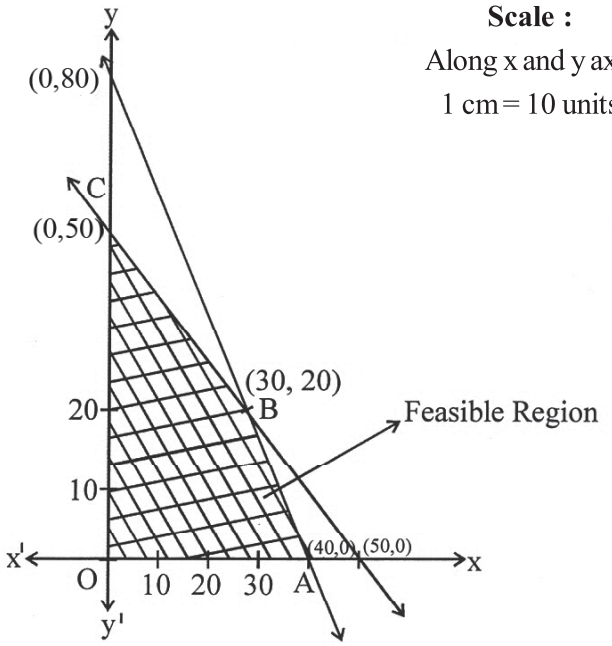
At B(30, 20), z = 495000

At C (0, 50) , z = 450000

$\therefore$ z is maximum at B(30, 20)

$\therefore$ land allocated to crop X = 30 hectare
land allocated to crop Y = 20 hectare

$\therefore$ maximum profit = ₹450000



Scale :

Along x and y axis

1 cm = 10 units

Construction of lines

$x + y = 50$

x	0	50
y	50	0

$x + y = 50$

x	0	40
y	80	0

Example 9 :

Arjun wants to invest at most ₹12,000 in Bonds A and B. According to the rule, he has to invest atleast ₹2000 in Bond A and atleast ₹4000 in Bond B. If the rates of interest on Bonds A and B respectively are 8% and 10% per annum formulate the problem as L.P.P. and solve if graphically for maximum interest.

Solution :

Let Arjun invest in bond A = ₹ x

Arjun invests in bond B = ₹ y

Then interest in bond A = 8% of x

$$= \frac{8}{100} \times x = \frac{8x}{100} = 0.08x$$

Interest in bond B = 10% of y

$$= \frac{10}{100} \times y = 0 \cdot 10y$$

Let Z be the yearly income of Arjun

Then L.P.P. is to maximize, $z = 0.08x + 0.10y$

subject to $x + y \leq 12000$

$x \geq 2000$

$y \geq 4000$

$x, y \geq 0$

Draw the lines $x = 2000$, $y = 4000$ and $x + y = 12000$

The shaded region ABC represents the feasible region.

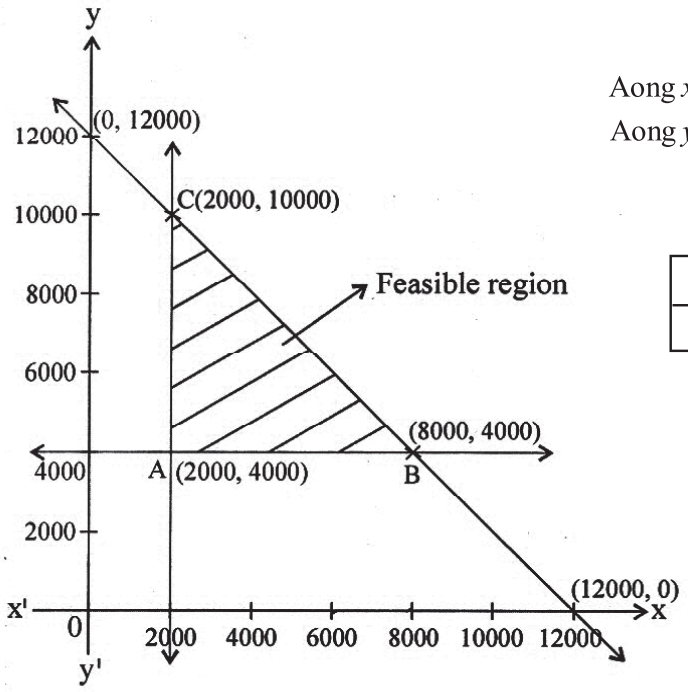
$I_A = \cdot 08(2000) + \cdot 10(4000) = ₹\ 560$

$I_B = \cdot 08(8000) + \cdot 10(4000) = ₹\ 1040$

$I_C = \cdot 08(2000) + \cdot 10(10000) = ₹\ 1160$

The interest is maximum at C(2000, 10000) and the maximum interest is ₹1160

Construction of lines using table



Scale

Aong x-axis 1 cm = 2000 units

Aong y-axis 1 cm = 2000 units

$x + y = 12000$

x	0	12000
y	12000	0

(0, 12000) (12000, 0)

Exercise 11.2

Five marks questions:

I. Solve the following linear programming problems using graphical method.

1) Maximise $z = 60x + 15y$ subject to

$x + y \leq 50$

$3x + y \leq 90$

$x, y \geq 0$

2) Maximise $z = 5x + 3y$ subject to

$3x + 5y \leq 15$

$5x + 2y \leq 10$

$x \geq 0,\ y \geq 0$

3) Minimise $z = 3x + 5y$ subject to

$x + 3y \geq 3$

$x + y \geq 2$

$x, y \geq 0$

4) Minimise $z = x - 7y + 190$ subject to

$x + y \leq 8$

$x \leq 5,\ y \leq 5,\ x + y \geq 4,\ x \geq 0,\ y \geq 0$

II.5) Kellogg is a new cereal formed from a mixture of bran and rice that contains atleast 88gm of protein and atleast 36 milligram of iron per kg. knowing that bran contains 80 gm of protein and 40 milligram of iron per kilogram and that rice contains 100gm of protein and 30 milligram of iron per kilogram, find the minimum cost of producing this new cereal if bran cost ₹5 per kilogram and rice cost ₹4 per kilogram.

6) A firm owned by Abhirami has to transport 1200 packages using large vans which can carry 200 packages each and small vans which can take 80 packages each. The cost for engaging each large van is ₹400 and each small van is ₹200. Not more than ₹3000 is to be spent on the job and the number of large vans cannot exceed the number of small vans. Solve this L.P.P. graphically to find the minimum cost.

7) There are two types of fertilisers F_1 and F_2. F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer Vaidhya finds that he needs atleast 14kg of nitrogen and 14kg of phosphoric acid for the crop. If F_1 costs ₹6 per kg and F_2 costs ₹5 per kg, determine how much of each type of fertilizer should be used so that nutrient requirements are met at a minimum cost. Solve graphically.

8) A Television company owned by Priyanka and Bhavana operates two assembly lines, line I and line II. Each of line is used to assemble the components of three types of television: colour standard and economy. The expected daily production on each line is as follows:

TV model	Line I	Line II
Colour	3	1
Standard	1	1
Economy	2	6

The daily running costs fro two lines average ₹6000 for line I and ₹4000 for line II. It is given that company must produce atleast 24 colour, 16 standard and 48 economy TV sets for which an order is pending. Formulate L.P.P. and solve graphically determining the number of days the two lines should be run to meet the requirements.

9) Old hens can be bought at ₹2 each and young ones at ₹5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen cost ₹1 per week to feed. Deepthi has only ₹80 to spend for hens. How many hens of each kind should Deepthi buy to give a profit of more than ₹6 per week assuming that Deepthi cannot house more than 20 hens. Solve graphically.

10) A company owned by Viswa Narayana concentrates on two grades of paper A and B, produced on a paper machine. Because of raw material restrictions, not more than 400 tonnes of grade A and 300 tonnes of grade B can be produced in a week. There are 160 production hours in a week. It requires 0.2 hour and 0.4 hour to produce one tonne of products A and B respectively with corresponding profits of ₹20 and ₹50 per tonne. Find the optimum product mix using the graphical method.

11) A company owned by Navya manufactures two types of cloth, using three different colours of wool. One yard length of type A cloth required 4 oz (ounce) of red wool, 5 oz of green wool, 3 oz (ounce) of yellow wool. One yard length of type B cloth requires 5 oz red wool, 2 oz of green wool and 8 oz of yellow wool. The wool available for manufacture is 1000 oz of red wool 1000 oz of green wool and 1200 oz of yellow wool. The manufactures can make a profit of ₹5 on one yard of type A cloth and ₹3 on one yard of type B cloth. Find the best combination of the quantities of type A and Type B cloth which gives him maximum profit by solving the L.P.P. by graphical method.

Answers 11.2

I. 1) $Z_{max} = 1800$ when $x = 30$ and $y = 0$

2) $Z_{max} = \frac{235}{19}$ when $x = \frac{20}{19}$ and $y = \frac{45}{19}$

3) Z_{min} 7 at $\left(\frac{3}{2}, \frac{1}{2}\right)$

4) Minimum value of z = 155 when $x = 0$ and $y = 5$

5) Minimum cost is ₹4.60 when 0.6kg of bran and 0.4 kg of rice are mixed

6) Min $z = 400x + 200y$ subject to $5x + 2y \geq 30,\ 2x + y \leq 15,\ x \leq y,\ x \leq 0,\ y \leq 0$

Minimum value of z = ₹2600 when $x = 4, y = 5$

7) Minimum value of z = 1000 when $x = 100$ and $y = 80$

8) $Z_{min} = 72000$ when $x = 4$ and $y = 12$

9) Z_{max} = ₹8 when $x = 0$ and $y = 16$

10) Z_{max} = ₹19,000 when $x = 200$ and $y = 300$

11) $Z_{max} = 1000$ when $x = 200$ and $y = 0$

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Chapter

12

SALES TAX AND VALUE ADDED TAX

12.1 Introduction :

The developmental activities done by the government to the public by collecting revenue from various sources from the public in the form of various type of taxes. Among them the sales tax and value added tax plays a vital role in revenue to the government. At each stage of the transaction of the goods from the manufactures, distributors, whole saler and retailer till to customer, **sales tax (ST)** and the **value added tax (VAT)** Exist. In this chapter we discuss and solve the problem in detail.

12.2 Sales Tax :

Definition : On the purchase of some items, we have to pay certain amount at a specified rate. This amount is called 'Sales Tax'.

- The rate of sales tax depends upon the nature of goods purchased. Some items of daily use are completely exempted from sales tax eg : Match boxes, salt etc.
- Sales Tax is one of the indirect taxes as it affects all individual indirectly. Other examples of indirect taxes are excise duty, custom duty etc. State Government levis tax on sales of goods within the state.
- Calculation of sales tax involves the use of the concept of percentage. The rate of sales tax is different for different items. Sales tax collected by the state government is used for the development of the state as well as to meet the expenses like payment to its employees, health, education etc.

Important formulae

1. Profit = Selling price - Cost price = SP – CP
2. Loss = Cost price - Selling price = CP – SP
3. $\text{Profit\%} = \frac{\text{Profit}}{\text{CP}} \times 100$
4. $\text{Loss\%} = \frac{\text{Loss}}{\text{CP}} \times 100$
5. $\text{SP} = \frac{100 + \text{Gain\%}}{100} \times \text{CP}$
6. $\text{SP} = \frac{100 - \text{Loss\%}}{100} \times \text{CP}$
7. Total amount to be paid = marked price + sales tax on marked price = MP + ST% on MP
8. Discount = discount % of MP
9. $\text{ST\%} = \frac{\text{ST}}{\text{MP}} \times 100$

Note :

(i) Marked price = Market price = Printed price = List price = Sale price = MP

(ii) Sales Tax is always found on MP

(iii) Discount is Calculated on MP

(iv) ST% is found on new marked price (after discount)

WORKED EXAMPLES

Example 1 :

Sharath paid ₹ 40 sales tax on a pair of shoes worth ₹ 500. Find the rate of sales tax?

Solution : $\text{S.T\%} = \frac{\text{ST}}{\text{MP}} \times 100$

$= \frac{40}{500} \times 100 = 8\%$

Example 2 :

The price of a washing machine inclusive of sales tax is ₹ 13,530. If the sales tax is 10%. Find the basic price.

Solution : Suppose the basic price (MP) of the washing machine = ₹ x

$\therefore$ total amount paid = MP + ST% of MP

$13530 = x + 10\%$ of x

$13530 = x + \frac{10x}{100}$

$13530 = x + \frac{x}{10}$

$13530 = \frac{10x + x}{10} = \frac{11x}{10}$

$x = \frac{13530 \times 10}{11}$ $\quad \therefore \ x = ₹12,300$

Example 3 :

Gopal purchased a scooter costing ₹ 32,450. If the rate of sales tax is 9%. Calculate the total amount payable by him?

Solution : Total amount paid = MP + ST% of MP

$= 32,450 + 9\%$ of $32,450$

$$= 32{,}450 + \frac{9}{100} \times 32450$$

$$= 32450 + 9 \times 32450$$

Total amount paid = ₹ 35,370.50

Example 4 :

A colour T.V is marked for sale for ₹17,600 which include sales tax at 10%. Calculate the sales tax in ₹?

Solution : SP = ₹ 17,600, ST% = 10%, ST = ?, MP = x

Total amount to be paid (SP) = MP + ST% of MP

$$17600 = x + 10\% \text{ of } x$$

$$17600 = x + \frac{10}{100}x$$

$$17600 = x + \frac{x}{10}$$

$$17600 = \frac{10x + x}{10} = \frac{11x}{10} \qquad \therefore x = \frac{17600 \times 10}{11}$$

$$x = ₹\ 16{,}000$$

$$\therefore \text{S.T} = \text{SP} - \text{MP}$$

$$= 17600 - 16000 = ₹\ 1{,}600$$

Example 5 :

Sanju goes to a shop to buy a Bicycle quoted at ₹ 2,000. The rate of sales tax is 12% on it. He asks the shopkeeper for a rebate on the price of the bicycle to such an extent that he has to pay ₹ 2,016 inclusive of sales tax. Find the rate of rebate percentage on the price of the bicycle.

Solution :

Let SP = ₹2016

ST = 12%

MP = ₹ x Rebate % = ?

Amount paid = SP = MP + 12% of MP

$$2016 = x + \frac{12}{100}x$$

$$2016 = \frac{100x + 12x}{100}$$

$$2016 \times 100 = 112x$$

$$\therefore x = \frac{2016 \times 100}{112} = ₹\ 1{,}800$$

$$\therefore \text{Rebate} = 2000 - 1800$$

$$= ₹200$$

$$\therefore \text{Rebate} = \frac{\text{Rebate}}{\text{Quoted price}} \times 100 = \frac{200}{2000} \times 100$$

$$= 10\%$$

Example 6 :

A shopkeeper announces a discount of 10% on a T.V set. The marked price of the T.V is ₹ 22,000. How much will a customer have to pay for buying the T.V set if the rate of sales tax is 8%.

Solution :

MP = ₹ 22,000

ST = 8%

Discount = 10% of M.P

Total amount paid = ?

$$= \frac{10}{100} \times 22000 = ₹\ 2{,}200$$

$$\therefore \text{New MP} = 22{,}000 - 2{,}200$$

$$= ₹19{,}800$$

$$\therefore \text{Amount to be paid} = \text{MP} + \text{ST\% on MP}$$

$$= 19800 + 8\% \text{ of } 19800$$

$$= 19800 + \frac{8}{100} \times 19800$$

$$= 19800 + (8 \times 198) = ₹\ 21{,}384$$

Example 7 :

Mr. Govind buys a tape recorder for ₹ 10,260 including sales tax. If the list price of the tape recorder in ₹ 9,500. Find the rate of sales tax charged?

Solution:

SP = 10,260 selling price = list price (MP) + ST% on MP

MP = ₹ 9500, ST = x%

$$10{,}260 = 9{,}500 + x\% \text{ of } 9{,}500$$

$$10{,}260 - 9500 = x\% \text{ of } 9500$$

$$760 = \frac{x \times 9500}{100}$$

$$760 = 95\,x$$

$$\therefore\ x = \frac{760}{95}$$

$$\therefore\ \text{ST} = 8\%$$

Example 8 :

Sohail goes to purchase a motorcycle which is priced at 35,640 including 10% as sales tax. How was the actual rate of sales tax at the time of purchase is 7%. Find the extra profit made by the shopkeeper if he still charges the original list price?

Solution:

Purchase price = S.P = ₹ 35,640

M.P = ₹ x

S.T = 10%

S.T at the time of purchase = 7%

Project = ?

SP = MP + ST% of MP

$35{,}640 = x + 10\%$ of x

$$35{,}640 = x + \frac{10x}{100}$$

$$35{,}640 = x + \frac{x}{10} = \frac{10x + x}{10} = \frac{11x}{10}$$

$$x = \frac{35640 \times 10}{11}$$

x = ₹ 32,400

Amount to be paid = 32,400 + 7% of 32400

$$= 32{,}400 + \frac{7 \times 32400}{100}$$

= 32400 + 2268

= ₹ 34,668

$\therefore$ profit made by the shopkeeper = 35640 – 34668 = ₹ 972

Example 9 :

A shopkeeper announces a discount of 10% on a washing machine set. The marked price of the washing machine is ₹ 12000. How much will a customer have to pay for buying the washing machine set if the rate of sales tax is 8%?

Solution: Discount = 10%

M.P = ₹ 12000

S.T = 8%

Amount paid = SP = ?

Discount of 10% on MP = 10% of 12000

$= \frac{10}{100} \times 12000 = ₹\ 1200$

New M.P = 12000 – 1200 = ₹10,800

∴ Total amount to be paid = 10,800 + 8% of 10,800

$= 10,800 + \frac{8}{100} \times 10,800$

$= 10,800 + 864 = ₹\ 11,664$

Example 10 :

A shopkeeper purchases an Audio system for ₹ 3,000 and sells it off at a gain of 15%. He also charges a sales tax of 10% on the selling price. Calculate the amount that the buyer will pay to the shopkeeper.

Solution: Costprice = ₹ 3,000

Gain (profit) = 15% of cost price

$= \frac{15 \times 3000}{100}$

$= ₹\ 450$

∴ S.P = C.P + Gain

$= 3000 + 450 = ₹\ 3450$

∴ Total amount to be paid = S.P + S.T% of SP

$= 3450 + 10\%$ of 3450

$= 3450 + \frac{10 \times 3450}{100}$

$= 3450 + 345 = ₹\ 3,795$

Example 11:

When the rate of sales tax is decreased from 9% to 7%. For a Radio, Rahul has to pay ₹632 less for it. What is the listed price of the radio?

Solution: Assume the M.P $= ₹\ x$

Total amount paid when S.T is 9% $= x$ + S.T% on x

$$= x + \frac{9x}{100}$$

Again total amount paid when ST is 7% $= x$ + S.T% on $x = x + \frac{7x}{100}$

$\therefore$ Total amount paid when ST is 9% = Total amount paid when ST is 7% + 632

$$x + \frac{9x}{100} = \left(x + \frac{7x}{100}\right) + 632$$

$$\frac{100x + 9x}{100} = \left(\frac{100x + 7x}{100}\right) + 632$$

$$\frac{109x}{100} = \frac{107x}{100} + \frac{632}{1} \text{ (Take LCM)}$$

$$= \frac{107x + 632(100)}{100}$$

$$\frac{109x}{\cancel{100}} = \frac{107x + 63200}{\cancel{100}}$$

$$109x - 107x = 63200$$

$$2x = 63200$$

$$\therefore\ x = \frac{63200}{2}$$

$$\boxed{\text{MP} = ₹31,600}$$

Example 12:

If the rate of sales tax is 5% Sushma has to pay ₹ 7,140 for the steel cupboard. What amount she has to pay if the sales tax is increased by 2%.

Solution: Let the MP of cupboard $= ₹\ x$

S.T $= 5\%$

Amount paid by Sushma at 5% ST = ₹ 7140

$$\therefore\ 7140 = \text{M.P} + \text{S.T\% on MP}$$

$$7140 = x + 5\%\ \text{of}\ x$$

$$7140 = x + \frac{5x}{100}$$

$$7140 = \frac{100x + 5x}{100}$$

$$714000 = 105\,x$$

$$x = \frac{714000}{105} = ₹\ 6{,}800$$

$$\because \boxed{\text{MP} = ₹6{,}800}$$

Sales tax charges $= 5 + 2 = 7\%$

Total amount to be paid by Sushma $= 6800 + 7\%$ of 6800

$$= 6800 + \frac{7 \times 68\cancel{00}}{1\cancel{00}} = 6800 + 476 = ₹\ 7276$$

Example 13:

Chandana purchases an article for ₹ 5,400 which include 10% rebate on the marked price and 20% sales tax on the remaining price. Find the marked price of the article?

Solution:

Let the marked price of the article = ₹ x

$$\text{Rebate} = \frac{10}{100}x = \boxed{\frac{x}{10}}$$

$$\therefore \text{Remaining price} = x - \frac{x}{10} = \frac{10x - x}{10} = \frac{9x}{10}$$

$\therefore$ Total amount to be paid (purchase price) = Remaining price + ST% on remaining price

$$5400 = \frac{9x}{10} + 20\%\ \text{of}\ \frac{9x}{10}$$

$$= \frac{9x}{10} + \left(\frac{\cancel{20}}{\cancel{100}_{5}} \times \frac{9x}{10}\right)$$

$$= \frac{9x}{10} + \frac{9x}{50}$$

$$5400 = \frac{45x + 9x}{50} = \frac{54x}{50}$$

$$54x = 5400 \times 50$$

$$\therefore x = \frac{\overset{100}{\cancel{5400}} \times 50}{\cancel{54}}$$

$$\therefore \text{M.P} = \boxed{x = ₹5,000}$$

Example 14 :

Sharath goes to a departmental store and purchase the following articles

(i) A raincoat for ₹ 300 S.T @ 10%

(ii) A pair of shoes for ₹ 460 S.T@ 9%

(iii) Food article for ₹ 450 S.T@5%

(iv) Cloth for ₹ 800 ST@1%

Calculate total amount of the bill.

Solution : Purchase price of Raincoat = 300 + S.T% 300

= 300 + 10% of 300

$$= 300 + \frac{10}{\cancel{100}} \times \overset{3}{\cancel{300}} = 300 + 30 = ₹ 330$$

Similary, purchase price of shoe $= 460 + \left(\frac{9}{100} \times 460\right)$

= 460 + 41.40 = ₹ 501.40

Purchase price of food article = 450 + 22.50 = ₹ 472.50

Purchase price of clothes $= 800 + \frac{1 \times \overset{8}{\cancel{800}}}{\cancel{100}}$

= 800 + 8 = ₹ 808

∴ Total amount of the bill = ₹ (330 + 501.40 + 4 72.50 + 808)

= ₹ $\boxed{2,111.90}$

Example 15 :

Following purchases were made from a store

Item	List price (₹)	No. of item	Rate of ST
Tape recorder	**10,900**	**2**	**10%**
Suitcase	**4,200**	**1**	**5%**
Raincoat	**400**	**3**	**7%**
Mixer	**2,500**	***x***	**2%**

Total amount paid by Raju is ₹ 34,774. Find the number of mixers bought.

Solution:

Amount to be paid for 2 tape recorder

$$= 2[10,900 + 10\%\ 10,900]$$

$$= 2\left[10,900 + \frac{10}{100} \times 10900\right]$$

$$= 2 \times [10,900 + 1090]$$

$$= 2 \times 11,990 = ₹\ 23,980$$

Similarly, amount to be paid for 1 suitcase

$$= 4200 + 5\%\ (4200)$$

$$= 4200 + \frac{5}{100} \times 4200 = 4200 + 210 = 4410$$

Amount to be paid for 3 Raincoats

$$= 3[400 + 7\%\ (400)]$$

$$= 3\left[400 + \frac{7 \times 400}{100}\right]$$

$$= 3 \times 428 = ₹1284$$

Amount to be paid for 'x' mixers

$$= x[2500 + 2\%\ (2500)]$$

$$= x\left(2500 + \frac{2 \times 2500}{100}\right)$$

$$= x\ (2500 + 50) = 2550\ x$$

$\therefore$ Total amount paid = total amount of all the articles

$$34,774 = 23980 + 4410 + 1284 + 2550\ x$$

$$34{,}774 = 29{,}674 + 2550\,x$$
$$\therefore\ 2550\,x = 34{,}774 - 29674$$
$$2550\,x = 5100$$
$$\therefore\ x = \frac{\cancel{5100}}{\cancel{2500}} = 2$$
$$\therefore \text{ number of mixers bought} = 2$$

Exercise 12.1

I. 2 Marks question:

1. Abhishek purchase a bicycle costing ₹12,000. If the rate of sales tax is 9%. Calculate the total amount payable by him.
2. Ramu paid ₹ 60 as sales tax on a Titan Raga watch worth ₹1200. Find the rate of sales tax.
3. Refrigerator is marked for sale for ₹ 17,000 which include sales tax at 10% Calculate the sales tax in ₹

II. 3 Marks question:

1. Bharath bought a shirt for ₹ 336, including 12% sales tax and a necktie for ₹110 including 10% sales tax. Find the printed price of shirt and necktie together.
2. A furniture dealers sold furniture for ₹ 21,000 and added 5% S.T to the quoted price. The customer agrees to buy it for ₹ 21,000 including S.T. find the discount he received.
3. A shopkeeper sells an item at the price of ₹ 810 including ST of 8% what should a customer pay for the same item if the ST is reduced to 6%.
4. The price of T.V set inclusive of sales tax of 9% is ₹ 13.407. Find its marked price. If the S.T is increased to 13% how much more does the customer Sneha pay for the T.V.

Answers 12.1

I. 1. ₹13,080 2. 5% 3. ₹1545.45

II. 1. ₹ 400 2. ₹1000 3. ₹ 795

4. ₹12,300, ₹ 492

12.3 Value Added Tax (VAT)

It is neither a new tax nor in addition to the existing sales tax. It is the replacement of sales tax.

Note:

(i) VAT is a form of sales tax only. The only difference in that it is collected in stages rather than at one point from the sales of goods.

(ii) In VAT, the firm seller pays the first point of tax to the government and sub sequent

seller pays tax to the government on the value added by them, leading to a total tax exactly to the last point tax paid by the customer.

(iii) Advantage of VAT : (i) It reduces the scope of under evalution

(ii) It provide a broad base tax system.

(iv) VAT paid by the shopkeeper = VAT% of (selling price - Shopkeeper purchase price)

(v) VAT = VAT% (selling price - cost price)

Example 1: Let the rate of sales tax be 10% suppose a trader Siddharth

Step 1 : Buys a Radio from a factory owner Venkatesh for ₹1000 then the tax paid by the trader to the factory owner.

$= 10\% \text{ of } 1000 = \frac{10}{100} \times 1000 = ₹100$. So the factory owner Venkatesh pay ₹100 to the government as a tax.

Step 2 : It is trader sell the Radio to a consumer for ₹ 1500 then the tax paid by the consumer to the trader $= 10\% \text{ of } 1500 = \frac{10}{100} \times 1500 = ₹150$

Hence the trader has added ₹500 in value of the radio so he will pay tax to the government only on the added value

i.e., 10% of ₹ 500 $= \frac{10 \times 500}{100} = 50$

Total tax paid to the government = ₹100 by factory owner + ₹ 50 by trader

= ₹150 tax paid by the consumer

Example 2 :

Let rate of sales tax be 5%

	Producer	trader 1	trader 2	consumer
Sale price	₹1000	₹1200	₹1600	
Tax charged	by producer $= \frac{5 \times 1000}{100} = 50$	by trader 1 $= \frac{5 \times 1200}{100} = 60$	by trade 2 $= \frac{5 \times 1600}{100} = ₹80$	Tax paid
Tax paid to Government	by producer ₹50	by trader ₹10 = (60 - 50)	by trader 2 ₹20 (80 – 60)	by the consumer ₹80

Total tax paid by the consumer = ₹80

Total tax collected by the govt = ₹ (50 + 10 + 20) = ₹80

Note : The tax is collected at different stages

Input tax : Tax paid by a dealer on the purchase of good for resale (subject to tax under VAT) during any fixed tax period is called input tax.

Output tax : The tax charged by dealers on the sales of good during any fixed period is called output tax.

Computation of VAT : Net tax to be paid by the dealer = (output tax) – (Input tax)

Worked Examples

Example 1 :

A shopkeeper purchase an article for 7,000 and sell it to a customer for 8,200. If the VAT rate is 6%. Find the VAT paid by the shopkeeper?

Solution: VAT paid by the shopkeeper = VAT% of (customer price - shopkeeper price)

$$= \text{VAT\% of (SP} - \text{CP)}$$

$$= 6\% \ (8200 - 7000)$$

$$= \frac{6 \times (1200)}{100} = ₹\ 72$$

Example 2 :

Mr. Arya purchase an article for ₹ 3,100 and sell it to Mr. Aravind for ₹ 4,250. Mr. Aravind in turn sells it to Mr. Anil for ₹ 5,000. If the VAT levied 10%. Find the VAT levied on Arya and Aravind?

Solution:

$$\text{VAT paid by Arya} = 10\% \text{ of } ₹\ (4{,}250 - 3{,}100)$$

$$= \frac{10 \times 115}{100} = ₹115$$

$$\text{VAT paid by Aravind} = 10\% \text{ of } ₹\ (5{,}000 - 4250)$$

$$= \frac{10 \times 750}{100} = ₹75$$

Example 3 :

A shopkeeper purchased an item of ₹100 at 8% VAT and sell it at ₹ 120 to a customer and the customer also pay 8% VAT to the shopkeeper. How much amount did the shopkeeper deposit to the government as VAT?

Solution:

	Shopkeeper	Customer
C.P	₹100	₹120
VAT = 8%, VAT	$\frac{8\times 100}{100}$ = ₹8	$\frac{8\times 120}{100} = 9.60$
Tax charged : VAT	= ₹8	₹$(9.60 - 8) = 1.60$

Total tax paid to government = (8 + 1.60) = ₹ 9.60

Example 4 :

A shopkeeper buys a mobile set at a discount rate of 20% from the wholesaler, the printed price of the mobile set being ₹1,600 and the rate of sales tax is 6%. The shopkeeper sells it to the buyer at the printed price and charges tax at the same rate.

Find (i) the price at which the mobile set can be bought from whole saler

(ii) the VAT paid by the shopkeeper.

Solution: (i) Price at which the mobile net can be bought

$$= 80\% \text{ of } 1,600 + 6\% \text{ VAT of } 1280$$

$$= \left(\frac{80\times 1600}{100}\right) + \left(\frac{6\times 1280}{100}\right)$$

$$= 1,280 + 76.8$$

$$= ₹\ 1356.80$$

(ii) for whole saler's VAT $= 6\%$ of $1,280$

$$= \frac{6\times 1280}{100}$$

$$= ₹\ 76.80$$

for buyer VAT $= 6\%$ of 1600

$$= \frac{6\times 1600}{100}$$

$$= ₹\ 96$$

VAT paid by shopkeeper = for buyer VAT for whole sale VAT

$$= 96 - 76.80$$

$$= ₹19.20$$

Example 5 :

Assume that the profit of 20 at each stage of the selling chain. The manufacture 'A' the sale distributer B, the whole saler C and the retailer D live in the same state where the rate of VAT is 8%. The cost price of 'A' in commodity is 200. Find the amount of VAT if A sells to B. B sells to C and C sells to D. Also find the selling price of D.

Solution:

	Manufacture	Distributer	Whole saler	Retailer
	A	B	C	D
cost price	₹200	₹220	₹240	₹200
selling price	₹220	₹240	₹260	₹280
$\frac{\text{VAT} = 8\%}{\text{Tax charged} \rightarrow}$	$\frac{8 \times 220}{100} = 17.60$	₹19.20	₹20.80	₹22.40
VAT	₹17.60	(19.20-17.60) ₹1.60	₹1.60	₹1.60

$\therefore$ Total VAT = (17.60 + 1.60 + 1.60 + 1.60) = ₹ 22.40

Actual selling price of D = ₹260 + ₹20 + 22.40 = ₹302.40

EXERCISE 12.2

I. 2 and 3 mark question

1. A shopkeeper purchase an article for ₹ 9,000 and sell it to a customer for 10,500. If the VAT rate is 4%. Find the VAT paid by the shopkeeper?
2. A shopkeeper purchased an electric iron of ₹1000 at 8% VAT from the whole saler and sell it to the customer of ₹1400 at 8% VAT.

   Find

   (a) amount paid by the customer

   (b) the VAT to be paid by the shopkeeper
3. A shopkeeper bought a TV at a discount of 30% of the listed price of ₹24,000. The shopkeeper offer a discount of 10% of the listed price to the customer. If the VAT is 10%. Find

   (i) the amount paid by the customer.

   (ii) the VAT to be paid by the shopkeeper.

4. Mohan a owner of a departmental store purchased an article of ₹ 1,500 at 4% VAT and sell it at ₹ 1,700 to the customer at 4% VAT. How much amount did the shopkeeper deposit to the government as VAT.

5. Sanju a owner of jewellary shop purchased a ear ring of ₹ 2,000 at 12% VAT and sells it at 2,300 to the customer Radhika. If Radhika also pays 12% VAT to the shopkeepr how much did the shopkeeper deposit to the government as VAT?

6. 'A' is manufacture of electric iron. The cost price of each electric iron in ₹1,600. He sells to 'B' and 'B' sells to 'C' and 'C' sells to 'D' the retailer. The tax rate is 12.5% and the profit is ₹ 150 at each stage of the selling chain. Find the

   (i) the total amount of VAT and

   (ii) the amount that the purchased will have to pay.

ANSWERS 12.2

1. ₹60
2. ₹1512 and ₹32
3. ₹23,760, ₹480
4. ₹ 8
5. ₹36
6. (i) ₹256.25 (ii) ₹ 2,475

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UNIT III - TRIGONOMETRY

Chapter	Title	No. of Teaching hrs.
13.	HEIGHTS AND DISTANCES	04 hrs
14.	COMPOUND ANGLES, MULTIPLE ANGLES, SUBMULTIPLE ANGLES AND TRANSFORMATION FORMULAE	08 hrs
	TOTAL TEACHING HOURS	**12 hrs**

Chapter

13

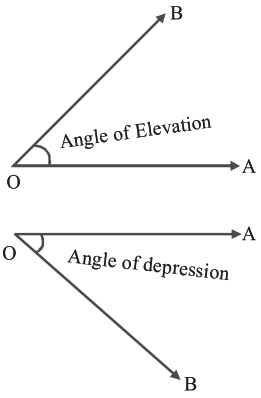
HEIGHTS AND DISTANCES

13.1 Introduction :

One of the important practical applications of Trigonometry is to find the heights and distances of visible inaccessible objects such as the top of a Tower, the summit of a hill, two ships at sea and so an, from the position of the observer without actually measuring it. To solve these, we often use the terms angle of elevation and angle of depression, which are defined below.

13.2 Angle of Elevation and Angle of Depression :

Angle of Elevation : The angle between the horizontal line drawn through the observer's eye and the line joining, the eye of the observer to any object is called the Angle of Elevation of the object, when the object is at a higher level than the eye.



Angle of depression : The angle between the horizontal line drawn through the observer's eye and the line joining the eye of the observer to any object is called the Angle of depression of the object when the object is at a lower level then the eye.

Note : In all the problems, objects such as mountains, towers, trees etc. are considered to be linear and unless, otherwise mentioned the height of the observer is neglected.

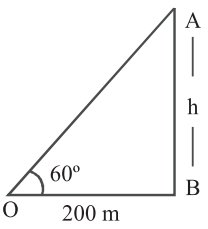
WORKED EXAMPLES

Example 1:

The angle of elevation of the top of a tower at a distance of 200 metres from its foot is 60°, find the height of the tower.

Solution:

Let 0 be the position of the observer and AB denote the Tower. Let the heights of the Tower be h.



We have $\angle AOB = 60^\circ$, OB = 200m

From the right angled triangle ABO, we get

$$\tan 60^\circ = \frac{h}{200} \Rightarrow \sqrt{3} = \frac{h}{200}$$

$$\Rightarrow h = 200\sqrt{3} \text{ metres}$$

Hence the height of the Tower is $200\sqrt{3}$ metres

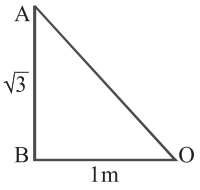
Example 2:

Find the angle of elevation of the sun's rays from a point on the ground at a distance of 1 metre from the foot of tower $\sqrt{3}$ m height.

Solution:

Let AB be the height of the tower & $AB = \sqrt{3}\ m$

$OB = 1m$ & $\angle AOB = ?$



From the right angled triangle AOB we have $\tan\theta = \frac{\sqrt{3}}{1}$

$$\tan\theta = \sqrt{3} \Rightarrow \theta = 60^o$$

$\therefore$ The angle of elevation is 60^o

Example 3:

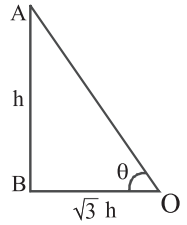
Find the angle of elevation of the Sun when the length of the shadow of a pole is $\sqrt{3}$ times the height of the pole.

Solution:

Let AB be the height of the pole

Let AB = h, OB = $\sqrt{3}$ h is the length of the shadow

From the right angled triangle



$$\tan\theta = \frac{h}{\sqrt{3}h} = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30^o$$

$\therefore$ The angle of elevation is 30^o

Example 4:

From a ship's mast head 50 metres high, the angle of depression of a boat is observed to be 30^o. Find its distance from the ship.

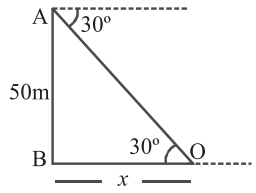
Solution:

Let AB is mast head of a ship AB = 50 metres

O be the position of the boat

we have $\angle AOB = 30^o$

From the right angled triangle



ABO we have $\tan 30^o = \frac{50}{x}$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{50}{x} \Rightarrow x = 50\sqrt{3} \text{ metres}$$

$\therefore$ The distance from the ship $= 50\sqrt{3}$ metres

Example 5:

A Ladder leaning against a wall makes an angle of 60° with the ground, the foot of the ladder 36m away from the wall. Find the length of the ladder.

Solution:

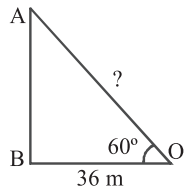
Let AO be the ladder

By data, BO = 36m, $\angle AOB = 60^\circ$

From a right angled triangle $\cos 60^\circ = \frac{36}{AO}$

$$\frac{1}{2} = \frac{36}{AO} \Rightarrow AO = 72m$$

$\therefore$ The length of the ladder = 72 metres.



Example 6:

A kite flying at a height of h is tied to a thread which is 50m long. Assuming that there is no kink in the thread and it makes an angle of 30° with the ground. Find the height of the kite.

Solution:

Let AB = h is the height of the kite

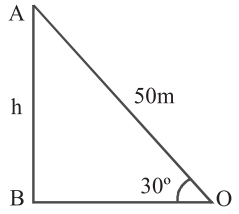
Give AO = 50m,

From the right angled triangle ABO

$$\sin 30^\circ = \frac{h}{50}$$

$$\frac{1}{2} = \frac{h}{50} \Rightarrow h = \frac{50}{2} = 25m$$

$\therefore$ The height of the kite is 25m



Example 7:

A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60°. When he returns 40 metres from the bank, he finds the angle to be 30°. Find the height of the tree and the breadth of the river.

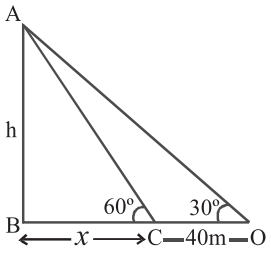
Solution:

Let AB = h be the height of the tree,

BC = x be the breadth of the river.

By data $\angle AOB = 30^\circ$, $\angle ACB = 60^\circ$

OC = 40m



From right angled triangle ACB we get

$$\tan 60^\circ = \frac{h}{2} \Rightarrow \sqrt{3} = \frac{h}{x}$$

$$\Rightarrow x = \frac{h}{\sqrt{3}} \quad . \quad \text{.....(1)}$$

Again from right angled triangle AOB

$$\tan 30^\circ = \frac{h}{40+x} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{40+x}$$

$$\Rightarrow h\sqrt{3} = 40 + x$$

$$\Rightarrow x = h\sqrt{3} - 40 \quad \text{.....(2)}$$

From equations 1 and 2 we get

$$\frac{h}{\sqrt{3}} = \frac{h\sqrt{3} - 40}{1}$$

$$h = \sqrt{3}\left(h\sqrt{3} - 40\right)$$

$$h = 3h - 40\sqrt{3}$$

$$40\sqrt{3} = 2h \Rightarrow h = 20\sqrt{3} \text{ metres}$$

But $x = \frac{h}{\sqrt{3}} = \frac{20\sqrt{3}}{\sqrt{3}} = 20$ metres

$\therefore$ The height of the tree is $20\sqrt{3}$ metres and the breadth of the river is 20 metres.

Example 8:

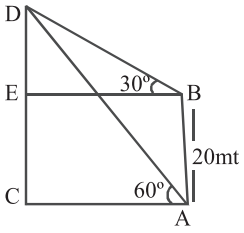
The angles of elevation of the top of a tower from the base and the top of a building are 60° and 30°. The building is 20 metres high. Find the height of the tower.

Solution:

Let AB be building and CD represent the tower

Let BE be the horizontal drawn through the point B

By data $\angle EBD = 30^\circ$, $\angle CAD = 60^\circ$ and AB = 20 metres



From the right angled triangle ADC. We have

$$\tan 60^o = \frac{DC}{AC} \Rightarrow \sqrt{3} = \frac{DC}{AC}$$

$$\Rightarrow AC = \frac{DC}{\sqrt{3}} \quad \text{.....(1)}$$

Again from the right angled triangle DEB we have, $\tan 30^o = \frac{DE}{EB} = \frac{DE}{AC} \quad (\because EB = AC)$

$$\frac{1}{\sqrt{3}} = \frac{DE}{AC} \Rightarrow AC = DE\sqrt{3} \quad \text{.....(2)}$$

[But, DE = DC – EC (EC = AB = 20)

DE = DC – 20]

From equations 1 and 2 we get

$$\frac{DC}{\sqrt{3}} = DE\sqrt{3}$$

$$\frac{DC}{\sqrt{3}} = (DC - 20)\sqrt{3}$$

$$\Rightarrow DC = 3(DC - 20)$$

DC = 3DC – 60

2DC = 60

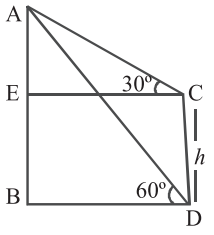
$\Rightarrow DC = 30$ metres

$\therefore$ The height of the tower = 30 metres.

Example 9:

The angles of elevation of the Summit of a hill from the top and the bottom of a tower are 30º and 60º respectively. If the height of the tower is h, show that the height of the hill is $\frac{3h}{2}$

Solution:



Let AB represents the height of the hill, CD represents the height of the tower

By data, $\angle ADB = 60^o, \angle ACE = 30^o$

DC = h

From the right angled triangle ABD we have

$$\tan 60^\circ = \frac{AB}{BD} \Rightarrow \sqrt{3} = \frac{AB}{BD} \Rightarrow BD = \frac{AB}{\sqrt{3}} \quad \ldots\ldots\ldots (1)$$

Again from the right angled triangle AEC

$$\tan 30^\circ = \frac{AE}{EC} = \frac{AE}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{AB - EB}{BD} \Rightarrow \frac{1}{\sqrt{3}} = \frac{AB - h}{BD}$$

$$BD = \sqrt{3}(AB - h)$$

$$\frac{AB}{\sqrt{3}} = \sqrt{3}(AB - h)$$

$$AB = 3\,(AB - h)$$

$$AB = 3\,AB - 3h$$

$$3h - 2\,AB \Rightarrow AB = \frac{3h}{2}$$

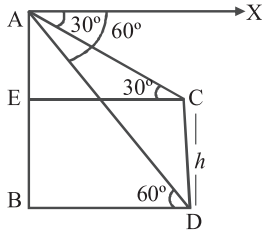
$\therefore$ The height of the hill $= \frac{3h}{2}$

Example 10:

A person is at the top of a tower 75 feet high, from there he observes a vertical pole and finds the angles of depressions of the top and the bottom of the pole which are 30° and 60° respectively. Find the height of the pole.

Solution:

Let *AB* represent the tower and CD represents the vertical pole. Let AX be the horizonatal drawn through A.



By data $\angle XAD = 60^\circ$

$\angle XAC = 30^\circ$

Also AB = 75 ft. Let CD = h = ?

Since AX is parallel to AC we have ACE = XAC

Again as AX is parallel to BD, we have $\angle ADB = \angle XAD = 60^\circ$

From the right angled triangle ADB $\tan 60^\circ = \frac{AB}{BD} \Rightarrow \sqrt{3} = \frac{75}{BD}$

$$\Rightarrow BD = \frac{75}{\sqrt{3}} \quad \ldots\ldots\ldots\ldots (1)$$

Again from the right angled Δle AEC, we have

$\tan 30^o \frac{AE}{EC} = \frac{AE}{BD}$ $(\therefore EC = BD)$

$\frac{1}{\sqrt{3}} = \frac{AE}{BD} = \frac{AB - EB}{BD} = \frac{75 - CD}{BD}$

$\frac{1}{\sqrt{3}} = \frac{75 - CD}{BD} \Rightarrow BD = \sqrt{3}\ (75 - CD)$ ..... (2)

From Eq^{ns} 1 and 2 we get

$\frac{75}{\sqrt{3}} = \sqrt{3}(75 - CD)$

$75 = 3\ (75 - h) \Rightarrow 3h = 75 \times 2$

$h = \frac{75 \times 2}{2} \Rightarrow h = 50\ ft$

$\therefore$ height of the vertical pole = 50ft

Example 11:

Two towers of height 14m and 25m stand on level ground. The angles of elevation of their tops from a point on the line joining their feet are 45^o and 60^o respectively. Find the distance between the towers.

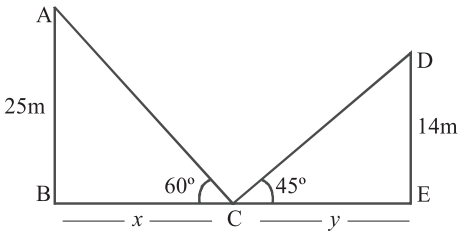
Solution:

Let AB = 25m & DE = 14m are two towers

By data $\angle ACB = 60^o$, $\angle DCE = 45^o$

Let BC = x & CE = y, $x + y = ?$

from right $\angle led$ Δle ABC



$\tan 60^o = \frac{25}{x} \Rightarrow \sqrt{3} = \frac{25}{x}$

$\Rightarrow x = \frac{25}{\sqrt{3}}$ .....(1)

Again from right angled Δle DCE

$\tan 45^o = \frac{14}{y} \Rightarrow 1 = \frac{14}{y}$

$\Rightarrow y = 14$ .....(2)

From Eq^n (1) and Eq^n (2) we get

$$x + y = \frac{25}{\sqrt{3}} + \frac{14}{1} = \frac{25 + 14\sqrt{3}}{\sqrt{3}}$$

Hence the distance between the towers = $\frac{25 + 14\sqrt{3}}{\sqrt{3}}$ metres.

Example 12:

The angles of depressions of two boats as observed from the mast head of a ship 50m high are 45^o and 30^o. What is the distance between the boats if they are on the same side of the mast head in line with it?

Solution:

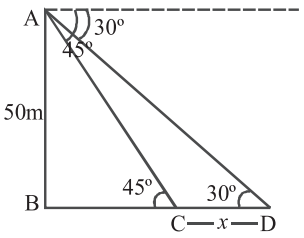
Let AB be the mast head, C and D denote the positions of the boats

Given $AB = 50m$ $\angle XAC = 45^o$

$\angle XAD = 30^o$

Also $\angle BCA = 45^o$ ($\therefore$ BD || AX)

$\angle BDA = 30^o$ ($\therefore$ BD || AX)



From the right angled triangle ABC

we get $\tan 45^o = \frac{AB}{BC} \Rightarrow BC = 50$ .....(1)

Again, from a right angled triangle ABD

$$\tan 30^o = \frac{AB}{BD} = \frac{50}{BC + x} \quad (\therefore CD = x)$$

$$\frac{1}{\sqrt{3}} = \frac{50}{BC + x} \Rightarrow BC + x = 50\sqrt{3}$$

$$\Rightarrow BC = 50\sqrt{3} - x \quad .....(2)$$

From equations 1 & 2 we get $50 = 50\sqrt{3} - x$

$x = 50\left(\sqrt{3} - 1\right)$ metres

Hence, the distance between the boats is $50(\sqrt{3} - 1)$ metres

Exercise 13.1

Two Mark Questions:

I 1) The angle of elevation of the top of a tower at a distance 500 metres from its foot is 30°. Find the height of the tower.

2) The angle of elevation of the top of a chimney at a distance of 100 metres from a foot is 30°. Find its height.

3) From a ship a mast head 40 metres high the angle of depression of a boat is observed to be 45°. Find its distance from the ship.

4) What is the angle of elevation of the sun when the length of the shadow of a pole is $\frac{1}{\sqrt{3}}$ times the height of the pole?

5) Find the angle of elevation of the sun when the shadow of a tower 75 metres high is $25\sqrt{3}$ metres long.

6) A kite flying at a height of h is tied to a thread which is 500m long. Assuming that there is no kink in the thread and H makes an angle of 30° with the ground. Find the height of the kite.

7) A ladder leaning against a wall makes an angle of 60° with the ground. The foot of the ladder is 6m away from the wall. Find the length of the ladder.

8) Find the angle of elevation of the Sun's rays from a point on the ground at a distance of $3\sqrt{3}$ m, from the foot of tower 3m high.

Four or Five marks questions:

9) The angles of elevation of the top of a tower from the base and the top of a building are 60° and 45°. The building is 20 metres high. Find the height of the tower.

10) The shadow of a tower standing on a level plane is found to be 50 metres longer when Sun's altitude is 30°. Than when it is 60°. Find the height of the tower.

11) An aeroplane when flying at a heights of 2000 metres passes vertically above another plane at an instant when their angles of elevation from the same point of observation are 60° and 45° respectively. Find the distance between the aeroplanes.

12) From a point on the line joining the feet of two poles of equal heights, the angles of elevation of the tops of the poles are observed to be 30° and 60°. If the distance between the poles is a Find (i) the height of the poles (ii) the position of the point of observation.

13) The angles of elevation of the top of a tower from two points distant a and b ($a < b$) from its foot and the same straight line from it are 30° and 60°. Show that the height of the tower is $\sqrt{ab}$.

14) A flag staff stands upon the top of a building. At a distance of 20 metres the angles of elevation of the top of the flag staff and building are 60° & 45° respectively. Find the height of the flag staff.

15) From the top of a cliff, the angles of depression of two boats in the same vertical plane as the observer are 30° and 45°. If the distance between the boats is 100 metres, find the height of the cliff.

16) From a point A due north of the tower, the elevation of the top of the tower is 60°. From a point B due south, the elevation is 45°, if AB = 100 metres. Show that the height of the tower is $50\sqrt{3}(\sqrt{3}-1)$ metres.

17) A person at the top of a hill observes that the angles of depression of two consecative kilometres stones on a road leading to the foot of the hill and in the same vertical plane containing the position of the observer are 30° and 60°. Find the height of the hill.

18) The angle of elevation of a tower from a point on the ground is 30°. At a point on the horizontal line passing through the foot of the tower and 100 metres nearer it, the angle of elevation is found to be 60° . Find the height of the tower and the distance of the first point from the tower.

20) A person is at the top of a tower 75 feet high from there he observes a vertical pole and finds the angles of depressions of the top and the bottom of the pole which are 30° and 60° respectively. Find the height of the pole.

ANSWERS 13.1

1) $\frac{500\sqrt{3}}{3}$ mts 2) $\frac{100}{\sqrt{3}}$ mts 3) 40 mts 4) 60° 5) 60° 6) 250 mts

7) 12 mts 8) 30° 9) $10\sqrt{3}+1$ mts 10) 43.3 mts 11) $\frac{2000\left(\sqrt{3}-1\right)}{\sqrt{3}}$ metres

12) (i) $\frac{a\sqrt{3}}{4}$ (ii) $\frac{3a}{4}$ 13) $\sqrt{ab}$ 14) $\frac{20\sqrt{3}}{3}$ mts 15) $+50\left(1+\sqrt{3}\right)$

16) $50\left(\sqrt{3}+1\right)$mts 17) $\frac{\sqrt{3}}{2}$ kms 18) $32\left(\sqrt{3}+1\right)$ mts, $32\sqrt{3}$ mts

19) $50\sqrt{3}$ mts, 150 mts 20) 50 feet

* * * * *

Chapter **14**

COMPOUND ANGLES, MULTIPLE ANGLES, SUB MULTIPLE ANGLES AND TRANSFORMATION FORMULAE

14.1 Introduction:

The algebraic sum of two of more angles is called compound angles. Thus A + B, A – B, A + B + C, A + B – C ..... etc are all Compound Angles.

A is a given angle, then the angles 2A, 3A, 4A ..... etc are called Multiple Angles.

A is a given angle then $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}$.....etc are called Sub Multiple angles.

The transformation formulae which we derive here transforms the product of two trigonometric function in to sum or difference of two trigonometric functions and vice versa.

14.2 Trigonometrical ratios of Compound Angles

If A and B are the two angles then (A + B) and (A – B) are called compound angles let us now find the trigonometric ratios of (A + B) and (A – B) and use them to find the T–ratios of other compound angles.

Trigonometric ratios of (A + B)

(wtihout proof)

1. $\sin(A+B) = \sin A \cos B + \cos A \sin B$
2. $\cos(A+B) = \cos A . \cos B - \sin A \sin B$
3. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

Trigonometric - Ratios of (A – B)

4. $\sin(A-B) = \sin A \cos B - \cos A \sin B$
5. $\cos(A-B) = \cos A \cos B + \sin A \sin B$
6. $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Note : (i) We can find the T- ratios of (A – B) by replacing B by - B in T–Ratio's of (A + B) using The results $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = +\cos\theta$

$\tan(-\theta) = -\tan\theta$

(ii) $\sin(A+B) \neq \sin A + \sin B$

But $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Worked Examples

Example 1:

Prove that $\cot(A+B) = \dfrac{\cot A.\cot B - 1}{\cot B + \cot A}$

Solution:

$$\text{L.H.S} = \cot(A+B) = \frac{1}{\tan(A+B)}$$

$$= \frac{1}{\left[\dfrac{\tan A + \tan B}{1 - \tan A \tan B}\right]}$$

$$= \frac{1 - \tan A.\tan B}{\tan A + \tan B}$$

Divide both Numerator and Denominator by tan A. tan B.

$$= \frac{\cot A.\cot B - 1}{\cot B + \cot A} = \text{RHS}$$

$$\boxed{\cot(A+B) = \frac{\cot A.\cot B - 1}{\cot B + \cot A}}$$

Similarly we can prove

$$\cot(A-B) = \frac{\cot A.\cot B + 1}{\cot B - \cot A}$$

Example 2:

Find the values of sin 75°, cos 75° and tan 75°

Solution:

(i) $\sin(75°) = \sin(45° + 30°)$

$= \sin 45°.\cos 30° + \cos 45° \sin 30°$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\therefore \sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $\cos 75° = \cos(45° + 30°)$

$= \cos 45 . \cos 30 - \sin 45 \sin 30$

$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} . \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(iii) $\tan 75° = \tan(45° + 30°)$

$= \frac{\tan 45 + \tan 30°}{1 - \tan 45 . \tan 30°}$

$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 . \frac{1}{\sqrt{3}}}$

$= \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \frac{(3+1)^2}{(\sqrt{3})^2 - (1)^2} = \frac{3+1+2\sqrt{3}}{3-1}$

$= \frac{4+2\sqrt{3}}{2} = 2 + \sqrt{3}$

$\therefore \tan 75° = 2 + \sqrt{3}$

Example 3:

Find the value of sin 15°, cos 15° and tan 15°.

Solution:

(i) $\sin 15° = \sin (45° - 30°)$

$= \sin 45°. \cos 30° - \cos 45°. \sin 30°$

$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\therefore \sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(ii) $\cos 15° = \cos (45° - 30°)$

$= \cos 45° \cos 30° + \sin 45° \sin 30$

$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\therefore \cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

(iii) $\tan 15° = \tan (45° - 30°)$

$= \frac{\tan 45 - \tan 30°}{1 + \tan 45.\tan 30°}$

$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1.\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$

$$= \frac{\left(\sqrt{3}-1\right)^2}{\left(\sqrt{3}\right)^2-(1)^2} = \frac{3+1-2\sqrt{3}}{3-1}$$

$$= \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$

$\therefore \tan 15° = 2-\sqrt{3}$

Example 4:

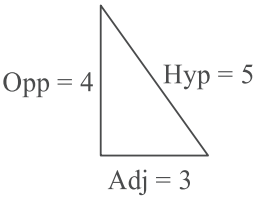
If $\sin A = \frac{4}{5}$ and $\sin B = \frac{5}{13}$ where A and B are acute angles. Find sin (A + B), sin (A – B), cos (A + B), cos (A – B), tan (A + B), tan (A – B)

Solution:

Given: $\sin A = \frac{4}{5}$ $\quad \therefore \cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$

Also given $\sin B = \frac{5}{13}$ $\quad \therefore \cos B = \sqrt{1-\sin^2 B} = \sqrt{1-\frac{25}{169}} = \frac{12}{13}$

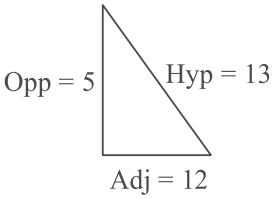
OR



$\text{Hyp}^2 = \text{Opp}^2 + \text{Adj}^2$

$\therefore \text{Adj} = \sqrt{\text{Hyp}^2 - \text{Opp}^2} = \sqrt{5^2-4^2} = 3$

|||ly adj $= \sqrt{13^2-5^2} = 12$



$\therefore \cos A = \frac{\text{Adj}}{\text{Hyp}} = \frac{3}{5}$

$\tan A = \frac{\text{Opp}}{\text{Adj}} = \frac{4}{3}$, $\cos B = \frac{12}{13}$, $\tan B = \frac{5}{12}$

(i) $\sin(A+B) = \sin A \cos B + \cos A . \sin B$

$$= \frac{4}{5}.\frac{12}{13} + \frac{3}{5}.\frac{5}{13} = \frac{48}{65} + \frac{15}{65} = \frac{63}{65}$$

$\therefore \sin(A+B) = \frac{63}{65}$

(ii) $\sin(A-B) = \sin A.\cos B - \cos A.\sin B$

$$= \frac{4}{5}.\frac{12}{13} - \frac{3}{5}.\frac{5}{13} = \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

$$\sin(A-B) = \frac{33}{65}$$

(iii) $\cos(A+B) = \cos A\cos B - \sin A.\sin B$

$$= \frac{3}{5}.\frac{12}{13} - \frac{4}{5}.\frac{5}{13}$$

$$= \frac{36-20}{65} = \frac{16}{25}$$

$$\therefore \cos(A+B) = \frac{16}{25}$$

(iv) $\cos(A-B) = \cos A.\cos B + \sin A.\sin B$

$$= \frac{3}{5}.\frac{12}{13} + \frac{4}{5}.\frac{5}{13}$$

$$= \frac{36}{65} + \frac{20}{65}$$

$$= \frac{36+20}{65} = \frac{56}{65}$$

$$\therefore \cos(A-B) = \frac{56}{65}$$

(v) $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$$= \frac{\frac{63}{\cancel{65}}}{\frac{33}{\cancel{65}}} = \frac{63}{33}$$

$$\therefore \tan(A+B) = \frac{63}{33}$$

$$\text{(vi) } \tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$$

$$= \frac{\frac{33}{\cancel{65}}}{\frac{56}{\cancel{65}}} = \frac{33}{56}$$

$$\therefore \tan(A-B) = \frac{33}{56}$$

Example 5:

If $\tan A = \frac{1}{2}$ and $\tan(A-B) = \frac{2}{7}$ find tan B and tan (A + B).

Solution:

$$\text{w.k.t. } \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A.\tan B}$$

$$\frac{2}{7} = \frac{\frac{1}{2} - \tan B}{1 + \frac{1}{2}\tan B}$$

$$\frac{2}{7} = \frac{1 - 2\tan B}{2 + \tan B}$$

$$4 + 2\tan B = 7 - 14\tan B$$

$$2\tan B + 14\tan B = 7 - 4$$

$$16\tan B = 3$$

$$\therefore \tan B = \frac{3}{16}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A.\tan B} = \frac{\frac{1}{2} + \frac{3}{16}}{1 - \frac{1}{2}.\frac{3}{16}} = \frac{16+6}{32-3}$$

$$\therefore \tan(A+B) = \frac{22}{29}$$

Example 6:

If A + B = 45° S.T (1 + tan A) (1 + tan B) = 2 and hence deduce tan A = $\sqrt{2}-1$. If A = B

Solution:

Given: $(A+B)=45°$

$\therefore \quad \tan(A+B)=\tan 45°$

$$\frac{\tan A+\tan B}{1-\tan A.\tan B}=1$$

$$\tan A+\tan B=1-\tan A.\tan B$$

$$\tan A+\tan B+\tan A.\tan B=1$$

Add 1 to both sides we get

$$1+\tan A+\tan B+\tan A.\tan B=1+1$$

$$(1+\tan A)+\tan B(1+\tan A)=2$$

$$(1+\tan A)(1+\tan B)=2$$

Given A = B

$\therefore \quad (1+\tan A)(1+\tan A)=2$

$$(1+\tan A)^2=2$$

$$1+\tan A=\pm\sqrt{2}$$

But A is an acute angle and therefore tan A is positive

$$1+\tan A=\sqrt{2}$$

$$\tan A=\sqrt{2}-1$$

Example 7:

Show that $\frac{\cos 2A}{\sec A}-\frac{\sin 2A}{\operatorname{cosec} A}=\cos 3A$

Solution:

$$\text{L.H.S.}=\cos 2A.\frac{1}{\sec A}-\sin 2A\frac{1}{\operatorname{cosec} A}$$

$$=\cos 2A.\cos A-\sin 2A.\sin A \quad \text{[using } \cos(A+B) \text{ formula]}$$

$$=\cos(2A+A)$$

$$=\cos 3A$$

$$=\text{RHS}$$

Example 8:

S.T cos (A – B) cos (A + B) – sin (A – B). sin (A + B) = cos2A

Solution:

Take A – B = X, A + B = Y in the LHS

$\therefore$ LHS = cos X cos Y – sin X. sin Y

= cos (X + Y)

$= \cos\left(A - \cancel{B} + A + \cancel{B}\right)$

= cos 2A = R.H.S

Example 9:

S.T $\cos\left[\frac{\pi}{3}+A\right].\cos\left[\frac{\pi}{3}-A\right]-\sin\left(\frac{\pi}{3}+A\right).\sin\left[\frac{\pi}{3}-A\right]=\frac{-1}{2}$

Solution:

Take $\frac{\pi}{3}+A=X \;\&\; \frac{\pi}{3}-A=Y$

$\therefore$ LHS = cos X. cos Y – sin X. sin Y

= cos (X + Y)

$= \cos\left[\frac{\pi}{3}+\cancel{A}+\frac{\pi}{3}-\cancel{A}\right]$

$= \cos\frac{2\pi}{3} = \cos 120° = \cos[180-60] = -\cos 60°$

$= \frac{-1}{2} = \text{RHS}$

Example 10:

S.T sin 2A. cos A + cos 2A. sin A = sin 4A.cos A – cos 4A – sin A

Solution:

LHS = sin2A.cosA + cos2A.sinA [Using Compound Angle Formula]

= sin(2A + A) = sin 3A ——— (1)

RHS = sin 4A. cos A – cos 4A. sin A

= sin (4A – A) = sin 3A ——— (2)

From (1) and (2) we get

LHS = RHS

Example 11:

S.T $\cot 2\theta + \tan\theta = \operatorname{cosec} 2\theta$

Solution:

$$\text{L.H.S.} = \cot 2\theta + \tan\theta$$

$$= \frac{\cos 2\theta}{\sin 2\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos 2\theta.\cos\theta + \sin 2\theta \sin\theta}{\sin 2\theta.\cos\theta}$$

$$= \frac{\cos(2\theta - \theta)}{\sin 2\theta.\cos\theta}$$

$$= \frac{\cancel{\cos\theta}}{\sin 2\theta.\cancel{\cos\theta}}$$

$$= \frac{1}{\sin 2\theta} = \operatorname{cosec} 2\theta = \text{R.H.S}$$

Example 12:

If $\tan A = \frac{1}{3}$, $\tan B = \frac{2}{7}$, then find $\cot(A - B)$.

Solution:

Given: $\tan A = \frac{1}{3} \Rightarrow \cot A = 3$

$$\tan B = \frac{2}{7} \Rightarrow \cot B = \frac{7}{2}$$

$$\text{w.k.t } \cot(A - B) = \frac{\cot B \cot A + 1}{\cot B - \cot A}$$

$$= \frac{\frac{7}{2}.3 + 1}{\frac{7}{2} - 3} = \frac{\frac{21}{2} + 1}{\frac{7 - 6}{2}}$$

$$= \frac{21 + 2}{7 - 6} = \frac{23}{1}$$

$$\therefore \cot(A - B) = 23$$

Example 13:

Prove that (i) $\tan(45^o + A) = \frac{1+\tan A}{1-\tan A}$ (ii) $\tan(45 - A) = \frac{1-\tan A}{1+\tan A}$

Solution:

(i) LHS = $\tan(45 + A)$

$$= \frac{\tan 45^o + \tan A}{1 - \tan 45^o . \tan A}$$

$$= \frac{1+\tan A}{1-\tan A} = \text{RHS}$$

(ii) Try yourself

Example 14:

If A + B + C = 180° and tan A = 1, tan B = 2, show that tan C = 3.

Solution:

Given: $A + B + C = 180°$

$\Rightarrow A + B = 180 - C$

$\therefore \tan(A + B) = \tan(180 - C)$

$$\frac{\tan A + \tan B}{1 - \tan A . \tan B} = -\tan C$$

$$\frac{1+2}{1-2} = -\tan C$$

$$\frac{3}{-1} = -\tan C \quad \Rightarrow \tan C = 3$$

Example 15:

P.T tan 2A. tan3A. tan5A = tan5A – tan3A – tan2A

Solution:

Consider, tan5A = tan (2A + 3A)

$$\tan 5A = \frac{\tan 2A + \tan 3A}{1 - \tan 2A . \tan 3A}$$

$\tan 2A + \tan 3A = \tan 5A (1 - \tan 2A . \tan 3A)$

$\tan 2A + \tan 3A = \tan 5A - \tan 2A . \tan 3A . \tan 5A$

$\tan 2A . \tan 3A . \tan 5A = \tan 5A - \tan 3A - \tan 2A$

Example 16:

If $\tan A = \frac{5}{6}$, $\tan B = \frac{1}{11}$ S.T. $A + B = \frac{\pi}{4}$

Solution:

$$\text{Consider, } \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{5}{6} + \frac{1}{11}}{1 - \frac{5}{6}.\frac{1}{11}} = \frac{\frac{55+6}{66}}{\frac{66-5}{66}}$$

$$\tan(A+B) = \frac{\cancel{61}}{\cancel{61}} = 1$$

$$\Rightarrow \quad A + B = \pi/4 \quad \left(\because \tan\frac{\pi}{4} = 1\right)$$

17. If $\sin A = \frac{5}{13}$ $\cos B = \frac{-4}{5}$, $\frac{\pi}{2} < A < \pi$ and $\pi < B < \frac{3\pi}{2}$

Find (i) sin (A + B) (ii) cos (A + B) (iii) sin (A – B) (iv) cos (A – B)
(v) tan (A + B) (vi) tan (A – B)

Solution:

By the given data angle A lies in II quadrant and angle B lies in III quadrant

$$\text{Given: } \sin A = \frac{5}{13} = \frac{\text{Opp}}{\text{Hyp}}, \therefore \text{adj} = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$$

$$\therefore \cos A = \frac{-12}{13} \text{ (–ve sign is taken) } (\because \cos A \text{ is negative in II quadrant})$$

$$\text{Also, } \cos B = \frac{-4}{5} \quad \text{opp} = \sqrt{5^2 - 4^2} = \sqrt{9} = 3$$

$$\sin B = \frac{-3}{5} \text{ (sin B is negative in III quadrant)}$$

(i) $\sin(A+B) = \sin A.\cos B + \cos A.\sin B$

$$= \frac{5}{13}.\frac{-4}{5} + \frac{-12}{13}, \frac{-3}{5}$$

$$= \frac{-20}{65} + \frac{36}{65} = \frac{16}{65}$$

(ii) $\cos(A+B) = \cos A.\cos B - \sin A.\sin B$

$$= \frac{-12}{13}.\frac{-4}{5} - \frac{5}{13}.\frac{-3}{5}$$

$$= \frac{48}{65} + \frac{15}{65} = \frac{63}{65}$$

(iii) $\sin(A-B) = \sin A.\cos B - \cos A \sin B$

$$= \frac{-12}{13}.\frac{-4}{5} + \frac{5}{13}.\frac{-3}{5}$$

$$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

(v) $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{16/\cancel{65}}{63/\cancel{65}} = \dfrac{16}{63}$

(vi) $\tan(A-B) = \dfrac{\sin(A-B)}{\cos A - B} = \dfrac{-56/65}{33/65} = \dfrac{-56}{33}$

Example 18:

P.T $\cos A + \cos(120° + A) + \cos(120° - A) = 0$

Solution:

LHS $= \cos A + \cos(120° + A) + \cos(120° - A)$

$= \cos A + \cos 120.\cos A - \sin 12\cancel{0} \sin A + \cos 120.\cos A + \cancel{\sin} 120 \sin A$

$= \cos A + 2.\cos 120°.\cos A$

$= \cos A + 2.\cos(180° - 60°).\cos A$

$= \cos A + 2.(-\cos 60°)\cos A$

$= \cos A - \cancel{2}.\dfrac{1}{\cancel{2}}.\cos A = \cancel{\cos A} - \cancel{\cos A} = 0 = \text{RHS}$

Example 19:

S.T $\tan(45 + A)\tan(45 - A) = 1$

Solution:

$$\text{LHS} = \left(\frac{\tan 45 + \tan A}{1 - \tan 45.\tan A}\right)\left(\frac{\tan 45 - \tan A}{1 + \tan 45.\tan A}\right)$$

$= 1 = \text{RHS} \qquad (\because \tan 45 = 1)$

Example 20:

P.T sin (A + B). sin (A – B) = $\sin^2 A - \sin^2 B$

Solution:

LHS = (sin A. cos B + cos A. sinB) (sinA.cosB – cosA.sinB)

= $\sin^2 A.\cos^2 B - \cos^2 A.\sin^2 B$ [using $(a+b)(a-b) = a^2 - b^2$]

= $\sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B$

= $\sin^2 A - \sin^2 A.\sin^2 B - \sin^2 B + \sin^2 A.\sin^2 B$

= $\sin^2 A - \sin^2 B$ = RHS

Note : It is always useful to remember the following

Results :(i) $\sin 15° = \cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$ (ii) $\cos 15° = \sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

(iii) $\cos 105° = \frac{1-\sqrt{3}}{2\sqrt{2}}$ (iv) $\sin 105° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

EXERCISE 14.1

Two marks questions:

I. 1. Obtain the values of the trigonometric functions of 75°, 15° and 105° and prove that

(i) $\tan 75° + \cot 75° = 4$

(ii) $\sin 105° + \cos 105° = \frac{1}{\sqrt{2}}$

(iii) $\sec 15° + \operatorname{cosec} 15° = 2\sqrt{6}$

2. If $\sin A = \frac{3}{5}$, $\cos B = \frac{4}{5}$ find sin (A + B) and cos (A – B). Where A and B are acute angles.

3. If $\cos A = \frac{5}{13}$, $\cos B = \frac{24}{25}$ find cos (A + B) and sin (A – B). A and B are acute angles.

4. If $\sec A = \frac{17}{8}$, $\operatorname{cosec} B = \frac{5}{4}$ find sec (A + B). cosec (A – B), A and B are acute angles.

5. If $\sin A = \frac{3}{5}$, $\cos B = \frac{-8}{17}$, $\frac{\pi}{2} < A < \pi$ and $\frac{\pi}{2} < B < \pi$

   Find the values of $\sin(A + B)$ and $\cos(A - B)$

6. If $\sin A = \frac{7}{25}$, $\cos B = \frac{-12}{13}$ where $\frac{\pi}{2} < A < \pi$ and $\pi < B < \frac{3\pi}{2}$

   Find the values of:

   (i) $\sin(A + B)$ (ii) $\cos(A + B)$ (iii) $\sin(A - B)$

   (iv) $\cos(A - B)$ (v) $\tan(A + B)$ (vi) $\tan(A - B)$

7. If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, Find $\tan(A + B)$, $\tan(A - B)$

8. $\tan(A - B) = \frac{1}{7}$, $\tan A = \frac{1}{2}$ show that $A + B = 45°$

9. If $\tan A = \frac{1}{3}$, $\tan(A + B) = \frac{1}{7}$ find $\tan B$.

10. $\tan A = \frac{3}{4}$ and $\tan B = \frac{1}{7}$ S.T $\tan(A + B) = 1$

11. If $\tan A = \frac{5}{6}$ and $\tan(A + B) = 1$ S.T $\tan B = \frac{1}{11}$

12. If $\tan \alpha = \frac{n}{n+1}$ and $\tan \beta = \frac{1}{2n+1}$ S.T $\alpha + \beta = \frac{\pi}{4}$

13. P.T $\frac{\cos 2A}{\sec A} + \frac{\sin 2A}{\operatorname{cosec} A} = \cos A$.

14. P.T $\sin(45° + A) + \cos(45° + A) = \sqrt{2}\cos A$

15. P.T $\cos\left(A + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(\cos A - \sin A)$

16. P.T $\cos\left(\frac{\pi}{6} + A\right).\cos\left(\frac{\pi}{6} - A\right) - \sin\left(\frac{\pi}{6} + A\right).\sin\left(\frac{\pi}{6} - A\right) = \frac{1}{2}$

Five marks questions:

II. Prove the following

17. $\frac{\sin(A+B)}{\cos A.\cos B} = \tan A + \tan B$

18. $\frac{\sin(A-B)}{\sin A \sin B} = \cot B - \cot A$

19. S.T $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

20. $\sin(A+B)\sin(A-B) = \cos^2 B - \cos^2 A$.

21. $\cos\left(\frac{\pi}{4} - A\right) - \sin\left(\frac{\pi}{4} + A\right) = 0$

22. $\sin\left(\frac{\pi}{3} - A\right).\cos\left(\frac{\pi}{6} + A\right) + \cos\left(\frac{\pi}{3} - A\right).\sin\left(\frac{\pi}{6} + A\right) = 1$

23. $\tan 15° + \cot 15° = 4$

24. $\sin 105° + \cos 105° = \cos 45°$

25. $\cot 2A + \tan A = \operatorname{cosec} 2A$

26. P.T $\tan A \tan 3A. \tan 4A = \tan 4A - \tan 3A - \tan A$.

27. P.T $\cos(45° - A).\cos(45° - B) - \sin(45° - A)\sin(45° - B) = \sin(A + B)$

28. S.T $\sum \frac{\sin(A-B)}{\cos A.\cos B} = 0$

29. $\cos(120° + A) + \cos(120° - A) = -\cos A$

30. $\frac{\sin(A+B)}{\sin(A-B)} = \frac{\tan A + \tan B}{\tan A - \tan B}$

Answers 14.1

I. 1. $\frac{\sqrt{3}+1}{2\sqrt{2}}, \frac{\sqrt{3}-1}{2\sqrt{2}}, 2+\sqrt{3}, \frac{\sqrt{3}-1}{2\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}, 2-\sqrt{3}, \frac{\sqrt{3}+1}{2\sqrt{2}}, \frac{1-\sqrt{3}}{2\sqrt{2}}, \frac{1+\sqrt{3}}{1-\sqrt{3}}$

2. $\frac{24}{25}, 1$

3. $\frac{36}{325}, \frac{253}{325}$

4. $\frac{-85}{36}, \frac{85}{13}$

5. $\frac{-84}{85}, \frac{77}{85}$

6. $\frac{36}{325}, \frac{323}{325}, \frac{-204}{325}, \frac{253}{325}, \frac{36}{323}, \frac{-204}{253}$

7. $1, \frac{1}{7}$

8. $A + B = 45°$

9. $-2/11$

14.3 Multiple angles :

If A is the given angle then the angles 2A, 3A, 4A - etc are called multiple angles.

I. Trigonometric Ratios of 2A :

Using compound angle formulae, we get multiple angles formula as follows

(i) w.k.t. $\sin(A+B) = \sin A.\cos B + \cos A \sin B$

put $B = A$

$\sin(A+A) = \sin A.\cos A + \cos A \sin A$

$\sin 2A = 2 \sin A.\cos A$

$$\boxed{\sin 2A = 2\sin A.\cos A}$$

(ii) w.k.t. $\cos(A+B) = \cos A.\cos B - \sin A.\sin B$

put $B = A$

$\cos(A+A) = \cos A.\cos A - \sin A.\sin A$

$\cos 2A = \cos^2 A - \sin^2 A$

$$\boxed{\cos 2A = \cos^2 A - \sin^2 A} \quad \text{——— (1)}$$

we can express cos 2A in to other useful forms using the identily $\sin^2 A + \cos^2 A = 1$

From eq^n (1) $\cos 2A = \cos^2 A - \sin^2 A$

$= \cos^2 A - (1 - \cos^2 A)$

$= \cos^2 A - 1 + \cos^2 A$

$= 2\cos^2 A - 1$

$$\therefore \boxed{\cos 2A = 2\cos^2 A - 1}$$

Again from (1) $\cos 2A = \cos^2 A - \sin^2 A$

$= (1 - \sin^2 A) - \sin^2 A$

$= 1 - 2\sin^2 A$

$$\boxed{\cos 2A = 1 - 2\sin^2 A}$$

Hence $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

(iii) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A.\tan B}$

put $B = A$

$$\tan(A+A) = \frac{\tan A + \tan B}{1 - \tan A.\tan A}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$\therefore \boxed{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$$

II. Trigonometric rotios of 3A

(i) Prove that $\sin 3A = 3\sin A - 4\sin^3 A$

$$\begin{aligned} LHS = \sin 3A &= \sin(A + 2A) \\ &= \sin A.\cos 2A + \sin 2A.\cos A \\ &= \sin A(1 - 2\sin^2 A) + (2\sin A.\cos A).\cos A \\ &= \sin A - 2\sin^3 A + 2\sin A(\cos^2 A) \\ &= \sin A - 2\sin^3 A + 2\sin A(1 - \sin^2 A) \\ &= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A \\ &= 3\sin A - 4\sin^3 A \end{aligned}$$

$$\therefore \boxed{\sin 3A = 3\sin A - 4\sin^3 A}$$

(ii) Prove that $\cos 3A = 4\cos 3A - 3\cos A$

$$\begin{aligned} LHS = \cos 3A &= \cos(A + 2A) \\ &= \cos A.\cos^2 A - \sin A.\sin 2A \\ &= \cos A(2\cos 2A - 1) - \sin A(2\sin A.\cos A) \\ &= 2\cos^3 A - \cos A - 2\sin^2 A.\cos A \\ &= 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A) \\ &= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A \\ &= 4\cos^3 A - 3\cos A \end{aligned}$$

$$\therefore \boxed{\cos 3A = 4\cos^3 A - 3\cos A}$$

(iii) Prove that $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

$$\text{LHS} = \tan 3A = \tan(A + 2A)$$

$$= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \dfrac{2\tan A}{1 - \tan^2 A}}{1 - \tan A.\dfrac{2\tan A}{1 - \tan^2 A}}$$

$$= \frac{\tan A(1 - \tan^2 A) + 2\tan A}{1 - \tan^2 A - 2\tan^2 A}$$

$$= \frac{\tan A - \tan^3 A + 2\tan A}{1 - 3\tan^2 A}$$

$$= \frac{3\tan A - \tan^2 A}{1 - 3\tan^2 A}$$

$$\therefore \boxed{\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}}$$

III. To express sin2A and cos2A in terms of tanA

Prove that $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$

$$\text{LHS} = \sin 2A = 2\sin A \cos A$$

$$= \frac{2\sin A}{\cos A}.\cos^2 A$$

$$= \frac{2\tan A}{1 + \tan^2 A} \qquad \left[\because \sec^2 A = 1 + \tan^2 A \;\&\; \frac{\sin A}{\cos A} = \tan A\right]$$

$$\therefore \boxed{\sin 2A = \frac{2\tan A}{1 + \tan^2 A}}$$

(ii) P.T $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

.

$$\text{LHS} = \cos 2A = \cos^2 A - \sin^2 A = \cos^2 A\left(1 - \frac{\sin^2 A}{\cos^2 A}\right)$$

$$= \cos^2 A\,(1 - \tan^2 A)$$

$$= \frac{1-\tan^2 A}{\sec^2 A}$$

$$= \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\therefore \boxed{\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}}$$

List of Multiple angles formulae

(1) $\sin 2A = \begin{cases} (i)\, 2\sin A \cos A \\ (ii)\, \dfrac{2\tan A}{1+\tan^2 A} \end{cases}$

(2) $\cos 2A = \begin{cases} (i)\, \cos^2 A - \sin^2 A \\ (ii)\, 2\cos^2 A - 1 \\ (iii)\, 1 - 2\sin^2 A \\ (iv)\, \dfrac{1-\tan^2 A}{1+\tan^2 A} \end{cases}$

(3) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

(4) $\sin 3A = 3\sin A - 4\sin^3 A$

(5) $\cos 3A = 4\cos^3 A - 3\cos A$

(6) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

14.4 Sub multiple angles :

If 'A' is an angle then $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}$ ..... are called submultiple angle

In multiple angle formulae, replace 2A by A and A by $\frac{A}{2}$, we get sub multiple angle formulae as follows:

$\therefore \sin 2A = 2 \sin A . \cos A$

$\therefore \sin A = 2 \sin\left(\frac{A}{2}\right) . \cos\left(\frac{A}{2}\right)$

Hence all the multiple angle formula of '2A' can be changed as follows

List of Half Angle Formulae or Submultiple Angles Formulae (Without Proof & Problem)

(1) $\sin A = \begin{cases} (i)\ 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) \\ (ii)\ \frac{2\tan\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)} \end{cases}$

(2) $\cos A = \begin{cases} (i)\ \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right) \\ (ii)\ 2\cos^2\left(\frac{A}{2}\right) - 1 \\ (iii)\ 1 - 2\sin^2\left(\frac{A}{2}\right) \\ (iv)\ \frac{1-\tan^2\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)} \end{cases}$

(3) $\tan A = \frac{2\tan\left(\frac{A}{2}\right)}{1-\tan^2\left(\frac{A}{2}\right)}$

Note : The above all formulae are very useful in the problems of differentiation and integration

Note : If $\sin 2A = 2 \sin A \cos A$

then for sin4A multiply 'A' by 2 on both sides

$\sin 4A = 2 \sin 2A \cos 2A$

|||ly $\sin 6A = 2 \sin 3A . \cos 3A$ Multiply 'A' by 3 on both sides

Similarly we can find cos 4A, cos 6A, tan 4A ..... etc by multiplying 'A' by respective positive number on both in all the above formula

WORKED EXAMPLES

Example 1:

P. T $\dfrac{\cos 2A}{1-\sin 2A} = \tan(45+A)$

Solution:

$$\text{LHS} = \frac{\cos 2A}{1-\sin 2A}$$

$$= \frac{\cos^2 A - \sin^2 A}{1 - 2\sin A \cos A}$$

$$= \frac{(\cos A - \sin A)(\cos A + \sin A)}{(\cos A - \sin A)}$$ by cos A to both NR andDNR

$$= \frac{1+\tan A}{1-\tan A}$$

$= \tan(45^\circ + A) = \text{RHS}$

Example 2:

P. T $\dfrac{\sin 3\theta}{1+2\cos 2\theta} = \sin\theta$**. Hence deduce the value of sin 15°**

Solution:

$$\text{LHS} = \frac{\sin 3\theta}{1+2\cos 2\theta}$$

$$= \frac{3\sin\theta - 4\sin^3\theta}{1+2(1-2\sin^2\theta)}$$

$$= \frac{3\sin\theta - 4\sin^3\theta}{3 - 4\sin^2\theta}$$

$$= \sin\theta \frac{\left(3 - 4\sin^2\theta\right)}{\left(3 - 4\sin^2\theta\right)}$$

$$= \sin\theta = \text{RHS}$$

Put $\theta = 15°$

Then $\sin 3\theta = \sin 45° = \frac{1}{\sqrt{2}}$

$$\cos 2\theta = \cos 30° = \frac{\sqrt{3}}{2}$$

$$\therefore \sin\theta = \frac{\sin 3\theta}{1 + 2\cos 2\theta}$$

$$\sin 15° = \frac{\sin 45°}{1 + 2\cos 30°}$$

$$= \frac{\frac{1}{\sqrt{2}}}{1 + \frac{2.\sqrt{3}}{2}} = \frac{1}{\sqrt{2}\left(\sqrt{3}+1\right)} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} \quad \text{(Multiply and divide by } \left(\sqrt{3}-1\right)$$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}-1}{(3-1)} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Example 3 :

P. T $\cos^4\theta - \sin^4\theta = \cos 2\theta$

Solution:

$$\text{LHS} = \cos^4\theta - \sin^4\theta$$

$$= (\cos^2\theta)^2 - (\sin^2\theta)^2$$

$$= (\cos^2\theta^2 - \sin^2\theta)\ (\cos^2\theta + \sin^2\theta)$$

$$= \cos 2\theta \times 1$$

$$= \cos 2\theta = \text{RHS}$$

Example 4 :

$$\textbf{P. T } \frac{1-\cos 2A}{\sin 2A} = \tan A$$

Solution:

$$\text{LHS} = \frac{1-\cos^2 A}{\sin^2 A} = \frac{1-(1-2\sin^2 A)}{2\sin A\cos A}$$

$$= \frac{1-1+2\sin^2 A}{2\sin A.\cos A}$$

$$= \frac{2\sin A}{2\sin A.\cos A}$$

$$= \frac{\sin A}{\cos A} = \tan A = \text{RHS}$$

Example 5 :

$$\text{P. T } \cos^6 A + \sin^6 A = 1 - \frac{3}{4}\sin^2(2A)$$

Solution:

$$\text{LHS} = (\cos^2 A)^3 + (\sin^2 A)^3$$

$$= (\cos^2 A + \sin^2 A)^3 - 3\cos^2 A.\sin^2 A\,(\cos^2 A + \sin^2 A)$$

$$= 1^3 - 3\cos^2 A.\sin^2 A\,(1)$$

$$= 1 - 3\,\frac{(2\cos A\sin A)^2}{4} \qquad \text{(Divide \& Multiply by 4)}$$

$$= 1 - \frac{3}{4}\sin^2 2A = RHS$$

Example 6:

$$\textbf{P. T } \frac{1-\cos 2A+\sin 2A}{1+\cos 2A+\sin 2A} = \tan A$$

Solution:

$$\text{LHS} = \frac{1-\cos 2A+\sin 2A}{1+\cos 2A+\sin 2A}$$

$$= \frac{2\sin^2 A+2\sin A.\cos A}{2\cos^2 A+2\sin A.\cos A}$$

$$= \frac{2\sin A(\cancel{\sin A + \cos A})}{2\cos A(\cancel{\cos A + \sin A})}$$

$$= \frac{\cancel{2}\sin A}{\cancel{2}\cos A}$$

$$= \tan A = \text{RHS}$$

Example 7:

P. T $(\sin A + \cos A)^2 = 1 + \sin 2A$

Solution:

$$\text{LHS} = (\sin A + \cos A)^2$$

$$= \sin^2 A + \cos^2 A + 2\sin A \cos A$$

$$= 1 + \sin 2A = \text{RHS}$$

Example 8:

P. T $\dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A$

Solution:

$$\text{LHS} = \frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \frac{\sin A + 2\sin A \cos A}{\cancel{1} + \cos A + 2\cos^2 A - \cancel{1}}$$

$$= \frac{\sin A(\cancel{1 + 2\cos A})}{\cos A(\cancel{1 + 2\cos A})}$$

$$\frac{\sin A}{\cos A} = \tan A = \text{RHS}$$

Example 9:

P. T $\dfrac{\cot A}{\cot A - \cot 3A} + \dfrac{\tan A}{\tan A - \tan 3A} = 1$

Solution:

$$\text{LHS} = \frac{\cot A}{\frac{1}{\tan A} - \frac{1}{\tan 3A}} + \frac{\tan A}{\tan A - \tan 3A}$$

$$= \frac{(\cot A . \tan A)\tan 3A}{\tan 3A - \tan A} + \frac{\tan A}{(\tan 3A - \tan A)}$$

$$= \frac{1 \times \tan 3A}{\tan 3A - \tan A} - \frac{\tan A}{\tan 3A - \tan A}$$

$$= \frac{\cancel{\tan 3A - \tan 3A}}{\cancel{\tan 3A - \tan A}}$$

$$= 1 = \text{RHS}$$

Example 10:

P.T sec $(45^o + A)$. sec $(45^o - A) = 2$ sec 2A

Solution:

LHS = sec $(45^o + A)$. sec $(90^o - (45^o + A)]$

$$= \frac{1}{\cos(45^o+A).\sin(45^o+A)}$$

$$= \frac{2}{2.\sin(45^o+A).\cos(45^o+A)} = \frac{2}{\sin 2(45^o+A)}$$

$$= \frac{2}{\sin(90^o+2A)} = \frac{2}{\cos 2A} = 2\sec 2A = \text{RHS}$$

Example 11:

Find the value of $4\cos^3 10^0 - 3\cos 10^0$

Solution:

Consider $\cos 3A = 4\cos^3 A - 3\cos A$

Put $A = 10°$ $\quad \therefore 4\cos^3 10° - 3\cos 10° = \cos 3.10° = \cos 30° = \frac{\sqrt{3}}{2}$

Example 12:

P.T $\tan\left(\frac{\pi}{4} + A\right) - \tan\left(\frac{\pi}{4} - A\right) = 2\tan 2A$

Solution:

$$\text{LHS} = \tan\left(\frac{\pi}{4} + A\right) - \tan\left(\frac{\pi}{4} - A\right)$$

$$= \frac{\tan\frac{\pi}{4} + \tan A}{1 - \tan \pi/4 . \tan A} - \frac{\tan\frac{\pi}{4} - \tan A}{1 + \tan \pi/4 . \tan A}$$

$$= \frac{1+\tan A}{1-\tan A} - \frac{1-\tan A}{1+\tan A} = \frac{(1+\tan A)^2 - (1-\tan A)^2}{1^2 - \tan^2 A}$$

$$= \frac{1+\tan^2 A + 2\tan A - 1 - \tan^2 A + 2\tan A}{1-\tan^2 A}$$

$$= \frac{4\tan A}{1-\tan^2 A} = 2.\frac{2\tan A}{1-\tan^2 A} = 2.\tan 2A = \text{RHS}$$

Example 13:

S.T $\dfrac{1+\tan^2(45^\circ-\theta)}{1-\tan^2(45^\circ-\theta)} = \text{cosec}2\theta$

Solution:

$$\text{LHS} = \frac{1+\dfrac{\sin^2(45^\circ-\theta)}{\cos^2(45^\circ-\theta)}}{1-\dfrac{\sin^2(45^\circ-\theta)}{\cos^2(45^\circ-\theta)}}$$

$$= \frac{\cos^2(45^\circ-\theta)+\sin^2(45^\circ-\theta)}{\cos^2(45^\circ-\theta)-\sin^2(45^\circ-\theta)}$$

$$= \frac{1}{\cos 2(45^\circ-\theta)}$$

$$= \frac{1}{\cos(90^\circ-2\theta)}$$

$$= \frac{1}{\sin 2\theta}$$

$$= \text{cosec}2\theta = \text{RHS}$$

Example 14:

P.T tan 2 A = $\dfrac{\tan(A+B)+\tan(A-B)}{1-\tan(A+B).\tan(A-B)}$

Solution:

$$\text{LHS} = \tan 2A = \tan[(A+B)+(A-B)]$$

$$= \frac{\tan(A+B)+\tan(A-B)}{1-\tan(A+B).\tan(A-B)} = \text{RHS}$$

Example 15:

If 2 cosθ $= \left(x+\frac{1}{x}\right)$ S.T 2 cos 2θ $= \left(x^2+\frac{1}{x^2}\right)$

Solution:

LHS = 2 cos 2θ = 2 (2 $\cos^2\theta - 1$)

$= 4\cos^2\theta - 2 = (2\cos\theta)^2 - 2$

$= \left(x+\frac{1}{x}\right)^2 - 2 = x^2 + \frac{1}{x^2} + 2.\cancel{x}.\frac{1}{\cancel{x}} - 2$

$= x^2 + \frac{1}{x^2} = \text{RHS}$

Example 16 :

P. T $\frac{\sin 3A}{1+2\cos 2A} = \sin A$ and hence find sin30°

Solution:

$$\text{LHS} = \frac{\sin 3A}{1+2\cos 2A} = \frac{3\sin A - 4\sin^3 A}{1+2\left(1-2\sin^2 A\right)}$$

$$= \frac{\sin A\cancel{\left(3-4\sin^2 A\right)}}{\cancel{\left(3-4\sin^2 A\right)}} = \sin A = \text{RHS}$$

$$\sin 30° = \frac{\sin 3\cdot(30°)}{1+2\cdot\cos 2\cdot(30°)} = \frac{\sin 90°}{1+2\cos 60°}$$

$$= \frac{1}{1+\cancel{2}.\frac{1}{\cancel{2}}} = \frac{1}{2}$$

Example 17 :

P. T $\frac{\sec 8A - 1}{\sec 4A - 1} = \frac{\tan 8A}{\tan 2A}$

Solution:

$$\text{LHS} = \frac{\sec 8A - 1}{\sec 4A - 1} = \frac{\frac{1-\cos 8A}{\cos 8A}}{\frac{1-\cos 4A}{\cos 4A}} = \frac{1-\cos 8A}{\cos 8A} \times \frac{\cos 4A}{1-\cos 4A}$$

$$= \frac{2\sin^2 4A}{\cos 8A} \cdot \frac{\cos 4A}{2\sin^2 2A}$$

$$= \frac{(2\sin 4A.\cos 4A)}{\cos 8A} \cdot \frac{\sin 4A}{2\sin 2A \sin 2A}$$

$$= \frac{\sin 8A}{\cos 8A} \times \frac{\cancel{2.\sin 2A}\cos 2A}{\cancel{2.\sin 2A}.\sin 2A}$$

$$= \tan 8A.\cot 2A$$

$$= \frac{\tan 8A}{\tan 2A} = \text{RHS}$$

Example 18 :

$$\textbf{P. T } \frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2$$

Solution:

$$\text{LHS} = \frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A}$$

$$= \left(\frac{3\sin A - 4\sin^3 A}{\sin A}\right) - \left(\frac{4\cos^3 A - 3\cos A}{\cos A}\right)$$

$$= 3 - 4\sin^2 A - 4\cos^2 A + 3$$

$$= 6 - 4(\cos^2 A + \sin^2 A)$$

$$= 6 - 4 = 2 = \text{RHS}$$

Example 19:

Express cos 4 A in terms of sin A

Solution:

$$\cos 4A = \cos 2.(2A) = 1 - 2\sin^2(2A)$$

$$= 1 - 2(2\sin A \cos A]^2$$

$$= 1 - 8\sin^2 A \cos^2 A$$

$$\cos 4A = 1 - 8\sin^2 A(1 - \sin^2 A)$$

$$\therefore \cos 4A = 1 - 8\sin^2 A + 8\sin^4 A$$

Example 20 :

If $\sin\theta = \frac{12}{13}$ and θ is acute find $\sin2\theta$, $\cos2\theta$, $\tan2\theta$

Find (i) $\sin2\theta$, (ii) $\cos2\theta$, (iii) $\tan2\theta$

Solution:

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2.\frac{12}{13}.\frac{5}{13} = \frac{120}{169}$$

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{144}{169}\right) = \frac{-119}{169}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{120}{-119}$$

EXERCISE 14.2

Two marks questions:

1. If $\sin A = \frac{1}{2}$ find $\sin 2A$
2. If $\cos A = \frac{\sqrt{3}}{2}$ find $\cos 2A$
3. If $\tan A = \frac{1}{\sqrt{3}}$ find $\tan 2A$
4. If $\sin A = \frac{3}{5}$ find $\sin 3A$
5. If $\cos A = \frac{4}{5}$ find $\cos 3A$
6. if $\tan A = \frac{3}{4}$ find $\tan 3A$
7. Find the value of $3\sin 10° - 4\sin^3 10°$
8. If $\cot A = \frac{12}{5}$ and A is acute find $\sin 3A$ and $\cos 3A$.

9. S.T $\tan A = \dfrac{\tan(A-B)+\tan B}{1-\tan(A-B)\tan B}$

10. PT $\dfrac{\cos 2A}{1+\sin 2A} = \dfrac{\cos A-\sin A}{\cos A+\sin A}$

11. P.T $\dfrac{1+\sin 2\theta}{\cos 2\theta} = \dfrac{1+\tan\theta}{1-\tan\theta}$

12. P.T $\dfrac{\sin A+\sin 2A}{1+\cos A+\cos 2A} = \tan A$

13. P.T $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

14. P.T $(\sin A-\cos A)^2 = 1-\sin 2A$

15. P.T $\cos^4\theta - \sin^4\theta = 2\cos^2\theta - 1$

16. P.T $\dfrac{\cos^3 A-\sin^3 A}{\cos A-\sin A} = 1+\dfrac{1}{2}\sin 2A$

Five marks questions:

Prove the following

1. $\dfrac{1+\cos 2A+\sin 2A}{1-\cos 2A+\sin 2A} = \cot A$

2. $\dfrac{\cos 3A}{2\cos 2A-1} = \cos A$ and hence find cos 15°

3. $\dfrac{1-\cos 2A+\sin 2A}{1+\cos 2A+\sin 2A} = \tan A$

4. $\cos^6 A + \sin^6 A = 1-\dfrac{3}{4}\sin^2(2A)$

5. P.T $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$

6. $\sec(45°+A).\sec(45°-A) = 2\sec 2A$

7. $\dfrac{\cot A}{\cot A-\cot 3A} + \dfrac{\tan A}{\tan A-\tan 3A} = 1$

8. P.T $\dfrac{\cos 2A}{1+\sin 2A} = \tan(45°-A)$

9. If $\tan\alpha = \frac{1}{3}$, $\tan\beta = \frac{1}{7}$ P.T $\tan(2\alpha + \beta) = 45°$

10. If $\tan^2(45° + \theta) = \frac{a}{b}$ P.T $\frac{b-a}{b+a} = -\sin 2\theta$

11. P.T $\tan 2\theta - \tan\theta = \tan\theta . \sec 2\theta$

12. P.T $\cos 2\alpha - \tan\alpha = \frac{\cos 3\alpha}{\cos\alpha . \sin 2\alpha}$

ANSWERS 14.2

I. 1. $\frac{\sqrt{3}}{2}$ 2. $\frac{1}{2}$ 3. $\sqrt{3}$ 4. $\frac{117}{125}$

5. $\frac{-44}{125}$ 6. $\frac{117}{-44}$ 7. $\frac{1}{2}$ 8. $\frac{-828}{2197}, \frac{2035}{2197}$

14.5 Transformation Formulae:

Transformation formulae are the formulae used to convert the product of two trigononetric functions into sum or of difference. And sum of difference into product of two function as follows.

w.k.t $\sin(A + B) = \sin A . \cos B + \cos A . \sin B$ —— (1)

$\sin(A - B) = \sin A . \cos B - \cos A . \sin B$ —— (2)

$\cos(A + B) = \cos A . \cos B - \sin A . \sin B$ —— (3)

$\cos(A - B) = \cos A . \cos B + \sin A . \sin B$ —— (4)

Adding (1) and (2) we get

$\sin(A + B) + \sin(A - B) = 2 \sin A . \cos B$ —— (5)

$$\therefore \sin A . \cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

Subtracting (2) from (1), we get

$2 \cos A . \sin B = \sin(A + B) - \sin(A - B)$ —— (6)

$$\therefore \cos A . \sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

Similarly adding (3) and (4), we get:

$2 \cos A . \cos B = \cos(A + B) + \cos(A - B)$ —— (7)

$$\cos A . \cos B = \frac{1}{2}\left[\cos(A + B) + \cos(A - B)\right]$$

subtracting (4) from (3), we get

$-2 \sin A. \sin B = \cos (A + B) - \cos (A - B)$ —— (8)

$\therefore \sin A. \sin B = -\frac{1}{2} [\cos (A + B) - \cos (A - B)]$

Hence the transformation formulae from product into sum or difference are

1. $\sin A. \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$
2. $\cos A. \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$
3. $\cos A. \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$
4. $\sin A. \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

In the above formulae 5, 6, 7 and 8. Let us take A + B = C and A – B = D. So we get, $A = \frac{C + D}{2}$ and $B = \frac{C - D}{2}$ then we get the sum or difference of trigonometric functions into product of two functions as follows:

5. $\sin C + \sin D = 2\sin\left(\frac{C + D}{2}\right).\cos\left(\frac{C - D}{2}\right)$
6. $\sin C - \sin D = 2 \cos \left(\frac{C + D}{2}\right).\sin \left(\frac{C - D}{2}\right)$
7. $\cos C + \cos D = 2\cos\left(\frac{C + D}{2}\right).\cos\left(\frac{C - D}{2}\right)$
8. $\cos C - \cos D = -2\sin\left(\frac{C + D}{2}\right).\sin\left(\frac{C - D}{2}\right)$

Trigonometrical identities in a triangle:

If A, B and C are the three angles of a triangle. Then $A + B + C = 180°$ and we have the following useful relation.

I $\quad A + B + C = 180^o$

$$\frac{A}{2} = \frac{B}{2} + \frac{C}{2} = 90°$$

$$\frac{A}{2} + \frac{B}{2} = 90^o - \frac{C}{2}$$

$$A + B = 180^o - C \qquad \sin\left(\frac{A+B}{2}\right) = \sin\left(90^o - \frac{C}{2}\right)$$

$$\sin(A + B) = \sin(180 - C) \qquad \sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$$

$$\sin(A + B) = \sin C \qquad \text{Similarly } \cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$$

Also $\quad \sin(B + C) = \sin A$

$\sin(C + A) = \sin B$

II $\quad A + B + C = 180^o$

$A + B = 180^o - C$

$\cos(A + B) = \cos(180^o - C)$ (cos is 2nd quadrant is negative)

$\cos(A + B) = -\cos C$

Also $\quad \cos(B + C) = -\cos A$

$\cos(C + A) = -\cos B$

III $\quad A + B + C = 180°$

$A + B = 180^o - C$

$\tan(A + B) = \tan(180^o - C)$ ($\because$ $\tan\theta$ in 2nd quadrant is negative)

$\tan(A + B) = -\tan C$

Also $\quad \tan(B + C) = -\tan A$

$\tan(C + A) = -\tan B$

Worked Examples

Example 1:

Express the following as sum or difference of two trigonometric functions.

(i) sin 4A.cos 2A **(ii) cos 65° cos 15°**

(ii) $\cos\frac{7\theta}{2}\sin\frac{3\theta}{2}$ **(iv) $\sin\frac{\pi}{5}.\sin\frac{3\pi}{5}$**

Solution:

(i) $\sin 4A.\cos 2A = \frac{1}{2}\left[\sin(4A+2A)+\sin(4A-2A)\right]$

$= \frac{1}{2}\left[\sin 6A+\sin 2A\right]$

(ii) $\cos 65°.\cos 15° = \frac{1}{2}\left[\cos(65°+15°)+\cos(65°-15°)\right]$

$= \frac{1}{2}\left[\cos 80°+\cos 50°\right]$

(iii) $\cos\left(\frac{7\theta}{2}\right).\sin\left(\frac{3\theta}{2}\right) = \frac{1}{2}\left[\sin\left(\frac{7\theta}{2}+\frac{3\theta}{2}\right)-\sin\left(\frac{7\theta}{2}-\frac{3\theta}{2}\right)\right]$

$= \frac{1}{2}\left[\sin 5\theta-\sin 2\theta\right]$

(iv) $\sin\left(\frac{\pi}{5}\right).\sin\left(\frac{3\pi}{5}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi}{5}-\frac{3\pi}{5}\right)-\cos\left(\frac{\pi}{5}+\frac{3\pi}{5}\right)\right]$

$= \frac{1}{2}\left[\cos\left(-\frac{2\pi}{5}\right)-\cos\left(\frac{4\pi}{5}\right)\right]$

$= \frac{1}{2}\left[\cos\left(\frac{2\pi}{5}\right)-\cos\left(\frac{4\pi}{5}\right)\right]$

Example 2:

Express the following as product of two functions.

(i) sin 5θ + sin θ **(ii) sin 4θ – sin 3 θ**

(iii) cos 4A + cos 2A **(iv) cos 10° – cos 50°**

Solution:

(i) $\sin 5\theta + \sin\theta = 2\sin\left(\frac{5\theta+\theta}{2}\right).\cos\left(\frac{5\theta-\theta}{2}\right)$

$= 2\sin 3\theta.\cos 2\theta$

(ii) $\sin 4\theta - \sin 3\theta = 2\cos\left(\frac{4\theta+3\theta}{2}\right).\sin\left(\frac{4\theta-3\theta}{2}\right)$

$= 2\cos\frac{7\theta}{2}.\sin\frac{\theta}{2}$

(iii) $\cos 4A + \cos 2A = 2\cos\left(\frac{4A+2A}{2}\right)\cos\left(\frac{4A-2A}{2}\right)$

$= 2\cos 3A.\cos A$

(iv) $\cos 10° - \cos 50° = -2\sin\left(\frac{10°+50°}{2}\right).\sin\left(\frac{10°-50°}{2}\right)$

$= -2.\sin 30°.\sin(-20°)$

$= -2\left(\frac{1}{2}\right)(-\sin 20°)$

$= \sin 20°$

Example 3:

Prove the following

$$\frac{\sin 4A + \sin 2A}{\sin 4A - \sin 2A} = \tan 3A.\cot A$$

Solution:

$$\text{LHS} = \frac{\sin 4A + \sin 2A}{\sin 4A - \sin 2A}$$

$$= \frac{\cancel{2}\sin\left(\frac{4A+2A}{2}\right).\cos\left(\frac{4A-2A}{2}\right)}{\cancel{2}\cos\left(\frac{4A+2A}{2}\right).\sin\left(\frac{4A-2A}{2}\right)}$$

$$= \frac{\sin 3A.\cos A}{\cos 3A.\sin A} = \tan 3A.\cot A = \text{RHS}$$

Example 4 :

P.T $\dfrac{\cos 2y - \cos 2x}{\sin 2x + \sin 2y} = \tan(x - y)$

Solution:

$$\text{LHS} = \frac{\cos 2y - \cos 2x}{\sin 2x + \sin 2y}$$

$$= \frac{\cancel{2}\sin\left(\frac{\cancel{2y} + 2x}{2}\right).\sin\left(\frac{2x - 2y}{2}\right)}{\cancel{2}\sin\left(\frac{2x + \cancel{2y}}{2}\right)\cos\left(\frac{2x - 2y}{2}\right)}$$

$$= \frac{\sin(x-y)}{\cos(x-y)} = \tan(x - y) = \text{RHS}$$

Example 5:

$$\frac{\cos 75° + \cos 15°}{\sin 75° - \sin 15°} = \sqrt{3}$$

Solution:

$$\text{LHS} = \frac{\cos 75° + \cos 15°}{\sin 75° - \sin 15°}$$

$$= \frac{\cancel{2}\cos\left(\frac{75+15}{\cancel{2}}\right).\cos\left(\frac{75-15}{2}\right)}{\cancel{2}\cos\left(\frac{75+15}{\cancel{2}}\right).\sin\left(\frac{75-15}{2}\right)}$$

$$= \frac{\cos 30}{\sin 30°} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} = \text{RHS}$$

Example 6:

$$\cos\left(\frac{\pi}{4}-\alpha\right)-\cos\left(\frac{\pi}{4}+\alpha\right)=\sqrt{2}\sin\alpha$$

Solution:

$$\text{LHS} = \cos\left(\frac{\pi}{4}-\alpha\right)-\cos\left(\frac{\pi}{4}+\alpha\right)$$

$$=-2\sin\left[\frac{\left(\frac{\pi}{4}-\alpha\right)+\left(\frac{\pi}{4}+\alpha\right)}{2}\right]\sin\left[\frac{\left(\frac{\pi}{4}-\alpha\right)-\left(\frac{\pi}{4}+\alpha\right)}{2}\right]$$

$$= -2\sin\frac{\pi}{4}.\sin(-\alpha)=2.\frac{1}{\sqrt{2}}\sin\alpha=\sqrt{2}\sin\alpha=\text{RHS}$$

Example 7:

cos 130° + cos 110° + cos 10° = 0

Solution:

LHS = (cos 130° + cos 10°) + cos 110°

$$= 2\cos\left(\frac{130^o+10^o}{2}\right).\cos\left(\frac{130^o-10^o}{2}\right)+\cos 110°$$

$$= 2.\cos 70°.\cos 60° + \cos 110°$$

$$= \cancel{2}.\cos 70°.\frac{1}{\cancel{2}}+\cos 110°$$

$$= \cos(180° - 110°) + \cos 110°$$

$$= -\cancel{\cos 110} + \cancel{\cos 110°} = 0 = \text{RHS}$$

Example 8:

4 sin A. sin (60° + A). sin (60° – A) = sin 3 A

Solution.

$$\text{LHS} = 4\sin A[\sin^2 60 - \sin^2 A]$$

$$= 4\sin A\left[\left(\frac{\sqrt{3}}{2}\right)^2-\sin^2 A\right] \qquad \left[\because \sin(A+B).\sin(A+B) = \sin^2 A-\sin^2 B\right]$$

$$= 4\sin A\left[\frac{3}{4}-\sin^2 A\right]$$

$$= 3\sin A - 4\sin^3 A = \sin 3A = \text{RHS}$$

Example 9:

P.T $\dfrac{\sin 6A + \sin 2A + 2\sin 4A}{\sin 7A + \sin 3A + 2\sin 5A} = \dfrac{\sin 4A}{\sin 5A}$

Solution:

$$\text{LHS} = \frac{(\sin 6A + \sin 2A) + 2\sin 4A}{(\sin 7A + \sin 3A) + 2\sin 5A}$$

$$= \frac{2\sin\left(\frac{6A+2A}{2}\right).\cos\left(\frac{6A-2A}{2}\right) + 2\sin 4A}{2\sin\left(\frac{7A+3A}{2}\right).\cos\left(\frac{7A-3A}{2}\right) + 2\sin 5A}$$

$$= \frac{2.\sin 4A.\cos 2A + 2\sin 4A}{2\sin 5A.\cos 2A + 2\sin 5A}$$

$$= \frac{\cancel{2}\sin 4A\cancel{(\cos 2A + 1)}}{\cancel{2}\sin 5A\cancel{(\cos 2A + 1)}}$$

$$= \frac{\sin 4A}{\sin 5A} = \text{RHS}$$

Example 10:

If $A + B + C = \pi$ **P.T** $\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right) + \tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right) + \tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right) = 1$

Solution.

Given: $A + B + C = \pi$

$$\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$$

$$\frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$$

$$\tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right)$$

$$\Rightarrow \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}.\tan\frac{B}{2}} = \cot\frac{C}{2} = \frac{1}{\tan\frac{C}{2}}$$

$$\Rightarrow \tan\frac{C}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\Rightarrow \tan\frac{A}{2}.\tan\frac{C}{2}+\tan\frac{B}{2}.\tan\frac{C}{2}=1-\tan\frac{A}{2}.\tan\frac{B}{2}$$

$$\Rightarrow \tan\frac{A}{2}.\tan\frac{B}{2}+\tan\frac{B}{2}.\tan\frac{C}{2}-\tan\frac{A}{2}.\tan\frac{C}{2}=1$$

Example 11:

$$\frac{\sin 5A+\sin 4A+\sin 2A+\sin A}{\cos 5A+\cos 4A+\cos 2A+\cos A}=\tan 3A$$

Solution:

$$\text{LHS}=\frac{(\sin 5A+\sin A)+(\sin 4A+\sin 2A)}{(\cos 5A+\cos A)+(\cos 4A+\cos 2A)}$$

$$=\frac{2\sin 3A.\cos 2A+2\sin 3A.\cos A}{2\cos 3A.\cos 2A+2\cos 3A.\cos A}\ \text{(By using transformation formula)}$$

$$=\frac{\cancel{2}\sin 3A\cancel{(\cos 2A+\cos A)}}{\cancel{2}\cos 3A\cancel{(\cos 2A+\cos A)}}$$

$$=\tan 3A=\text{RHS}$$

Example 12:

$$\sin 20^\circ.\sin 40^\circ.\sin 60^\circ.\sin 80^\circ=\frac{3}{16}$$

Solution:

$$\text{LHS}=(\sin 40^\circ.\sin 20^\circ).\frac{\sqrt{3}}{2}.\sin 80^\circ$$

$$=\frac{\sqrt{3}}{2}\left[-\frac{1}{2}(\cos 60^\circ-\cos 20^\circ)\right]\sin 80^\circ$$

$$=\frac{-\sqrt{3}}{4}[\cos 60^\circ.\sin 80^\circ]+\frac{\sqrt{3}}{4}[\cos 20^\circ.\sin 80^\circ]$$

$$=\frac{-\sqrt{3}}{4}.\frac{1}{2}\sin 80^\circ+\frac{\sqrt{3}}{4}.\frac{1}{2}[\sin(80^\circ+20^\circ)+\sin(80^\circ-20^\circ)]$$

$$= \frac{-\sqrt{3}}{8}.\sin 80^o + \frac{\sqrt{3}}{8}\left[\sin 100^o + \sin 60^o\right]$$

$$= \frac{-\sqrt{3}}{8}.\sin 80^o + \frac{\sqrt{3}}{8}.\sin 100^o + \frac{\sqrt{3}}{8}.\sin 60^o$$

$$= \frac{-\sqrt{3}}{8}.\sin 80^o + \frac{\sqrt{3}}{8}.\sin\left(180^o - 80^o\right) + \frac{\sqrt{3}}{8}.\frac{\sqrt{3}}{2}$$

$$= \frac{-\sqrt{3}}{8}.\cancel{\sin 80^o} + \frac{\sqrt{3}}{8}.\cancel{\sin 80^o} + \frac{3}{16}$$

$$= \frac{3}{16} = \text{RHS}$$

Example 13:

$$\cos 20^o.\cos 40^o.\cos 80^o = \frac{1}{8}$$

Solution:

$$\text{LHS} = (\cos 40^o.\cos 20^o)\cos 80^o$$

$$= \frac{1}{2}\left[\cos 60^o + \cos 20^o\right].\cos 80^o$$

$$= \frac{1}{2}\left[\frac{1}{2} + \cos 20^o\right]\cos 80^o$$

$$= \frac{1}{4}.\cos 80^\circ + \frac{1}{2}.\cos 80^\circ.\cos 20^\circ$$

$$= \frac{1}{4}.\cos 80^o + \frac{1}{2}.\frac{1}{2}\left[\cos 100^o + \cos 60^o\right]$$

$$= \frac{1}{4}.\cos 80^o + \frac{1}{4}.\cos 100^o + \frac{1}{4}.\cos 60^o$$

$$= \frac{1}{4}.\cancel{\cos 80^o} - \frac{1}{4}.\cancel{\cos 80^o} + \frac{1}{4}.\frac{1}{2} \qquad \left[\because \cos 100^o = \cos\left(180^o - 80^o\right) = -\cos 80^o\right]$$

$$= \frac{1}{8} = \text{RHS}$$

Example 14:

S.T $\cos^2\theta + \cos^2(60^o + \theta) + \cos^2(60^o - \theta) = \frac{3}{2}$

Solution:

$$\text{LHS} = \cos^2\theta + \cos^2(60^o + \theta) + \cos^2(60^o - 2)$$

$$= \frac{1+\cos 2\theta}{2} + \frac{1+\cos 2(60+\theta)}{2} + \frac{1+\cos 2(60-\theta)}{2}$$

$$= \frac{1}{2}\left[1+\cos 2\theta + 1 + \cos(120^o + 2\theta) + 1 + \cos(120^o - 2\theta)\right]$$

$$= \frac{1}{2}\left[3 + \cos 2\theta + \cos(120 + 2\theta) + \cos(120^o - 2\theta)\right]$$

$$= \frac{1}{2}\left[3 + \cos 2\theta + 2\cos 120^o.\cos 2\theta\right]$$

$$= \frac{1}{2}\left[3 + \cos 2\theta + \cancel{2}\,\frac{-1}{\cancel{2}}\cos 2\theta\right]$$

$$= \frac{1}{2}\left[3 + \cancel{\cos 2\theta} - \cancel{\cos 2\theta}\right] = \frac{3}{2} = \text{RHS}$$

Example 15:

In any ΔABC P.T sin2A + sin2B – sin2C=4cosA.cosB.sinC

Solution:

LHS = sin2A + sin2B – sin2C

= 2sin(A + B).cos (A – B) – sin 2C [$\because$ sin 2C = 2sinC.cosC and sin (A + B) = sin C]

= 2 sin C cos (A – B) – 2 sinC.cosC

= 2 sin C [cos (A – B) – cos C]

= 2 sin C [cos (A – B) + cos (A + B)] [$\because$ cos (A + B) = –cosC]

= 2 sin C. 2 cos A. cos B

= 4 cos A. cos B. sin C

= RHS

Example 16:

If A + B + C = 180°

P.T cos 2A + cos 2B + cos 2C = –1 –4 cos A.cos B.cos C

Solution:

LHS = cos 2A + cos 2 B + cos2C

$$= 2\cos\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)+\cos 2C$$

$$= 2\cos(A+B).\cos(A-B)+\cos 2C$$

$$= -2\cos C.\cos(A-B)+2\cos^2 C-1$$

$$= -1-2\cos C.\cos(A-B)+2\cos^2 C$$

$$= -1-2\cos C\,[\cos(A-B)-\cos C]$$

$$= -1-2\cos C\,[\cos(A-B)+\cos(A+B)]$$

$$= -1-2\cos C\,[2\cos A\cos B]$$

$$= -1-4\cos A.\cos B.\cos C$$

= RHS

Example 17:

$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A\cos B\cos C$

Solution:

LHS = $\sin^2 A + \sin^2 B + \sin^2 C$

$$= \frac{1-\cos 2A}{2}+\frac{1-\cos 2B}{2}+\frac{1-\cos 2C}{2}$$

$$= \frac{1}{2}\left[3-(\cos 2A+\cos 2B+\cos 2C)\right]$$

(Using the result of example 16)

$$= \frac{1}{2}\left[3-(-1-4\cos A\cos B.\cos C)\right]$$

$$= \frac{1}{2}\left[3+1+4\cos A\cos B\cos C\right]$$

$$= \frac{1}{2}\left[4+4\cos A\cos B\cos C\right]$$

= 2 + 2 cosA cosB cos C = RHS

Example 18:

$\cos^2A + \cos^2B + \cos^2C = 1 - 2\cos A \cos B \cos C$

Solution:

$$\text{LHS} = \cos^2A + \cos^2B + \cos^2C$$

$$= \frac{1+\cos 2A}{2} + \frac{1+\cos 2B}{2} + \frac{1+\cos 2C}{2}$$

$$= \frac{1}{2}[3 + \cos 2A + \cos 2B + \cos 2C]$$

$$\left[\begin{array}{l}\text{since} \\ \cos 2A + \cos 2B + \cos 2C \\ = -1 - 4\cos A.\cos B.\cos C\end{array}\right]$$

$$= \frac{1}{2}[3 - 1 - 4\cos A.\cos B.\cos C]$$

$$= \frac{1}{2}[2 - 4\cos A\cos B\cos C]$$

$$= 1 - 2\cos A\cos B\cos C$$

$$= \text{RHS}$$

EXERCISE 14.3

One marks questions:

I. 1. Express each of the following as sum or difference of two trigonometric functions.

i. $\sin 5A.\cos 3A$ ii. $\cos 4A \sin 2A$

iii. $\cos\left(\frac{5\theta}{2}\right).\sin\left(\frac{\theta}{2}\right)$ iv. $2\cos 70° \cos 10°$

2. Express each of the following as the product of two trigonometric ractions.

i. $\sin 12x + \sin 4x$ ii. $\sin 7A - \sin 3A$

iii. $\cos 2\theta + \cos 6\theta$ iv. $\sin 80° - \sin 40°$

Two marks questions :

II. Prove the following:

1. $\frac{\cos 2A - \cos 12A}{\sin 12A - \sin 2A} = \tan 7A$

2. $\frac{\sin x - \sin y}{\sin x + \sin y} = \tan\left(\frac{x-y}{2}\right).\cot\left(\frac{x+y}{2}\right)$

3. $\frac{\sin 2\alpha + \sin 3\alpha}{\cos 2\alpha - \cos 3\alpha} = \cot\left(\frac{\alpha}{2}\right)$

4. $\sin\left(\frac{\pi}{3} + A\right) - \sin\left(\frac{\pi}{3} - A\right) = \sin A$

5. $\cos A + \cos(120 - A) + \cos(120 + A) = 0$

6. $\sin 65° + \cos 65° = \sqrt{2}\cos 20°$

7. If $A + B + C = \pi$ P.T. $\tan A + \tan B + \tan C = \tan A.\ \tan B\ .\tan C$

8. P.T $\frac{\sin 2A + \sin 5A - \sin A}{\cos 2A + \cos 5A + \cos A} = \tan 2A$

9. If $A + B + C = 180°$. P.T $\cot B.\cot C + \cot C.\ \cot A + \cot A \times \cot B = 1$.

10. IF $A + B + C = \frac{\pi}{2}$. P.T $\tan A.\tan B + \tan B.\tan C + \tan C.\ \tan A = 1$

Five marks questions:

III. Prove the following

1. $\frac{\cos 7x + \cos 3x - \cos 5x - \cos x}{\sin 7x - \sin 3x - \sin 5x + \sin x} = \cot 2x$

2. $\cos 10°.\ \cos 30°.\ \cos 50°.\ \cos 70° = \frac{3}{16}$

3. $\cos 20.\ \cos 40.\ \cos 60.\ \cos 80 = \frac{1}{16}$

If $A + B + C = 180°$, prove that

4. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A.\ \sin B.\ \sin C$

5. $\tan 2A + \tan 2B + \tan 2C = \tan 2A.\ \tan 2B.\ \tan 2C$

6. $\sin 4A + \sin 4B + \sin 4C = -4\sin 2A.\ \sin 2B.\ \sin 2C$

7. $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A.\ \sin B.\ \sin C$

8. $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin 2A + \sin 2B - \sin 2C} = \tan A.\tan B$

9. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$.

10. $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A . \sin B . \cos C$

11. If $A + B + C = \pi$. Prove that:

    $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A . \sin B . \cos C$

12. If $A + B + C = 180°$. Prove that:

    $\sin 2A - \sin 2B + \sin 2C = 4 \cos A . \sin B . \cos C$

13. $\cos 2A - \cos 2B + \cos 2C = 1 - 4 \sin A . \cos B . \sin C$

Answers 14.3

I. 1. $\frac{1}{2}(\sin 8A + \sin 2A)$ 2. $\frac{1}{2}(\sin 6A - \sin 2A)$

3. $\frac{1}{2}[\sin 3\theta - \sin 2\theta]$ 4. $\cos 80° + \frac{1}{2}$

II. 1. $2 \sin 8x . \cos 4x$ 2. $2 \cos 5A . \sin 2A$

3. $2 \cos 4\theta . \cos 2\theta$ 4. $\sin 20°$

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UNIT IV - ANALYTICAL GEOMETRY

Chapter	Title	No. of Teaching hrs.
15.	CIRCLES	06 hrs
16.	PARABOLA	04 hrs
	TOTAL TEACHING HOURS	**10 hrs**

Chapter

15

CIRCLES

15.1 Introduction :

Of all the curves, the circle is the simplest one and occurs most frequently in nature, architecture, science and industry. We see applications of circles everywhere. Various objects round (circular) in shape are used by us in our daily life. Few examples are coins, wheels, compact disc etc. Therefore it is necessary to pay attention to the study of circles.

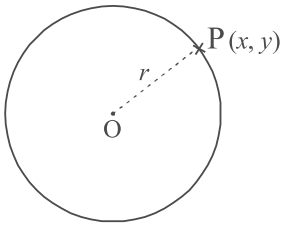
15.2 Definition :

A circle is the locus of a point which moves in a plane so that its distance from a fixed point always remain constant. This fixed point is called the **centre** of the circle and the constant distance is called the **radius** of the circle.

Equation of the circle in different forms:

(a) Equation of a circle whose centre is the origin and the radius being '*r*'

Consider a circle with centre at the origin O(0,0) and radius '*r*'. Let P(x,y) be a point on the circle. Join OP, by given data OP = r



Using distance formula

$$OP = \sqrt{(x-0)^2 + (y-0)^2}$$

$$r = \sqrt{x^2 + y^2}$$

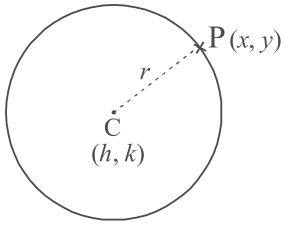
$$\boxed{r^2 = x^2 + y^2}$$ which is the required equation.

Example 1 : The equation of the circle with centre at the origin and radius 5 unit is given by

$x^2 + y^2 = 5^2$ i.e. $x^2 + y^2 = 25$

(b) Equation of the circle whose centre is (h,k) and radius being 'r'

Consider a point p(x,y) on the circle. Let the centre of the circle be $c(h,k)$ Join CP which is by definition equal to '*r*'.



By distance formula

$$CP = r = \sqrt{(x-h)^2 + (y-k)^2}$$

i.e. $$\boxed{r^2 = (x-h)^2 + (y-k)^2}$$

This is the equation of the circle in standard form with centre (h,k) and radius r

Note : In particular if $h = 0$ and $k = 0$ then the equation of the circle becomes $x^2 + y^2 = r^2$

Worked Examples

Example 1:

Find the equation of the circle with centre (–2, –1) and the radius is 2.

Solution:

Given $(h,k) = (-2,-1)$ and $r = 2$

we have the equation of the circle given by

$$r^2 = (x-h)^2 + (y-k)^2$$

$$2^2 = (x-(-2))^2 + (y-(-1))^2$$

$$4 = (x+2)^2 + (y+1)^2$$

Simplifying the above equation we get the equation of the circle as

$$x^2 + y^2 + 4x + 2y + 1 = 0$$

Example 2:

Find the equation of the circle with the centre as (3, –4) and radius is 5 units. Also show that the circle passes through origin.

Solution:

Given (h,k) = (3, –4) and r = 5

we have the equation of the circle given by

$$r^2 = (x-h)^2 + (y-k)^2$$

$$5^2 = (x-3)^2 + (y-(-4))^2$$

$$25 = (x-3)^2 + (y+4)^2$$

Simplifying the above equation we get the equation of the circle as

$$x^2 + y^2 - 6x + 8y = 0$$

The co-ordinates of the origin i.e., $x = 0$ and $y = 0$ satisfies the above equation and so the circle passes through the origin.

Example 3:

Find the equation of the circle whose centre is (2, –3) and passes through the point of intersection of the lines $3x - 2y = 1$ and $4x + y = 27$.

Solution:

$$3x - 2y = 1 \quad \ldots(1)$$

$$4x + y = 27 \quad \ldots(2)$$

Solving (1) and (2) we get $x = 5$ and $y = 7$

point of intersection is $(x,y) = (5,7)$. If C $(2, -3)$ and P (x, y)

Using distance formula

$\therefore \quad CP = \sqrt{(5-2)^2 + (7+3)^2} = \sqrt{109}$ = Radius

Hence the equation of the circle is

$$(x-h)^2 + (y-k) = r^2$$

$$(x-2)^2 + (y+3)^2 = \left(\sqrt{109}\right)^2$$

$$x^2 + y^2 - 4x + 6y - 96 = 0$$

Example 4:

Find the equation of the circle with two of whose diameters $x + y = 6$ and $x + 2y = 4$ having radius $2\sqrt{5}$ units.

Solution:

Solving $x + y = 6$ and $x + 2y = 4$ we get $y = -2$ and $x = 8$.

The centre of the circle is the meeting point of the two diameters which is given by $(8, -2)$

Given $r = 2\sqrt{5}$

Equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$

i.e. $(x-8)^2 + (y+2)^2 = \left(2\sqrt{5}\right)^2$

$$x^2 + y^2 - 16x + 4y + 48 = 0$$

Example 5:

If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along the diameters of a circle of circumference 10π, find the equation of the circle

Solution :

Solving $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ we get $x = 1, y = -1$ $\therefore$ centre $(1, -1)$

Circumference of the circle $= 2\pi r = 10\pi$

$\therefore r = 5$

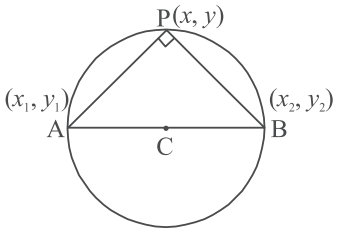
Thus the equation of the circle is given by

$$(x-1)^2 + (y+1)^2 = 5^2$$

i.e. $x^2 + y^2 - 2x + 2y - 23 = 0$ is the required equation.

(c) Equation of the circle with A(x_1, y_1) and B(x_2, y_2) as the co-ordinates of the end points of a diameter

Let $p(x,y)$ be a point on the circle. Join PA and PB.



We find $A\hat{P}B = 90^o$ (Angle in a semi circle)

$\therefore$ slope of AP × slope PB = –1 $(\because AP \perp BP)$

i.e. $\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right) = -1$

$(y-y_1)(y-y_2) = -1(x-x_1)(x-x_2)$

i.e. $\boxed{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0}$ is the required equation of the circle.

Example 6:

Find the equation of the circle with (–4, 3) and (12, –1) as the extremities of a diameter.

Solution :

The equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

$$(x+4)(x-12)+(y-3)(y+1)=0$$

i.e. $x^2+y^2-8x-2y-51=0$

(d) Equation of the point circle

The circle whose **radius is zero** is called as the **point circle.**

Note: Consider the equation $r^2=(x-h)^2+(y-k)^2$ where centre is (h,k) and radius is r.

(i) If $r>0$, we say it is a real circle.

(ii) If $r<0$, then the circle is called as an **imaginary circle or virtual** circle.

(iii) If $r=0$ then the circle is said to be a **point circle**

(iv) If $r=1$, then the circle is said to be a **unit circle**

Example 7:

Write the equation of the point circle with centre at (3, –5)

Solution :

Centre $(h,k) = (3,-5)$

Since it is a point circle, radius $r=0$

The equation of the point circle is given by

$$r^2=(x-h)^2+(y-k)^2$$

$$0=(x-3)^2+(y+5)^2$$

i.e., $x^2+y^2-6x+10y+34=0$

Exercise 15.1

One mark questions:

I. Find the equation of the circle with centre 'C' and radius 'r' in each of the following:

a) $c\,(-3,2)$ and $r = 5$ units

b) $c\,(0,0)$ and $r = 4$ units

c) $c\,(-1,-2)$ and diameter $r = 25$ units

d) $c\,(1,1)$ and $r = \sqrt{2}$ units

e) $c(a\cos\theta, a\sin\theta)$ and $r = a$ units

II. Find the centre of the circle, two of the diameters are

a) $x + y = 2,\ x - y = 0$

b) $2x - 3y = 1,\ 3x - 2y = 2$

c) $y = 0$ and $y = x - 5$

III. Write the equation of the point circle with centre at (a) $(4, -5)$ (b) $(-3, 2)$ (c) $(1,0)$

Two marks questions:

IV. Find the equation of the circle.

a) two of the diameters are $x + y = 6$ and $x + 2y = 4$ and its radius is 10 units.

b) two of the diameters $x + y = 4$ and $x - y = 2$ and passing through the point $(2, -1)$

c) two of the diameters are $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 sq.cm

V. a) Find the equation of the circle with centre at $(-2, 1)$ and passing through the origin.

b) Find the equation of the circle with centre at $(2,1)$ and passing through $(0, -1)$

VI. Find the equation of the circle described on the line joining the points A and B as diameter where

a) A$(-5, 1)$ and B$(1,3)$

b) A$(2,0)$ and B$(0,2)$

c) A$(3,4)$ and B$(1,-2)$

Answers 15.1

One mark questions:

I. a) $x^2 + y^2 + 6x - 4y = 12$

b) $x^2 + y^2 = 16$

c) $4x^2 + 4y^2 + 8x + 16y = 605$

d) $x^2 + y^2 - 2x - 2y = 0$

e) $x^2 + y^2 - (2a\cos\theta)x - (2a\sin\theta)y = 0$

II. a) centre $(1,1)$ b) centre $\left(\frac{4}{5}, \frac{1}{5}\right)$ c) centre $(5,0)$

III. a) $x^2 + y^2 - 8x + 10y + 41 = 0$

b) $x^2 + y^2 + 6x - 4y + 13 = 0$

c) $x^2 + y^2 - 2x + 1 = 0$

Two marks questions:

IV. a) $x^2 + y^2 - 16x + 4y - 32 = 0$ b) $x^2 + y^2 - 6x - 2y + 5 = 0$

c) $x^2 + y^2 - 2x + 2y - 47 = 0$

V. a) $x^2 + y^2 + 4x - 2y = 0$ b) $x^2 + y^2 - 4x - 2y = 3$

VI. a) $x^2 + y^2 + 4x - 4y = 2$ b) $x^2 + y^2 - 2x - 2y = 0$

c) $x^2 + y^2 - 4x - 2y = 5$

15.3 General equation of the circle

Show that the second degree equation $x^2 + y^2 + 2gx + 2fy + c = 0$ always represents a circle

Proof: Given $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(1)

i.e. $x^2 + y^2 + 2gx + 2fy = -c$

Adding $g^2 + f^2$ on both sides of the above equation we get

$$x^2 + y^2 + 2gx + 2fy + g^2 + f^2 = g^2 + f^2 - c$$

$$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c$$

$$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$$

$$\Rightarrow (x--g)^2 + (y--f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

which is of the form $(x-h)^2 + (y-k)^2 = r^2$

Hence the given equation (1) always represent a circle where centre $c(h,k) = (-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$

Definition : The equation of the circle in the form $x^2 + y^2 + 2gx + 2fy + c = 0$ is called the standard form of the equation of the circle.

Consider the general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. The equation represents a circle if $a = b$ and $h = 0$ i.e. the equation $ax^2 + ay^2 + 2gx + 2fy + c = 0$ always represent a circle. This equation is called the general equation of the circle. In particular if $a = b = 1$, we get the standard form of the equation of the circle. Which is given by $x^2 + y^2 + 2gx + 2fy + c = 0$, the centre of the circle is $(-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$

Note:

* x co-ordinate of the centre of the circle $= \dfrac{-(\text{coefficient of } x)}{2}$

* y coordinate of the centre of the circle $= \dfrac{-(\text{coefficient of } y)}{2}$

WORKED EXAMPLES

Example 1:

Find the centre and radius of the following circles.

(a) $x^2 + y^2 - 2x + 6y = 20$ **(b)** $3x^2 + 3y^2 - 6x + 4y + 1 = 0$

(c) $2x^2 + 2y^2 + 6x - 10y + 9 = 0$

Solution :

(a) The equation of the circle is given by

$$x^2 + y^2 - 2x + 6y - 20 = 0$$

Here $2g = -2,\ 2f = 6$ and $c = -20$

$\Rightarrow\ g = -1,\ f = 3$ and $c = -20$

$$c = (-g, -f) = (1, -3), \textit{radius} = \sqrt{g^2 + f^2 - c} = \sqrt{1 + 9 + 20} = \sqrt{30}$$

(b) The equation of the circle is given by $3x^2 + 3y^2 - 6x + 4y + 1 = 0$

In this equation, the coefficient of x^2 and y^2 are not 1. Thus, we have to divide, throughout by 3.

we get, $x^2 + y^2 - 2x + \dfrac{4}{3}y + \dfrac{1}{3} = 0$

Here $2g = -2,\ 2f = \dfrac{4}{3},\ c = \dfrac{1}{3}$

$\Rightarrow\ g = -1,\ f = \dfrac{2}{3},\ c = \dfrac{1}{3}$

centre $C(-g, -f) = \left(1, \dfrac{-2}{3}\right)$

radius $= \sqrt{g^2 + f^2 - c} = \sqrt{1 + \dfrac{4}{9} - \dfrac{1}{3}} = \dfrac{\sqrt{10}}{3}$

(c) The equation of the given circle is

$2x^2 + 2y^2 + 6x - 10y + 9 = 0$

Here also we have to make the coefficient of x^2 and y^2 as 1. To this, we shall divide throughout by 2. We get

$$x^2 + y^2 + 3x - 5y + \frac{9}{2} = 0$$

Here $2g = 3,\ 2f = -5,\ c = \frac{9}{2}$

$$\Rightarrow\ g = \frac{3}{2},\ f = \frac{-5}{2},\ c = \frac{9}{2}$$

$\therefore$ centre $c = (-g, -f) = \left(\frac{-3}{2}, \frac{5}{2}\right)$

and radius $r = \sqrt{g^2 + f^2 - c} = \sqrt{\frac{9}{4} + \frac{25}{4} - \frac{9}{2}} = 2$ units

Example 2:

If the radius of the circle $x^2 + y^2 - 2x + 3y + k = 0$ is $\frac{5}{2}$, find k.

Solution:

The given equation is $x^2 + y^2 - 2x + 3y + k = 0$

By data $r = \frac{5}{2} = \sqrt{g^2 + f^2 - c}$, $\quad 2g = -2 \quad \therefore\ g = -1$

i.e. $\left(\frac{5}{2}\right)^2 = (+1)^2 + \left(\frac{-3}{2}\right)^2 - (+k) \quad 2f = 3 \quad \therefore\ f = \frac{3}{2},\ c = k$

$$\frac{25}{4} = 1 + \frac{9}{4} - k$$

simplifying, $k = -3$

Example 3:

If one end of the diameter of the circle $x^2 + y^2 + 2x + 6y - 22 = 0$ is (3, –7), find the co-ordinate of the other end.

Solution:

The equation of the circle is $x^2 + y^2 + 2x + 6y - 22 = 0$

Now centre $C = (-g, -f) = C(-1, -3)$

One end of the diameter is A(3, –7)

Let the other end of the diameter be B(x,y)

$$C = \text{Midpoint of } AB \Rightarrow (-1,-3) = \left(\frac{3+x}{2}, \frac{-7+y}{2}\right)$$

$$-1 = \frac{3+x}{2}, \quad -3 = \frac{-7+y}{2} = -2 = 3+x, \quad -6 = -7+y$$

$$x = -5 \text{ and } y = 1$$

$\therefore$ the other end of the diameter is (–5,1)

Example 4:

If one end of the diameter of the circle $x^2 + y^2 - 4x - 2y - 5 = 0$ is (3,4), find co-ordinate of the other end.

Solution:

The equation of the circle is $x^2 + y^2 - 4x - 2y - 5 = 0$

Now, centre $C = (-g, -f) = (2,1)$. One end of the diameter is A = (3,4). Let the other end be B = (h,k)

$$\therefore C = \text{midpoint of } AB \Rightarrow (2,1) = \left(\frac{3+h}{2}, \frac{4+k}{2}\right) \text{ by equating we get}$$

$$\Rightarrow 2 = \frac{3+h}{2}, \; 1 = \frac{4+k}{2}$$

$$\Rightarrow 4 = 3+h, \; 2 = 4+k \Rightarrow h = 1, \; k = -2$$

$\therefore$ the other end of the diameter is (1, –2)

Example 5:

If $x^2 + y^2 - 14x + by + 53 = 0$ represents a point circle, find 'b'

Solution:

By data $x^2 + y^2 - 14x + by + 53 = 0$ represents a point circle. Thus, the radius = 0

$$\Rightarrow \sqrt{g^2 + f^2 - c} = 0 \rightarrow \sqrt{(-7)^2 + \left(\frac{b}{2}\right)^2 - 53} = 0$$

$$\begin{cases} 2g = -14 \therefore g = -7 \\ 2f = b \therefore f = \dfrac{b}{2} \\ c = 53 \end{cases}$$

$$\Rightarrow 49 + \frac{b^2}{4} - 53 = 0$$

$$\Rightarrow \frac{b^2}{4} = 4 \rightarrow b^2 = 16 \rightarrow b = \pm 4$$

Note : Consider the equation of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad .....(1)$$

Any circle concentric with this circle, will have the same centre. Thus, in its equation the coefficiants of x^2 and y^2 will be same as that of (1). Hence we have the following:

The equation of the circle concentric with the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$x^2 + y^2 + 2gx + 2fy + k = 0$.

For different values of k, we get different circles concentric with the given circle.

Example 6:

Find the equation of the circle concentric with the circle $x^2 + y^2 - 6x - 2y + 4 = 0$ and passing through the point (3,2).

Solution:

The equation of the circle concentric with

$x^2 + y^2 - 6x - 2y + 4 = 0$ is of the form

$x^2 + y^2 - 6x - 2y + k = 0$

By data this must pass through (3,2). Thus the equation (1) must satisfy

$$x = 3, y = 2 \rightarrow 9 + 4 - 18 - 4 + k = 0 \rightarrow k = 9$$

Hence, the equation of the required circle is $x^2 + y^2 - 6x - 2y + 9 = 0$

Example 7:

Find the equation of the circle whose centre is (–3,1) and passing through the centre of the circle $x^2 + y^2 + 2x - 4y + 4 = 0$.

Solution:

Let C = (–3,1). The centre of the given circle $x^2 + y^2 + 2x - 4y + 4 = 0$ is A = (–1,2)

$(\because \ 2g = 2, \ 2f = -4)$

The required circle passes through A. Thus,

$$\text{radius} = \text{CA} \rightarrow \sqrt{(-3+1)^2 + (1-2)^2} = \sqrt{5}$$

Now, the equation of the circle with centre (–3, 1) and radius $\sqrt{5}$ units is given by

$$(x+3)^2 + (y-1)^2 = \left(\sqrt{5}\right)^2 \Rightarrow x^2 + y^2 + 6x - 2y + 5 = 0$$

2nd method

By data the centre of the required circle is (–3, 1), $\therefore -g = -3$ and $-f = 1 \rightarrow g = 3, f = -1$

Thus, the required circle is of the form $x^2 + y^2 + 6x - 2y + c = 0$

This must pass through the centre of $x^2 + y^2 + 2x - 4y + 4 = 0$, i.e., it must pass through $(-1, 2) \rightarrow 1 + 4 - 6 - 4 + c = 0 \rightarrow c = 5$. Putting c = 5, we get the required equation $x^2 + y^2 + 6x - 2y + 5 = 0$

Note: The standard form of the equation of the circle has three parameters *g, f* and *c*. Thus to find the equation of a circle, we have to find corresponding values of g,f, and c. Hence, we require at least three conditions to determine the equation of a circle.

Example 8:

Find the equation of the circle passing through the points (0,0) and (1,1) and has its centre on *x*-axis.

Solution:

Let the equation of the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ .....(1)

Using the conditions in the problem, we shall determine the values of *g, f* and *c*

By data the circle passes through the point (0,0). Thus the equation (1) must satisfy $x = 0$ and $y = 0$

$$\Rightarrow \quad 0 + 0 + 2g(0) + 2f(0) + c = 0 \quad \Rightarrow \quad c = 0$$

Again by data the circle passes through the point (1,1). Thus the equation (1) must satisfy $x = 1, \ y = 1$

$$1 + 1 + 2g + 2f + c = 0 \Rightarrow 2g + 2f + 2 = 0 \qquad (\because \ c = 0)$$

$$\Rightarrow g + f + 1 = 0 \ .....(2)$$

Again by data, the centre $(-g, -f)$ of (1) lies on x - axis

$\Rightarrow$ y - coordinate of the centre = 0 $\Rightarrow$ $f = 0$

putting $f = 0$ in (2), we get $g + 1 = 0 \Rightarrow g = -1$

putting g = 1, $f = 0$, c = 0 in (1) we get the equation of the circle $x^2 + y^2 - 2x = 0$

Note: When the circle passes through the origin 0(0,0) then the standard form of the equation of the circle is given by $x^2 + y^2 + 2gx + 2fy = 0$

Example 9:

Find the equation of the circle passing through the points (–1, 2) and (3, –2), and has its centre on $x = 2y$.

Solution:

Let the equation of the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ .....(1)

By data this circle passes through the point (–1, 2) and (3, –2)

$\Rightarrow \quad -2g + 4f + c + 5 = 0$ .....(2)

and $6g - 4f + c + 13 = 0$ .....(3)

Again by data the centre (–g, –f) lies on $x = 2y$

$\Rightarrow \quad -g = -2f \quad \Rightarrow \quad g = +2f$

Solving the equations (1), (2) and (3) we get g, f and c

consider (2) - (3) we get $-8g + 8f - 8 = 0 \Rightarrow g - f + 1 = 0$

putting $g = 2f$, we get $2f - f + 1 = 0 \Rightarrow f = -1$

Now, $g = 2f \quad \Rightarrow \quad g = -2$

From (2) we have , $c = 2g - 4f - 5 \quad \Rightarrow \quad c = 4 + 4 - 5 \Rightarrow c = -5$

Thus the equation of the required circle is $x^2 + y^2 - 4x - 2y - 5 = 0$

Example 10:

Find the equation of the circle passing through the points (5,3), (1,5) and (3, –1).

Solution:

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(1)

By data this passes through the point

(5, 3), (1,5) and (3, –1) Thus we have

$10g + 6f + c + 34 = 0$ .....(2)

$2g + 10f + c + 26 = 0$ .....(3)

$6g - 2f + c + 10 = 0$ .....(4)

Solving these three equation we get f, g and c. To this end consider,

$(2) - (3) \rightarrow 8g - 4f + 8 = 0 \rightarrow 2g - f + 2 = 0$ .....(5)

$(3) - (4) \rightarrow -4g + 12f + 16 = 0 \rightarrow g - 3f - 4 = 0$ .....(6)

Solving (5) and (6), we get $g = -2, f = -2$

Now, from (1) $\Rightarrow \quad c = -10g - 6f - 34 \Rightarrow c = 20 + 12 - 34 \Rightarrow c = -2$

Thus, the equation of the required circle is $x^2 + y^2 - 4x - 4y - 2 = 0$

Example 11:

Show that the following points (2,0), (–1, 3), (–2, 0) and (1, –1) are concyclic.

Solution:

We shall find the equation of the circle passing through the points (2, 0), (–1, 3) and (–2, 0) and then we shall show that the fourth point (1, –1) lies on it.

Let the equation of the circle passing through (2, 0), (–1, 3) and (–2, 0) be

$x^2 + y^2 + 2gx + 2fy + c = 0$ .....(1)

Since it passes through (2,0), (–1,3) and (–2, 0)

we have $4g + c + 4 = 0$ .....(2)

$-2g + 6f + c + 10 = 0$ .....(3)

$-4g + c + 4 = 0$ .....(4)

Solving these equations we get f, g and c.

consider, (2) - (4) we get, $8g = 8g = 0 \rightarrow g = 0$

putting $g = 0$ in (2) we get, $c = -4$

Now $(3) \Rightarrow 6f = 2g - c - 10 \rightarrow 6f = +4 - 10 \Rightarrow f = -1$

Thus, the equation of the circle is $\boxed{x^2 + y^2 - 2y - 4 = 0}$

Now, putting $x = 1$ and $y = -1$ in LHS of (1)

we get, 1 + 1 + 2 – 4 = 4 – 4 = 0 = RHS

Thus, the point (1, –1) lies on the circle.

Hence, the given four points are concyclic

Example 12:

Find the equation of the diameter of the circle $x^2 + y^2 + 2x + 4y = 4$ which when produced passes through the point (–6, 2).

Solution : Equation of the circle is $x^2 + y^2 + 2x + 4y = 4$

centre C = (–1, –2)

Required diameter is the line joining the points C(–1, –2) and the point (–6, 2)

Its equation is $\frac{y+2}{x+1} = \frac{2+2}{-6+1}$

$$\frac{y+2}{x+1} = \frac{4}{-5} \quad \Rightarrow \quad -5y - 10 = 4x + 4$$

$$\Rightarrow \quad 4x + 5y + 14 = 0$$

Example 13:

Show that the line $2x+y=4$ passes through the centre of the circle $x^2+y^2-3x-2y+2=0$

Solution:

The centre of the circle $x^2+y^2-3x-2y+2=0$ is $C=\left(\frac{3}{2},1\right)$. Put $x=\frac{3}{2}$, $y=1$ in the left hand side of $2x+y=4$, we get $2\left(\frac{3}{2}\right)+1=3+1=4=\text{RHS}$

Thus $\left(\frac{3}{2},1\right)$ lies on $2x+y=4$

15.4 Length of the chord of the circle $x^2+y^2+2gx+2fy+c=0$ intercepted by the co-ordinate axes.

(a) To find the length of the chord of the circle $x^2+y^2+2gx+2fy+c=0$ intercepted by the x axis

Solution:

The equation of the circle is $x^2+y^2+2gx+2fy+c=0$. Let the circle cut the x axis at A and B.

Draw perpendicular CM from the centre onto B. Clearly M is the midpoint of AB.

Now x intercept 2 AM

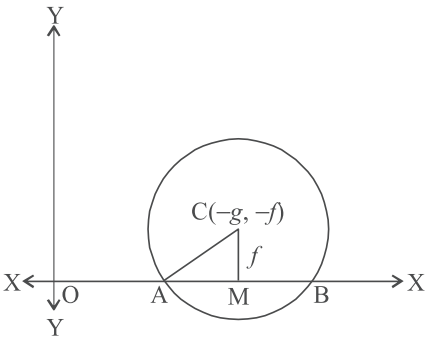
x intercept $= 2\sqrt{AC^2-CM^2}$

$[\because$ AMC is a right triangle$]$

x intercept $= 2\sqrt{g^2+f^2-c-f^2}$

$[\because$ AC = radius and CM $= -f]$

$\boxed{x\text{ intercept} = 2\sqrt{g^2-c}}$



(b) To find the length of the chord of the circle $x^2+y^2+2gx+2fy+c=0$ intercepted by the y axis

Solution:

Let the circle cut the y axis at D and E. Draw perpendicular CN from the centre on DE clearly N is the midpoint of DE

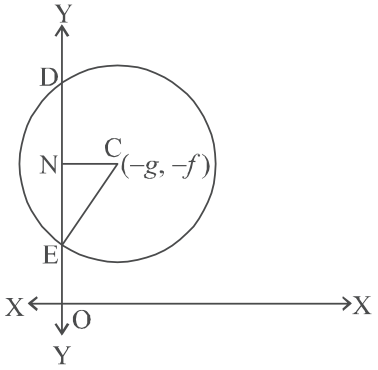
Now y intercept = 2 EN

y intercept = $2\sqrt{CE^2 - CN^2}$

y intercept = $2\sqrt{g^2 + f^2 - c - g^2}$

$(\because CE = \text{radius}, CN = g)$

$\boxed{y \text{ intercept } 2\sqrt{f^2 - c}}$



Example 14:

Find the length of the chord of the circle $x^2 + y^2 - 6x + 4y + 5 = 0$ intercepted by x axis.

Solution:

The equation of the circle $x^2 + y^2 - 6x + 4y + 5 = 0$

x intercept = $2\sqrt{g^2 - c} = 2\sqrt{9 - 5} = 2\sqrt{4} = 4$

Example 15:

Find the length of the chord of the circle $x^2 + y^2 + 3x - 2 = 0$ intercepted by y axis

Solution:

The equation of the circle is

$x^2 + y^2 + 3x - 2 = 0$

y intercept = $2\sqrt{f^2 - c} = 2\sqrt{0 + 2} = 2\sqrt{2}$

Example 16:

Find the equation of the circle passing through the origin and making positive intercepts 3 and 5 on the co-ordinate axes.

Solution:

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

The circle passes through the origin $\Rightarrow$ C = (0, 0)

By data x intercept = 3

i.e. $2\sqrt{g^2 - c} = 3$

$2\sqrt{g^2} = 3 \Rightarrow -2g = 3 \Rightarrow g = \frac{-3}{2}$

we take negative sign, as the centre must lie in the first quadrant.

$$2\sqrt{f^2 - c} = 5$$

$$2\sqrt{f^2} = 5$$

$$-2f = 5 \Rightarrow f = \frac{-5}{2}$$

we take negative sign, as the centre must lie in the first quadrant.

Thus the equation of the circle is $x^2 + y^2 - 3x - 5y = 0$

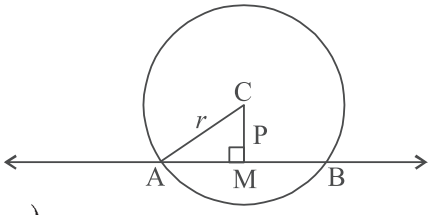
Note: To find the length of the chord intercepted by a line with the given circle whose radius is '*r*' and the length of the perpendicular from the centre on to the line.

Solution:

Let the given line cut the circle at A and B. Clearly the length of the chord is AB

Let C be the centre of the circle.

Draw CM perpendicular to the line AB



Now AB = 2 AM

$$= 2\sqrt{CA^2 - CM^2}$$

$$AB = 2\sqrt{r^2 - p^2} \qquad (\because CA = r, CM = p)$$

Thus length of the chord = $2\sqrt{r^2 - p^2}$

Example 17 : Find the length of the chord intercepted by the circle, $x^2 + y^2 + 4x + 6y = 12$ and $x + 4y = 6$

Solution: The equation of the circle is $x^2 + y^2 + 4x + 6y - 12 = 0$

Equation of the line is $x + 4y - 6 = 0$

Now centre C and radius *r* of the circle are C(–2, –3) $r = \sqrt{4 + 9 + 12} = 5$

p = length of perpendicular from C onto the line

$$p = \left|\frac{-2 + 4(-3) - 6}{\sqrt{1 + 16}}\right| = \frac{20}{\sqrt{17}}$$

Length of the chord = $2\sqrt{r^2 - p^2}$

$$= 2\sqrt{25 - \frac{400}{17}} = \frac{10}{\sqrt{17}}$$

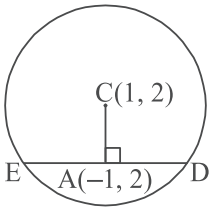
Example 18:

Find the equation of the chord of the circle $x^2 + y^2 - 2x + 4y - 17 = 0$ bisected at (–1, 2)

Solution:

The equation of the circle is $x^2 + y^2 - 2x + 4y - 17 = 0$ centre = (1, –2). The mid point of the chord A = (–1, 2). The chord is perpendicular to AC

$\therefore$ (slope of the chord) (slope of AC) = –1



$$\text{slope of the AC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2+2}{-1-1}$$

$$\text{slope of the chord} = \frac{-1}{\left(\frac{2+2}{-1-1}\right)} = \frac{1}{2}$$

The chord ED passes through (–1, 2). Thus the equation of the chord ED is

$$y - 2 = \frac{1}{2}(x+1) \Rightarrow 2y - 4 = x + 1$$

$$\Rightarrow x - 2y + 5 = 0$$

EXERCISE 15.2

One mark questions:

1. Find the centre and the radius of the circle.
   a) $x^2 + y^2 - 4x - y - 5 = 0$
   b) $3x^2 + 3y^2 - 6x - 12y - 2 = 0$
   c) $(x-2)(x-4) + (y-1)(y-3) = 0$
   d) $x^2 + y^2 - 2x\cos\alpha - 2y\sin\alpha = 1$
2. If the radius of the circle $x^2 + y^2 + 4x - 2y - k = 0$ is 4 units find k.
3. Find the other end of the diameter, if one end of the diameter of the circle
   a) $x^2 + y^2 = 25$ is (5,0)
   b) $x^2 + y^2 + 4x - 6y - 12 = 0$ is (–5, –1)
   c) $x^2 + y^2 - 6x + 2y = 31$ is (7, 4)
4. If (a,b) and (–5, 1) are the two end points of diameter of the circle $x^2 + y^2 + 4x - 4y = 2$. Find the values of a and b.
5. If $x^2 + y^2 - 4x - 8y + k = 0$ represents a point circle find k.

6. If $x^2 + y^2 + ax + 2y + 5 = 0$ represent a point circle find a.
7. If $x^2 + y^2 + ax + by = 3$ represents a circle with centre at $(1, -3)$, find a and b.
8. Find the unit circle concentric with the circle $x^2 + y^2 - 8x + 4y = 8$.

Two marks questions:

1. Find the equation of the circle whose centre is same as the centre of the circle $x^2 + y^2 + 6x + 2y + 1 = 0$, and passing through the point $(-2, -1)$.
2. Find the equation of the circle whose centre is same as the centre of the circle $x^2 + y^2 - 6x + 4y + 9 = 0$ and passing through the point $(-2,3)$.
3. Find the equation of the circle whose centre is $(-2, 3)$ and passing through the centre of the circle $x^2 + y^2 - 6x + 4y + 9 = 0$.
4. Find the equation of the circle passing through the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ and centre at $(4, -2)$.
5. Find the equation of the circle two of whose diameters are $x + y = 3$ and $2x + y = 2$ and passing through the centre of the circle $x^2 + y^2 - 4x + 2y - 1 = 0$.
6. Find the equation of the circle concentric with the centre of the circle $x^2 + y^2 - 2x + 2y - 1 = 0$ and having double its area.
7. Find the equation of the circle concentric with the centre of the circle $3x^2 + 3y^2 - 6x + 9y - 2 = 0$ and having $\frac{2}{3}$ of its area.
8. Find the equation of the diameter of the circle $x^2 + y^2 + 6x - 2y = 6$ which when produced passes through the point $(1, -2)$.
9. Find the equation of the diameter of the circle $2x^2 + 2y^2 + 3x - 5y - 1 = 0$, which when produced passes through the point $(-1,2)$.
10. Show that the line $4x - y = 17$ passes through the centre of the circle $x^2 + y^2 - 8x + 2y = 0$.
11. Find the length of the chord of the circle $x^2 + y^2 - 6x + 15y - 16 = 0$ intercepted by the $x-$axis.
12. Find the length of the chord of the circle $x^2 + y^2 + 3x - y - 6 = 0$ intercepted by the $y-$axis.
13. Find the length of the chord of the circle $x^2 + y^2 - 6x - 4y - 12 = 0$ on the coordinate axes.

Five marks questions:

1. Find the equation of the circle.
   a) passing through the origin, having its centre on the x - axis and radius 2 units.
   b) passing through (2,3) having its centre on the x - axis and radius 5 units.
   c) passing through the points (5,1), (3,4) and has its centre on the x - axis.
   d) passing through the points (1,2) and (2,1) and has its centre on the y - axis.
   e) passing through the points (0,5) and (6,1) and has its centre on the line $12x + 5y = 25$
   f) passing through the points (1,–4) and (5,2) and has its centre on the line $x - 2y + 9 = 0$.
   g) passing through the points (0, –3) and (0,5) and whose centre lies on $x - 2y + 5 = 0$
   h) passing through (1,1) and (2,2) and having radius 1.
2. Find the equation of the circle passing through the points.
   a) (0,2), (3,0), (3,2)
   b) (1,1), (–2,2), (–6,0)
   c) (1,1), (5,–5), (6,–4)
   d) (1,0), (3,0), (0,2)
   e) (5,7, (6,6), (2,–2)
   f) (p,q), (p,0), (0,q)
   g) (0,1), (2,3), (–2,5)
   h) (0,0), (a,0), (0,b)
3. Find the equations of the circles whose radius is 5 and which passes through the points on x - axis at distances 3 from the origin.
4. A circle has radius 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle if it passes through (7,3).
5. Find the equation of the circle which cuts intercepts of the lengths $'a'$ and $'b'$ on axes and passing through the origin.
6. Find the equation of the circle passing through the origin and making positive x - intercept $'a'$ units and negative y - intercepts $'b'$ units.
7. Find the length of the chord intercepted by the
   a) circle $x^2 + y^2 - 8x - 6y = 0$ and the line $x - 7y - 8 = 0$
   b) circle $x^2 + y^2 = 9$ and the line $x + 2y = 3$.
   c) circle $x^2 + y^2 - 6x - 2y + 5 = 0$ and the line $x - y + 1 = 0$
8. Find the points of intersection of the circle
   a) $x^2 + y^2 = 9$ and the line $x + 2y = 3$.
   b) $x^2 + y^2 - 6x - 2y + 5 = 0$ and the line $x - y + 1 = 0$.
   c) $x^2 + y^2 + 4x + 6y - 12 = 0$ and the line $x + 4y - 6 = 0$

   Also find the length of the chord.

Six marks questions:

3. Show that the following points are concyclic

a) (0,0), (1,1), (5,–5), (6,–4) b) (2,–4), (3,–1), (3,–3), (0,0)

c) (1,0), (2,–7), (8,1), (9,–6)

ANSWERS 15.2

I. 1. a) $\left(2,\frac{1}{2}\right),\frac{\sqrt{37}}{2}$ b) $(1,2)\sqrt{\frac{17}{3}}$

c) $(3,-1)\sqrt{5}$ d) $(\cos\alpha,\sin\alpha),\sqrt{2}$

2. k = 11

3. a) (–5,0) b) (1,7) c) (–1,–6)

4. $a = 1, b = 3$ 5. $k = 20$ 6. $a = \pm 4$ 7. $a = -2, b = 6$

8. $x^2 + y^2 - 8x + 4y + 19 = 0$

II. 1. $x^2 + y^2 + 6x + 2y + 9 = 0$ 2. $x^2 + y^2 - 6x + 4y - 37 = 0$

3. $x^2 + y^2 + 4x - 6y - 37 = 0$ 4. $x^2 + y^2 - 8x + 4y - 5 = 0$

5. $x^2 + y^2 + 2x - 8y - 17 = 0$ 6. $x^2 + y^2 - 2x + 2y - 4 = 0$

7. $36x^2 + 36y^2 - 72x + 108y + 23 = 0$ 8. $3x + 4y + 5 = 0$

9. $3x + y + 1 = 0$ 11. 10 units

12. 5 units 13. $2\sqrt{21}$, 8 units

III. 1. a) $x^2 + y^2 \pm 4x = 0$ b) $x^2 + y^2 - 12x + 11 = 0$, $x^2 + y^2 + 4x - 21 = 0$

c) $2x^2 + 2y^2 - x - 47 = 0$ d) $x^2 + y^2 - 5 = 0$

e) $3x^2 + 3y^2 - 10x - 6y - 45 = 0$ f) $x^2 + y^2 + 6x - 6y - 47 = 0$

g) $x^2 + y^2 + 6x - 2y - 15 = 0$ h) $x^2 + y^2 - 2x - 4y + 4 = 0$, $x^2 + y^2 - 4x - 2y + 4 = 0$

2. a) $x^2 + y^2 - 3x - 2y = 0$ b) $x^2 + y^2 + 4x + 6y - 12 = 0$

c) $x^2 + y^2 - 6x + 4y = 0$ d) $2x^2 + 2y^2 - 8x - 7y + 6 = 0$

e) $x^2 + y^2 - 4x - 6y - 12 = 0$ f) $x^2 + y^2 - px - qy = 0$

g) $3x^2 + 3y^2 + 2x - 20y + 17 = 0$ h) $x^2 + y^2 - ax - by = 0$

3. $x^2 + y^2 - 8y - 9 = 0,\ x^2 + y^2 + 8y - 9 = 0$

4. $x^2 + y^2 - 14x - 12y + 76 = 0,\ x^2 + y^2 - 8x - 6y + 16 = 0$

5. $x^2 + y^2 \pm ax \pm by = 0$

6. $x^2 + y^2 - ax + by = 0$

7. a) $5\sqrt{2}$ b) $\frac{12\sqrt{5}}{5}$ c) $\sqrt{2}$

8. a) $(3,0), \left(\frac{-9}{5}, \frac{12}{5}\right), \frac{12}{\sqrt{5}}$ b) $(2,3), (1,2), \frac{8}{\sqrt{5}}$

c) $(-2,2), \left(\frac{6}{17}, \frac{24}{17}\right), \frac{10}{\sqrt{17}}$

15.5 Point of Intersection of a line and a circle

With a line and a circle, we have the following three cases, namely

Case (i) : The line may cut the circle at two points A and B as shown in fig (i)

In this case the length of the perpendicular from the centre is less than the radius of the circle. AB is length of the chord of the circle.

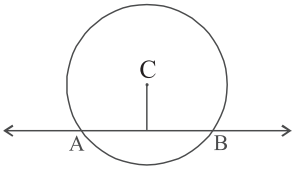


Fig. (i)

Case (ii) : The line may be tangent to the circle ie the line and the circle have only one point in common A as shown in fig(ii)

In this case the length of the perpendicular from the centre on to the line is equal to the radius of the circle.

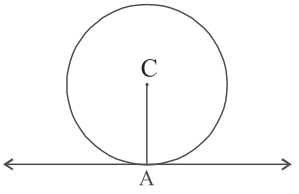


Fig. (ii)

Case (iii): The line and the circle have no points in common as shown in fig(iii)

In this case the length of the perpendicular from the centre on the line is greater than the radius of the circle.

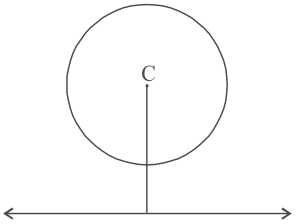


Fig. (iii)

Important Observation : A line will be a tangent to the given circle if and only if the length of the perpendicular from the centre on to the line is equal to radius of the circle.

Let $y = mx + c$ be the equation of a line and $x^2 + y^2 + 2gx + 2fy + k = 0$ be the equation of a circle. Substituting the value of $'y'$ in the equation of the circle, we get

$$x^2 + (mx + c)^2 + 2gx + 2f(mx + c) + k = 0$$

i.e. $$x^2 + m^2x^2 + c^2 + 2mxc + 2gx + 2fmx + 2fc + k = 0$$

$$\Rightarrow (1 + m)^2 x^2 + (2mc + 2g + 2fm)x + (c^2 + 2fc + k) = 0 \quad .....(1)$$

which is a quadratic equation in x.

1) The equation (1) has two real and distinct roots if the discriminant of the equation is greater than zero and in this case the line cuts the circle in two distinct points say A and B as shown in fig(i).

   The length **AB is called the length of the chord.**

2) The roots of the equation (1) are real and coincident if the discriminant of the equation is equal to zero and in this case the line touches the circle at only one point as shown in fig(ii). Thus the line will be a tangent to the circle.

3) The roots of the equation (i) are imaginary if the discriminant of the equation is less than zero and in this case, there will be no common points between the line and the circles, as shown in fig (iii).

Note :

1) **Length of the chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ intercepted by x - axis is $2\sqrt{g^2 - c}$**

2) **Length of the chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ intercepted by y - axis is $2\sqrt{f^2 - c}$**

3) **Length of the chord intercepted by the line with the given circle is $2\sqrt{r^2 - p^2}$**

4) **Condition for the circle to touch x - axis is $g^2 = c$**

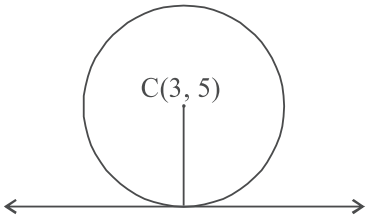
5) **Condition for the circle to touch y - axis is $f^2 = c$**

6) **Condition for the circle to touch both the coordinate axes is $g^2 = f^2 = c$**

Example 19:

Show that the line $3x-4y+6=0$ touches the circle $x^2+y^2-6x+10y-15=0$.

Solution:



We have $x^2+y^2-6x+10y-15=0$.

$2g=-6 \Rightarrow g=-3,\ \ 2f=10 \Rightarrow f=5$

$\therefore$ centre = C = $(-g, -f) = (3, -5)$

radius = $r=\sqrt{g^2+f^2-c}=\sqrt{9+25+15}=\sqrt{49}=7$

we know that a line touches the circle if and only if the length of the perpendicular from the centre on to the line is equal to radius of the circle.

i.e. $\left|\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|=r$ i.e., $\dfrac{3(3)-4(-5)+6}{\sqrt{9+16}}=7$

$$\Rightarrow \frac{35}{5}=7$$

$\therefore$ The line $3x-4y+6=0$ touches the circle.

Example 20:

Show that the line $2x+y+2=0$ is a tangent to the circle $x^2+y^2+6x+2y+5=0$. Also find the point of contact.

Solution :

We have $x^2+y^2+6x+2y+5=0$

$2g=6 \Rightarrow g=3,\ 2f=2 \ \Rightarrow f=1 \ \ c=5$

centre = C = (–g, –f) = (–3, –1);

radius $= r=\sqrt{g^2+f^2-c}=\sqrt{9+1-5}=\sqrt{5}$

Radius = length of the perpendicular from centre to the line

i.e., $= r=\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}} \rightarrow \sqrt{5}=\left|\dfrac{2(-3)+1(-1)+2}{\sqrt{4+1}}\right|$

Thus we find $r=\sqrt{5}$ and length of perpendicular from centre to the line $=\sqrt{5}$

Therefore the line $2x+y+2=0$ is a tangent to the given circle.

To find the contact : We shall find the equation of the line through the centre (–3, –1) and perpendicular $2x + y + 2 = 0$. We know any line perpendicular to $2x + y + 2 = 0$ is of the from $x - 2y + k = 0$. This line passes through (–3,–1). Therefore put $x = -3,\ y = -1$ we get $-3 + 2 + k = 0 \rightarrow k = 1$

So equation becomes $x - 2y + 1 = 0$. Now solving $2x + y + 2 = 0$ and $x - 2y + 1 = 0$ we get $x = -1,\ y = 0$

Hence the point of contact is (–1, 0)

Example 21:

Find ***k*, such that the straight line $x - 2y + r = 0$ may be tangent to the circle $x^2 + y^2 + 3x - 2y + 2 = 0$**

Solution:

We have $x^2 + y^2 + 3x - 2y + 2 = 0$

$$2g = 3 \rightarrow g = \frac{3}{2};\ 2f = -2 \rightarrow f = -1,\ c = 2$$

$$\text{centre} = \text{C} = (-g, -f) = \left(-\frac{3}{2}, 1\right);$$

$$\text{radius} = r = \sqrt{g^2 + f^2 - c} = \sqrt{\frac{9}{4} + 1 - 2} = \sqrt{\frac{9}{4} - 1} = \frac{\sqrt{5}}{2}$$

radius = length of the perpendicular from the centre to the line $x - 2y + r = 0$

$$\text{ie } \frac{\sqrt{5}}{2} = \left|\frac{1\left(-\frac{3}{2}\right) - 2(1) + r}{\sqrt{1+4}}\right|$$

$$\frac{\sqrt{5}}{2} = \left|\frac{\frac{-7}{2} + \lambda}{\sqrt{5}}\right| \rightarrow \pm\ \frac{5}{2} = \frac{-7}{2} + \lambda$$

$$\rightarrow \lambda = \frac{5}{2} + \frac{7}{2} = \frac{12}{2} = 6$$

Also $\lambda = \frac{-5}{2} + \frac{7}{2} = \frac{2}{2} = 1$

Thus $\lambda = 1$ or 6

Example 22:

Find the equation of the tangents to the circles $x^2 + y^2 + 2x + 4y - 4 = 0$ which are parallel to the line $5x + 12y + 6 = 0$

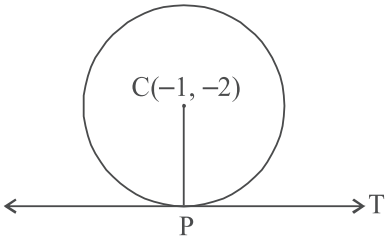
Solution :

Given circle is

$$x^2 + y^2 + 2x + 4y - 4 = 0$$

$$2g = 2,\ g = 1,\ 2f = 4,\ f = 2,\ c = -4$$

centre = $(-g, -f) = (-1, -2)$



Radius = $\sqrt{g^2 + f^2 - c} = \sqrt{1 + 4 + 4} = 3$

Equation of the tangent parallel to the given line can be taken as $5x + 12y + k = 0$ .....(1)

Length of perpendicular from (–1, –2) to the line (1)

$$CP = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right| = \frac{5(-1) + 12(-2) + k}{\sqrt{25 + 144}} = \left|\frac{-5 - 24 + k}{13}\right|$$

i.e., $3 = \left|\frac{-29 + k}{13}\right| \Rightarrow \frac{k - 29}{13} = \pm 3$

i.e., $k - 29 = \pm 39 \Rightarrow k = 68,\ k = -10$

$\therefore$ Equations of the tangents parallel to the given line are $5x + 12y + 68 = 0$ and $5x + 12y - 10 = 0$

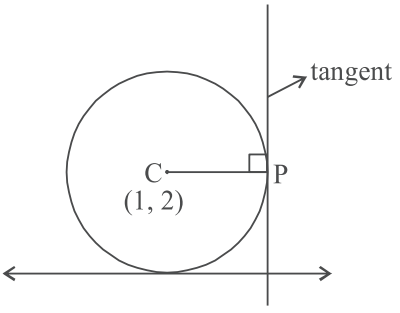
Example 23:

Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y + 1 = 0$ which are perpendicular to the line $3x - 4y + 7 = 0$

Solution:

The equation of a tangent $\perp$ to the given line can be taken as $4x + 3y + k = 0$ .....(1)

Given circle $x^2 + y^2 - 2x - 4y + 1 = 0$, $g = -1, f = -2$, $c = 1$ centre (1,2), radius = 2



Length of $\perp$ from the centre (1,2) to the line (1) is

$$CP = \left|\frac{4(1)+3(2)+k}{\sqrt{16+9}}\right| = \left|\frac{10+k}{5}\right|$$

i.e., $2 = \left|\frac{10+k}{5}\right| \Rightarrow \frac{10+k}{5} = \pm 2$

$10 \pm k = +10$

$\therefore\ k = 0, -20$

$\therefore$ The equation of the tangents $\perp$ to the given line are $4x+3y=0$ and $4x+3y-20=0$

Example 24:

Find the value of k for which the line $x+ky-5=0$ may touch the circle $x^2+y^2-2x-6y-6=0$

Solution:

$g=-1,\ f=-3,\ c=-6$; centre = (1,3)

Radius = $\sqrt{g^2+f^2-c} = \sqrt{1+9+6} = 4$

The line $x+ky-5=0$ touches the circle if the length of the perpendicular from the centre (1,3) to the line is equal to the radius of the circle.

ie., $\left|\frac{1+3k-5}{\sqrt{1+k^2}}\right| = 4$ ie., $|3k-4| = 4\sqrt{1+k^2}$

or $9k^2+16-24k = 16+16k^2$

or $7k^2+24k=0$ ie., $k(7k+24)=0$

$\Rightarrow k=0,\ k=-\frac{24}{7}$

Example 25:

Show that the circle $x^2+y^2+4x-3y+4=0$ touches x - axis

Solultion:

$2g=4 \Rightarrow g=2,\ c=4$

Condition for the circle to touch

x - axis is $g^2=c$

ie., $4=4$ $\therefore$ given circle touches x - axis.

Example 26:

Show that the circle $x^2 + y^2 - 3x + 8y + 16 = 0$ touches y - axis.

Solution:

$2f = 8 \rightarrow f = 4,\ c = 16$, Condition for the circle to touch y - axis is $f^2 = c$

Given circle touches y - axis, since $4^2 = 16$

Example 27:

Show that the circle $x^2 + y^2 - 2x + 2y + 1 = 0$ touches both the co-ordinate axes.

Solution :

Condition for the circle to touch both the axes is $g^2 = f^2 = c$. Here $g = 1, f = -1$ and $c = 1$.
Since $(1)^2 = (-1)^2 = 1$, we find the given circle touches both the axes.

EXERCISE 15.3

One mark questions:

I. Show that the circle

1. $x^2 + y^2 - 8x + 2y + 16 = 0$ touches x - axis
2. $x^2 + y^2 - 4x + 4y + 4 = 0$ touches x - axis
3. $x^2 + y^2 + 8x + 6y + 9 = 0$ touches y - axis
4. $x^2 + y^2 - x + 4y + 4 = 0$ touches y - axis

II. 5. Show that the circle $x^2 + y^2 + 4x - 3y + 4 = 0$ touches x - axis

6. Show that the circle $x^2 + y^2 - 3x + 8y + 16 = 0$ touches y - axis

Two marks questions:

III. 7. Find the length of the chord of the circle $x^2 + y^2 - 6x + 4y + 5 = 0$ intercepted by x - axis

8. Find the length of the chord of the circle $x^2 + y^2 - 6x + 15y - 16 = 0$ intercepted by the y - axis.
9. Find the length of the chord of the circle $x^2 + y^2 + 3x - 2 = 0$ intercepted by y - axis.
10. Find the length of the chord of the circle $x^2 + y^2 + 3x - y - 6 = 0$ intercepted by the y - axis.
11. Find the length of the chord of the circle $x^2 + y^2 + 4x + 6y - 12 = 0$ and $x + 4y - 6 = 0$

12. Find the length of the chord intercepted by the circle $x^2 + y^2 - 8x - 6y - 6 = 0$ and the line $x - 7y - 8 = 0$

13. Find the length of the chord intercepted by the circle $x^2 + y^2 - 6x - 2y + 5 = 0$ and the line $x - y + 1 = 0$

14. Find the equation of the chord of the circle $x^2 + y^2 - 2x + 4y - 17 = 0$ bisected at $(-1, 2)$

15. Find the equation of the chord of the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ bisected at $(2,3)$

16. Prove that the length of the intercept made by the circle $x^2 + y^2 + 10x - 6y + 9 = 0$ on the x - axis is 8 units.

17. Prove that the length of the chord $x + 2y = 5$ of the circle $x^2 + y^2 = 9$ is 4 units.

Five marks questions:

18. Show that the line $3x + 4y + 7 = 0$ touches the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ and find the point of contact.

19. Find k if the line $3x + y + k = 0$ touches the circle $x^2 + y^2 - 2x - 4y - 5 = 0$

20. Find k if the line $2x + y + k = 0$ touches the circle $x^2 + y^2 + 6x + 2y + 5 = 0$

21. Find k if the line $4x - y + k = 0$ touches the circle $x^2 + y^2 + 4x - 8y + 3 = 0$

22. Find k such that the line $x - 2y + k = 0$ touches the circle $x^2 + y^2 + 3x - 2y + 2 = 0$

Answers 15.3

III. 7) 4 8) 10 9) $2\sqrt{2}$ 10) 5

11) $\frac{10}{\sqrt{17}}$ 12) $5\sqrt{2}$ 13) $\sqrt{2}$ 14) $x - 2y + 5 = 0$

15) $y = 3$ 18) $(-1, -1)$ 19) $k = 5, -15$ 20) $k = 2$

21) $k = -5$ 22) $k = 6, 1$

* * * * *

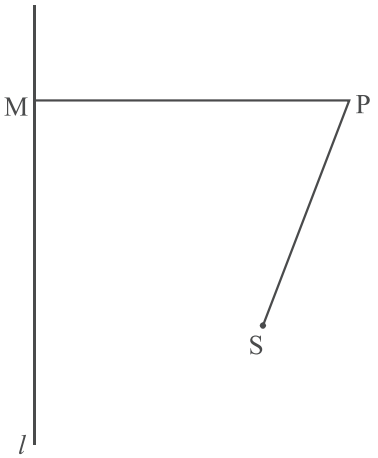
Chapter

16

PARABOLA

16.1 Introduction to Conic Section - Parabola:

Definition : A conic section is the locus of the point which moves such that the ratio of its distance from the fixed point S to its distance from the fixed line l, is always a fixed positive constant.



The fixed point S is called the focus of the conic section.

The fixed line l is called the directrix of the conic section.

If P is any point on the locus (conic section) then by definition we have $\frac{PS}{PM} = a$ fixed constant

This constant ratio $\frac{PS}{PM}$ is called the eccentricity of the conic section and it is denoted by e.

If the eccentricity e = 1, the conic section is called a parabola.

If the eccentricity e < 1, the conic section is called a ellipse.

If the eccentricity e > 1, the conic section is called a hyperbola.

Now we shall see the conic section parabola and study few properties and results associated to them Parabola.

16.2 Definition of Parabola and other forms of Parabola:

Parabola is a conic section whose eccentricity is 1. That is, Parabola is the locus of the point which moves such that its distance from the **fixed point (called the focus)** is equal to its distance from the **fixed line (called the directrix)**

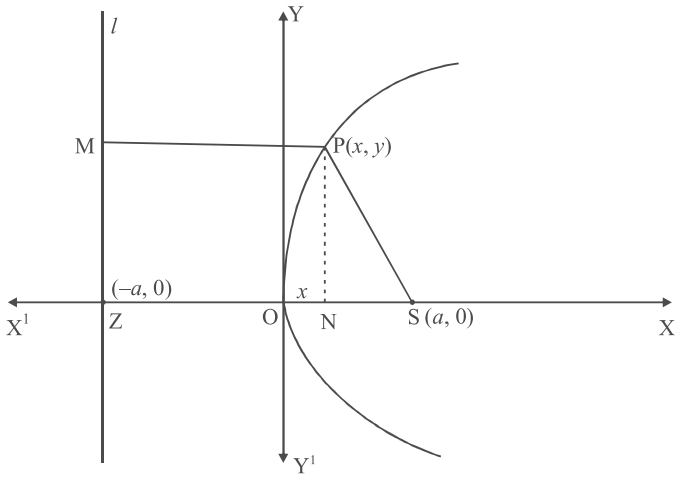
We shall derive the equation of the parabola in its standard form.

Theorem: The equation of a Parabola with proper choice of co-ordinates axes is $y^2 = 4ax$

Proof : Let S be the focus and the line l be the directrix. Draw SZ through S and perpendicular to the directrix. Let O be the midpoint of the line segment SZ.

Let SZ = 2a (a > 0)

$\Rightarrow OZ = OS = a$



We shall choose O as the origin and the line ZOSX as x axis. The line YOY^1 through O, perpendicular to the x axis will be y axis.

With this choice of co-ordinate axes we have $S(a, o)$ and $Z(-a, o)$

The equation of the directrix is l is $x = -a$

Let $P(x,y)$ be any point on the Parabola. Then by the definition of the parabola we have Distance of P from S = Distance of P from the line 'l'

$$|SP| = |PM|$$

$$SP^2 = ZN^2 \qquad \text{since PM = ZN}$$

$$SP^2 = (OZ + ON)^2$$

$$(x-a)^2 + y^2 = (a+x)^2 \quad \left(\because \ |OZ| = a,\ |ON| = x\right)$$

$$x^2 - 2ax + a^2 + y^2 = a^2 + 2ax + x^2$$

$\boxed{y^2 = 4ax}$ which is the equation of the parabola.

Note : The equation $y^2 = 4ax$ is also called standard form of the equation of the parabola.

Shape of the Parabola $y^2 = 4ax$

We shall note few observations from the equation $y^2 = 4ax$, which will help us to trace the curve parabola.

1. If y is replaced by $-y$ in the equation remains same, ie., $(-y)^2 = 4ax \rightarrow y^2 = 4ax$. This shows that if (x, y) is any point on the curve $y^2 = 4ax$, then $(x, -y)$ is also a point on the

curve. Thus, the curve is symmetric about the x - axis, ie the shape of the curve above the x axis is the mirror image (about the x axis) of the shape of the curve below the x - axis.

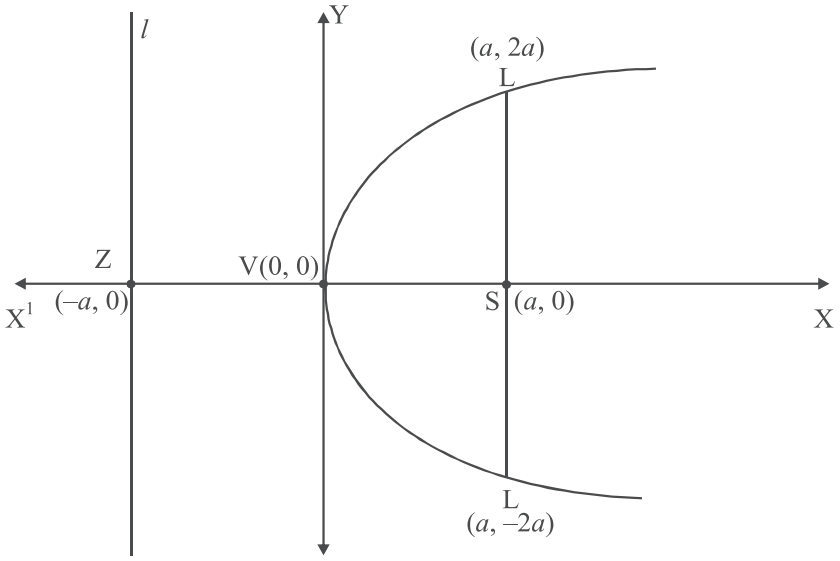
2. If $x < 0$, then y^2 will be a negative quantity (note that a > 0) and therefore $y^2 = 4ax$ will have no real solution for y. This shows that no part of the curve lies to the left side of the x - axis.
3. If $y = 0$, the only value of x we get is zero. Thus the curve cuts the x - axis at the origin(0, 0)
4. If $x = 0$, we get $y^2 = 0$, which gives $y = 0$. Thus the curve cuts the y - axis at the origin and further the y - axis meets the curve only at the origin. That is, y - axis is a tangent to the curve at the origin.
5. For any point P(x,y) on the parabola we have

$$y^2 = 4ax \quad \Rightarrow \quad y = \pm\, 2\sqrt{ax}$$

$$y = 2\sqrt{ax} \quad \text{and} \quad y = -2\sqrt{ax}$$

This shows that, as x increases from 0 to ∞, y also increases from 0 to ∞ or y decreases from 0 to $-\infty$. Thus, the two branches of the parabola, laying on opposite sides of the x - axis, will extend to infinity towards the positive directions of the x - axis.

From the above discussion and by plotting few points, whose coordinates satisfy $y^2 = 4ax$, it is found the shape of the parabola is as shown in the following figure.



The origin 0 is called the **Vertex** of the parabola, $y^2 = 4ax$, it is also denoted by V(0,0).

The line ZSX is called the **axis of the parabola** and its equation is $y = 0$

The focus $S = (a, 0)$ and the equation of the directrix is $x = -a$ i.e., $x + a = 0$

The y - axis is called the **tangent at the vertex** - its equation is $x = 0$.

Note: The distance between the vertex and the focus is the distance between the vertex and the directrix is equal to a and the distance between the directrix and the focus is $2a$.

Definition : The chord passing through the focus and perpendicular to the axis of the parabola is called the latus rectum of the parabola.

In the figure LSL^1 is the latus rectum.

The points L and L^1 on the parabola are called end points of the latus rectum, the length LL^1 is called the length of the latus rectum.

Clearly, the x - coordinates of L and L^1 is a because $OS = a$. To find the corresponding y - coordinates, we shall put $x = a$ in $y^2 = 4ax$, we get $y = \pm\ 2a$. Thus, y - coordinates of L and L^1 respectively.

$$L = (a,\ 2a) \text{ and } L^1 = (a, -2a)$$

$\therefore$ consider $\left|LL^1\right| = \sqrt{(a-a)^2 + (2a+2a)^2} = \sqrt{(4a)^2} = 4a$

Thus, the length of the latus rectum, $LL^1 = 4a$

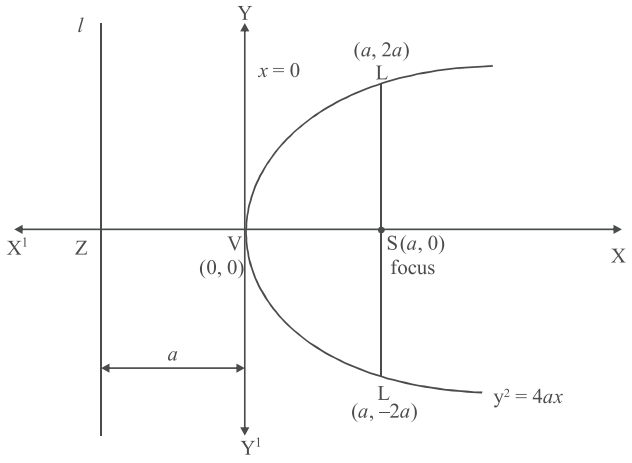
Note: Any chord passing through the focus is called a **focal chord.** Focal chord need not be perpendicular to the axis of the parabola.

16.3 Four Standard forms of Parabola and their Graphs

There are four standard forms of parabola viz $y^2 = 4ax$, $y^2 = -4ax$, $x^2 = 4ay$ and $x^2 = -4ay$, depending upon the choice of the axes along either of coordinate axes and vertex at the origin.

I. Right handed parabola

ie., parabola opening to right is of the form $y^2 = 4ax,\ a > 0$



Charactristics of Right Handed Parabola

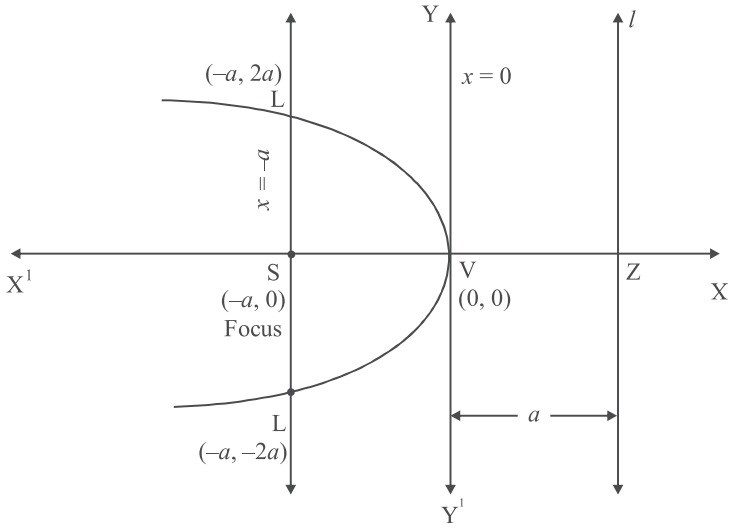
1) Vertex is V (0,0)
2) Focus is S (a,0)
3) Equation of directrix l is $x = -a$ or $x + a = 0$
4) Equation of the axis is $y = 0$ ie x - axis
5) Equation of the tangent at the vertex is $x = 0$ ie y - axis
6) Length of latus rectum = $LL^1 = 4a$
7) Equation of LR is $x - a = 0$ or $x = a$
8) Coordinates of ends of latus rectum are $L(-a, 2a)$ and $L^1(-a, -2a)$
9) Parametric coordinates is $(at^2, 2at)$
10) It is symmetric about x - axis

II. Left handed parabola

ie., parabola opening to left is of the form $y^2 = -4ax,\ a > 0$

Charactristics of Left Handed Parabola

1) Vertex is V (0,0)
2) Focus is S ($-a$, 0)
3) Equation of directrix MZ is $x = a$ or $x - a = 0$



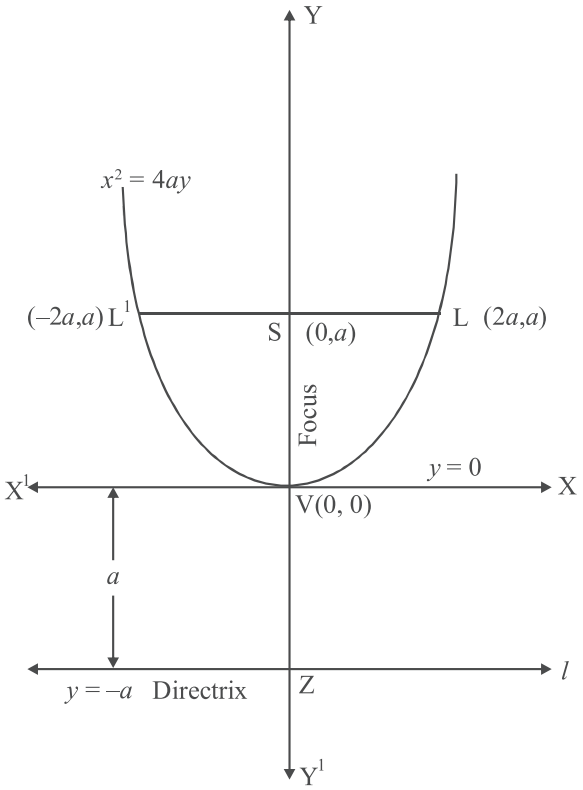
4) Equation of axis $y = 0$ is x - axis
5) Equation of the tangent at the vertex is $x = 0$ ie., y - axis
6) Length of latus rectum $LL^1 = 4a$
7) Equation of latus rectum = $x = -a$ or $x + a = 0$
8) Co-ordinates of ends of latus rectum is $L(-a, 2a)$ and $L^1(-a, -2a)$

9) Parametric coordinates is $(-at^2, 2at)$

10) It is symmetric about x - axis

III. Upward parabola

ie., parabola opening upwards is of the form $x^2 = 4ay, (a > 0)$



Charactristics of Upward Parabola

1) Vertex is A(0,0)

2) Focus is S $(0,a)$

3) Equation of directrix MZ is $y = -a$ or $y + a = 0$

4) Equation of axis is $x = 0$ ie, y - axis

5) Equation of tangent at the vertex is $y = 0$ ie, x - axis

6) Length of latus rectum $LL^1 = 4a$

7) Equation of latus rectum $y = a$ or $y - a = 0$

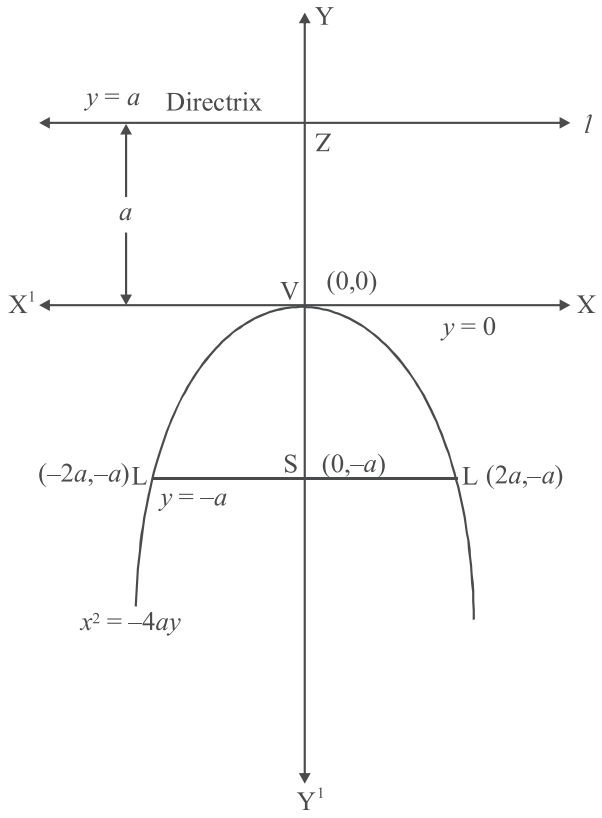
8) Co-ordinate of the ends of latus rectum are $L(2a, a)$ and $L^1(-2a, a)$

9) Parametric coordinates is $(2at, at^2)$

10) It is symmetric about the y - axis

IV Downward parabola

ie, parabola opening downwards is of the form $x^2 = -4ay,\ (a > 0)$



Charactristics of Downward Parabola

1) Vertex is V(0, 0)
2) Focus is S $(0, -a)$
3) Equation of directrix MZ is $y = a$ or $y - a = 0$
4) Equation of axis is $x = 0$ ie, y - axis
5) Equation of tangent at the vertex is $y = 0$ ie, x - axis
6) Length of latus rectum $LL^1 = 4a$
7) Equation of latus rectum $y = -a$ or $y + a = 0$
8) Coordinate of ends of latus rectum are $L(2a, -a)$ and $L^1(-2a, -a)$
9) Parametric coordinates are $(2at, -at^2)$
10) It is symmetric about the y - axis

Equation	$y^2 = +4ax$	$y^2 = -4ax$	$x^2 = +4ay$	$x^2 = -4ay$
Directrix	$x = -a$	$x = +a$	$y = -a$	$y = +a$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0,-a)$
Vertex	(0,0)	(0,0)	(0,0)	(0,0)
Axis	x - axis ($y = 0$)	x - axis ($y = 0$)	y - axis ($x = 0$)	y - axis ($x = 0$)
Equation of tangent vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Equation and length of latus rectum	$x = a$ and $4a$	$x = -a$ and $4a$	$y = a$ and $4a$	$y = -a$ and $4a$
Coordinates of ends of LL	$(a, 2a)$ and $(a, -2a)$	$(-a, 2a)$ and $(-a, -2a)$	$(2a, a)$ and $(-2a, a)$	$(-2a, -a)$ and $(-2a, a)$

WORKED EXAMPLES

Type I : Finding the focus, directrix, latus rectum, Axis etc. for a given Parabola in one of the standard forms.

Find the characteristics of the following parabolas

1) $y^2 = 8x$ 2) $x^2 = 6y$

3) $y^2 = -12x$ 4) $x^2 = -16y$

Solution :

1) $y^2 = 8x$ It is in the form of $y^2 = 4ax$, where $4a = 8 \rightarrow a = 2$

$\therefore$ the characteristics are given by

Directrix	$x = -a \Rightarrow x = -2$
Focus	$(a, 0) \Rightarrow (2, 0)$
Vertex	$(0, 0) \Rightarrow (0, 0)$
Axis	$x - axis\ (y = 0) \Rightarrow y = 0$
Equation of tangent at vertex	y - *axis* ie., $x = 0$
Equation of L.R.	$x = a \Rightarrow x = 2$
Length of L.R.	$4a \Rightarrow 4 \times 2 = 8$
Co-ordinates of ends of L.R.	$(a, 2a) = (2, 4)$ $(a, -2a) = (2, -4)$

2) $x^2 = 6y$

It is in the form of $x^2 = 4ay$ where $4a = 6 \Rightarrow a = \frac{3}{2}$ the characteristics are

Directrix	$y = -a \Rightarrow y = -\frac{3}{2}$
Focus	$(0, a) \Rightarrow \left(0, \frac{3}{2}\right)$
Vertex	$(0,0) \Rightarrow (0,0)$
Axis	$y - axis \Rightarrow x = 0$
Equation of tangent at vertex	$x - axis \Rightarrow y = 0$
Equation of L.R.	$y = a \Rightarrow y = \frac{3}{2}$
Length of L.R.	$4a \Rightarrow 4 \times \frac{3}{2} = 6$
Co-ordinates of ends of L.R	$(2a, a) = \left(2 \times \frac{3}{2}, \frac{3}{2}\right) = \left(3, \frac{3}{2}\right)$ and $(-2a, a) = \left(2 \times \frac{3}{2}, \frac{3}{2}\right) = \left(-3, \frac{3}{2}\right)$

3) It is the form of $y^2 = -4ax$ where $4a = 12 \Rightarrow a = 3$ the characteristics are

Directrix	$x = a \Rightarrow x = 3$
Focus	$(-a, 0) \Rightarrow (-3, 0)$
Vertex	$(0,0)$
Axis	$x - axis \Rightarrow y = 0$
Equation of tangent at vertex	$y - axis \Rightarrow x = 0$
Equation of L.R.	$x = -a \Rightarrow x = -3$ or $x + 3 = 0$
Length of L.R.	$4a \Rightarrow 4 \times 3 = 12$
Co-ordinates of ends of L.R	$(-a, 2a) = (-3, 6)$ and $(-a, -2a) = (3 - 6)$

4) $x^2 = -16y$

It is in the form of $x^2 = -4ay$ where $4a = 16 \Rightarrow a = 4$.

Characteristics are given by

Directrix	$y = a \Rightarrow y = 4$
Focus	$(0, -a) \Rightarrow (0, -4)$
Vertex	$(0, 0)$
Axis	$y - axis \Rightarrow x = 0$
Equation of tangent at vertex	$x - axis \Rightarrow y = 0$
Equation of L.R.	$y = -a \Rightarrow y = -4$
Length of L.R.	$4a \Rightarrow 4 \times 4 = 16$
Co-ordinates of ends of L.R.	$(2a, -a) = (8, -4)$ and $(-2a, -a) = (-8, -4)$

Type II : Find the standard equation of parabola when vertex and focus are given.

Find the equation of parabola given that

a) Vertex is (0, 0) and focus (3, 0)
b) Vertex is (0, 0) and focus (4, 0)
c) Vertex is (0, 0) and focus (–6, 0)
d) Vertex is (0, 0) and focus (–4, 0)
e) Vertex is (0, 0) and focus (0,3)
f) Vertex is (0, 0) and focus (0,4)
g) Vertex is (0, 0) and focus (0, –8)
h) Vertex is (0, 0) and focus (0,16)

Solution :

a) Given vertex A = (0,0) and focus S = (3,0) clearly focus is to the right side of vertex.

$\therefore$ required parabola is in the form $y^2 = 4ax \Rightarrow a = 3$

Thus equation given by $y^2 = 4 \times 3x \Rightarrow y^2 = 12x$

b) Given vertex is (0,0) and focus (4,0). Clearly focus is to the right side of vertex

$\therefore$ required parabola is $y^2 = 4ax \Rightarrow a = 4$

Thus equation is given by $y^2 = 4 \times 4x \Rightarrow y^2 = 16x$

c) Given vertex is (0,0) and focus (–6, 0) clearly focus is left side of vertex

$\therefore$ required parabola is in the focus

$y^2 = -4ax \Rightarrow -a = -6 \Rightarrow a = 6$

Thus equation is given by $y^2 = -4(6)x \Rightarrow y^2 = -24x$

d) Vertex (0,0) and focus (–4, 0) clearly focus is to the left side of the vertex

$\therefore$ required parabola is in the form $y^2 = -4ax$

$\Rightarrow -a = -4 \Rightarrow a = 4$

Thus equation is given by $y^2 = -4ax \Rightarrow y^2 = -4 \times 4 \times x \Rightarrow y^2 = -16x$

e) Vertex (0,0) and focus (0,3) clearly focus is to the left side of the vertex

$\therefore$ required parabola is in the form of $x^2 = 4ay \Rightarrow a = 3$

Thus equation is given by $x^2 = 4ay \Rightarrow x^2 = 4 \times 3y \Rightarrow x^2 = 12y$

f) Vertex (0,0) and focus (0,4) clearly focus is above the vertex

$\therefore$ required parabola is of the focus $x^2 = 4ay \Rightarrow a = 4$

Thus equation is given by $x^2 = 4ay \Rightarrow x^2 = 4 \times 4 \times y \Rightarrow x^2 = 16y$

g) Vertex (0,0) and focus (0,–8) clearly focus is below the vertex.

$\therefore$ required parabola is in the form

$x^2 = -4ay \Rightarrow -a = -8 \Rightarrow a = 8$

Thus equation is given by

$x^2 = -4ay \Rightarrow x^2 = -4 \times 8y \Rightarrow x^2 = -32y$

Type III : Finding the equation of parabola when vertex and directrix are given.

Find the equation of parabola given that

a) Vertex is (0, 0) and directrix $y = -4$

b) Vertex is (0, 0) and directrix $y = -2$

c) Vertex is (0, 0) and directrix $y = 6$

d) Vertex is (0, 0) and directrix $y = 8$

e) Vertex is (0, 0) and directrix $x = -3$

f) Vertex is (0, 0) and directrix $x = -5$

g) Vertex is (0, 0) and directrix $x = 4$

h) Vertex is (0, 0) and directrix $x = 7$

Solution :

a) Let the vertex of the parabola be A (0,0) and directrix is $y = -4$

Thus, the directrix is a line parallel to the x - axis at distance of 4 units below the x - axis. So the focus of the parabola lies on the y-axis consequently the focus is S(0,4). Hence the equation of the parabola is $x^2 = 4ay$ where $a = 4 \Rightarrow x^2 = 16y$

b) Let the vertex of the parabola be a (0,0) and directrix is $y = -2$.

Thus directrix is a line parallel to x - axis at 'a' distance of 2 units below the axis. So the focus of parabola lies on y - axis consequently the focus is S(0,2). Hence the equation of the parabola is $x^2 = 4ay$ where $a = 2 \Rightarrow x^2 = 8y$.

c) Let vertex is (0,0) and directrix is $y = 6$.

Thus directrix is a line parallel to the x-axis at a distance of 6 units above the vertex. So the focus of the parabola lie on y-axis consequently the focus is $(0, -6) \Rightarrow a = 6$. Hence the equation the parabola is $x^2 = 4ay$ where $a = 6$ then $x^2 = 4 \times 6y \Rightarrow x^2 = -24y$

d) Let the vertex of the parabola be (0,0) and directrix is $y = 8$.

Thus directrix is a line parallel to axis at a distance of 8 units above the vertex. So the focus of the parabola lies on y - axis consequently the focus is (0, –8). Hence the equation of the parabola is $x^2 = -4ay$ where $a = 8 \Rightarrow x^2 = -32y$

e) Let the vertex of the parabola be (0,0) and directrix $x = -3$.

Thus directrix is a line parallel to y axis at a distance of 3 units to the left side of the vertex. So the focus is (3, 0). Hence the equation of the parabola is $y^2 = 4ax$ where $a = 3 \Rightarrow y^2 = 12x$

f) Let the vertex of the parabola be A(0,0) and directrix is $x = -5$.

Thus the directrix is a line parallel to y - axis at a distance of 5 units to the left side of the vertex so the focus of the parabola lies on the x - axis. Consequently the focus is S(5,0). Hence the equation of the parabola is $y^2 = 4ax$ where $a = 5 \Rightarrow y^2 = 20x$

g) Let the vertex of the parabola be A (0,0) and directrics is $x = 4$.

Thus the directrix is line parallel to y - axis at distance of 4 units to the right side of the vertex. So the focus of the parabola lies on the x - axis. Consequently the focus is S(–4, 0). Hence the equation of parabola is $y^2 = 4ax$ where $a = 4 \Rightarrow y^2 = -16x$

h) Let the vertex of the parabola be A(0,0) and directrix is $x = 7$.

Thus the directrix is a line parallel to the y axis at a distance of 7 units to the right side of the vertex.

So the focus of the parabola lies on the x - axis, consequently the focus is S(–7, 0). Hence the equation of a parabola is $y^2 = 4ax$ where $a = 7 \Rightarrow y^2 = -28x$

Type IV : Find the equation of parabola when focus and directrix are given.

Find the equation of parabola given that

a) Focus (3,0) and directrix is $x = -3$

b) Focus (4,0) and directrix is $x = -4$

c) Focus (–3,0) and directrix is $x = 3$
d) Focus (–5,0) and directrix is $x = 5$
e) Focus (0,2) and directrix is $y = -2$
f) Focus (0,3) and directrix is $y = -3$
g) Focus (0,–4) and directrix is $y = 4$
h) Focus (0,–5) and directrix is $y = 5$

Solution :

a) Since focus lies on the $x-axis$, $\therefore$ $x-axis$ is the axis of the parabola.
Focus S(3,0) lies right side of the origin. So it is a right hand parabola.

$\therefore$ required equation of parabola is $y^2 = 4ax$

Since focus is S$(a,0)$ = (3,0) $\Rightarrow$ $a = 3$

Hence the required equation is $y^2 = 4 \times 3x \Rightarrow y^2 = 12x$

or

Let S = (3,0) and P(x,y) be any point on the parabola

$\therefore$ According definition of parabola

PS = $\perp^{lr}$ distance, from point P(x,y) to the directrix $y + 3 = 0$ is given by

$$\sqrt{(x-3)^2 + (y-0)^2} = \frac{|x+3|}{\sqrt{1+0}}$$

$$\Rightarrow x^2 + 9 - 6x + y^2 = x^2 + 9 + 6x \Rightarrow y^2 = 12x$$

b) Since focus lies on the x - axis $\therefore$ x - axis is the axis of the parabola
Focus S(3,0) lies rightside of the origin, so it is a right hand parabola.

$\therefore$ required equation of parabola is $y^2 = 4ax$

Since focus S$(a,0)$ = (4,0) $\Rightarrow$ $a = 4$

Hence the required equation is $y^2 = 4 \times 4x \Rightarrow y^2 = 16x$

or

Let S(4,0) and P(x,y) be any point of parabola

$\therefore$ According to definition of parabola

PS = $\perp^{lr}$ distance from point P(x,y) to the directrix $y + 4 = 0$ is given by

$$\sqrt{(x-4)^2 + (y-0)^2} = \frac{|x+4|}{\sqrt{1+0}}$$

$$\Rightarrow x^2 + y^2 - 8x + 16 = (x+4)^2$$

$$x^2 + y^2 - 8x + 16 = x^2 + 8x + 16 \Rightarrow y^2 = 16x$$

c) Since Focus lies on the x-axis, x-axis is the of the parabola Focus S(–3,0) lies to the left side of the origin.

So it is a left handed parabola

$\therefore$ required equation of parabola is $y^2 = 4ax$

Since focus is $(-a, 0) = (-3, 0) \Rightarrow a = 3$

Hence the required equation is $y^2 = -4ax \Rightarrow y^2 = -12x$

or

Let S = (–3, 0) and P(x,y) be any point on the parabola

$\therefore$ According definition of parabola

PS = $\perp^{lr}$ distance from P(x,y) to the directrix $x - 3 = 0$

$$\Rightarrow \sqrt{(x-(-3))^2 + (y-0)^2} = \frac{|x-3|}{\sqrt{1+0}}$$

$$\Rightarrow (x+3)^2 + (y-0)^2 = (x-3)^2 \Rightarrow x^2 + y^2 + 6x + 9 = x^2 - 6x + 9 \Rightarrow y^2 = -12x$$

d) Since focus lies on the x - axis; x - axis is the axis of the parabola

Focus S(–5, 0) lies to the left of the origin.

So it is a left hand parabola

$\therefore$ required equation of parabola is $y^2 = -4ax$

Since focus is $(-a, 0) = (-3,0) \Rightarrow a = 5$

Hence the required equation is $y^2 = -4 \times 5x \Rightarrow y^2 = -20x$

e) Since focus lies on the y - axis

$\therefore$ y axis is the axis of parabola. Focus S(0, 2) lies above the origin.

$\therefore$ required equation of parabola is $x^2 = 4ay$

since focus is $S(0,a) = (0, 2) \Rightarrow a = 2$

Hence the required equation is $x^2 = 4ay \Rightarrow x^2 = 4 \times 2y \Rightarrow x^2 = 8y$

f) Since focus lies on y axis

$\therefore$ y axis is the axis of parabola. Focus S(0,3) lies above the origin.

$\therefore$ required equation of parabola is $x^2 = 4ay$

since focus is $S(0,a) = (0,3) \Rightarrow a = 3$

Hence the required equation of is $x^2 = 4ay \Rightarrow x^2 = 4 \times 3y \Rightarrow x^2 = 12y$

g) Since focus lies on y - axis

$\therefore$ y axis is the axis of parabola. Focus S(0, –4) lies below the origin.

$\therefore$ required equation of parabola is $x^2 = -4ay$

Since focus is $S(0,-a) \Rightarrow (0,-4) \Rightarrow a = 4$

Hence the required equation is $x^2 = -4ay = 4 \times 4y \Rightarrow x^2 = -16y$

h) Since focus lies on y axis

$\therefore$ y axis is the axis of parabola. Focus S(0, –5) lies below the origin.

$\therefore$ required equation of parabola is $x^2 = -7ay$

since focus is $S(0,-a) \Rightarrow (0,-5) \Rightarrow a = 5$

Hence the required equation is $x^2 = -4ay \Rightarrow x^2 = -4 \times 5y \Rightarrow x^2 = -20y$

Type V : Find the equation of parabola if the vertex and axis of parabola are given.

Find the equation of parabola given that

a) Vertex is the origin and passing through the point P(3, –4) and symmetric about the y - axis.

b) Vertex at the origin and axis along x - axis and passing through the point P(2, 3).

c) Vertex at the origin and passing through the point P(5, 2) and symmetric with respect to the y - axis.

d) Vertex at the origin and passing through the point P(2, –3) and which is symmetric about the y - axis.

e) Vertex at the origin and symmetric about y - axis and passing through (–1, –3)

f) Vertex at the origin, axis is y - axis and passes through $\left(\frac{1}{2}, 2\right)$

Solution :

a) It is given that the vertex of the parabola is V(0,0) and symmetric about the y-axis so, its equation is $x^2 = 4ay$ or $x^2 = -4ay$, since the parabola passes through the point P(3, –4). So, it lies in the 4th quadrant.

$\therefore$ It is a downward parabola

Let its equation be $x^2 = -4ay$, since it passes through the point P(3, –4),

we have $3^2 = -4 \times a \times (-4) \Rightarrow a = \frac{+9}{16}$

So its equation is $x^2 = -4\left(\frac{9}{16}\right) y \Rightarrow x^2 = -\frac{9}{4}y \Rightarrow 4x^2 + 9y = 0$

b) It is given that vertex of the parabola is V(0,0) and symmetric about the x - axis so its equation is $y^2 = 4ax$ or $y^2 = -4ax$, since the parabola passes through the point P(2,3), so it lies in the 1st quadrant.

$\therefore$ It is a right handed parabola. Let its equation be $y^2 = 4ax$. Since it passes through the point P(2,3), we have $3^2 = 4 \times a(2) \Rightarrow a = \frac{9}{8}$. So its equation is

$$y^2 = 4 \times \frac{9}{8}x \Rightarrow y^2 = \frac{9}{2}x \Rightarrow 2y^2 - 9x = 0$$

c) It is given that vertex of the parabola is V(0,0) and symmetric about the y - axis, so its equation is $x^2 = 4ay$ or $x^2 = -4ay$, since the parabola passes through the point P(5,2). So it lies in the 1st quadrant

$\therefore$ It is a upward parabola. Let its equation be $x^2 = 4ay$, since it passes through the point P(5,2) we have $5^2 = 4 \times a \times 2 \Rightarrow a = \frac{25}{8}$ so its equation is given by

$$x^2 = 4ay \Rightarrow x^2 = 4 \times \frac{25}{8}y \Rightarrow 2x^2 - 25y = 0$$

d) It is given that the vertex of the parabola is V(0,0) and symmetric about the y-axis so its equation is $x^2 = 4ay$ or $x^2 = -4ay$ since the parabola passes through the point P(2, –3) so it lies in the 4th quadrant.

$\therefore$ It is a downward parabola. Let its equation be $x^2 = -4ay$, since it passes through the point P(2, –3) we have $2^2 = -4 \times a(-3) \Rightarrow a = \frac{1}{3}$ so its equation is

$$x^2 = -4 \times \frac{1}{3}y \Rightarrow x^2 = -\frac{4}{3}y \Rightarrow 3x^2 + 4y = 0$$

e) It is given that vertex of the parabola is V(0,0) and symmetric about the y - axis so its equation is $x^2 = -4ay$ or $x^2 = 4ay$ since parabola passes through the point P(–1, –3). So it lies in the 3rd quadrant.

$\therefore$ It is a downward parabola, let its equation be $x^2 = -4ay$, since it passes through the point P(–1,–3), we have $(-1)^2 = -4 \times a \times (-3) \Rightarrow a = \frac{1}{12}$. So its equation is

$$x^2 = -4\left(\frac{1}{12}\right)y \Rightarrow x^2 = \frac{-y}{3} \Rightarrow 3x^2 + y = 0$$

f) It is given that vertex of the parabola is V(0,0) and symmetric about y - axis so, its equation is $x^2 = 4ay$ or $x^2 = -4ay$, since the parabola passes through the point $p\left(\frac{1}{2}, 2\right)$ so it lies in the 1st quadrant.

$\therefore$ It is a upward parabola. Let its equation be $x^2 = 4ay$

Since it passes through the point $P\left(\frac{1}{2}, 2\right)$ we have $\left(\frac{1}{2}\right)^2 = 4 \times a \times 2 \Rightarrow a = \frac{1}{32}$

so its equation is $x^2 = 4\left(\frac{1}{32}\right)y \Rightarrow x^2 = \frac{1}{8}y$ or $8x^2 - y = 0$

Exercise 16.1

One mark questions:

I. 1. Write the characteristics of the following parabolas

a) $y^2 = 16x$ b) $y^2 = -8x$ c) $3x^2 = -8y$

d) $x^2 = 8y$ e) $y^2 = 8x$ f) $y^2 + 4x = 0$

g) $3x^2 + 4y = 0$ h) $x^2 + 16y = 0$

2. If the length of the latus rectum of $y^2 = 8kx$ is 4 find k.

3. If the length of the latus rectum of $x^2 = 4ky$ is 8, find the value of k.

Two marks questions:

II. 1. Find the equation of the parabola given that its

a) vertex is (0,0) and focus is (4,0)

b) vertex is (0,0) and directrix is $y = -3$

c) focus is (1,0) and directrix is $x = -1$

d) focus is (–4,0) and directrix is $x = 4$

e) focus is (0,–3) and directrix is $y = 3$

f) focus is (0,6) and vertex is (0,0)

g) vertex is (0,0), axis is y axis and passes through $\left(\frac{1}{2}, 2\right)$

h) vertex is (0,0), axis is y axis and passes through (–1, –3)

ANSWERS 16.1

	Vertex	Focus	Directrix	Axis	Tangent	L and L¹ Co-rodinates	LR length	Equation of LR
I.	a) (0,0)	(4,0)	$x = -4$	$y = 0$	$x = 0$	(4,8), (4, –8)	16	$x = 4$
	b) (0,0)	(–2,0)	$x = 2$	$y = 0$	$x = 0$	(–2,4), (–2,–4)	8	$x = -2$
	c) (0,0)	$\left(0,-\frac{2}{3}\right)$	$y = \frac{2}{3}$	$x = 0$	$y = 0$	$\left(\frac{4}{3},-\frac{2}{3}\right)\left(\frac{-4}{3},\frac{-2}{3}\right)$	$\frac{8}{3}$	$y = \frac{2}{3}$
	d) (0,0)	(0,2)	$y = -2$	$x = 0$	$y = 0$	(4,2), (–4,2)	8	$y = 2$
	e) (0,0)	(0,2)	$x = -2$	$y = 0$	$x = 0$	(2,4), (2,–4)	8	$x = 2$
	f) (0,0)	(–1,0)	$x = 1$	$y = 0$	$x = 0$	(–1,2), (–1,–2)	4	$x = 1$
	g) (0,0)	$\left(0,-\frac{1}{3}\right)$	$y = \frac{1}{3}$	$x = 0$	$y = 0$	$\left(\frac{2}{3},-\frac{1}{3}\right),\left(-\frac{2}{3},-\frac{1}{3}\right)$	$\frac{4}{3}$	$y = -\frac{1}{3}$
	h) (0,0)	(0,–4)	$y = 4$	$x = 0$	$y = 0$	(8,–4)(–8,–4)	16	$y = -4$

2. a) $k = \frac{1}{2}$ 3. $k = 2$

II. 1. a) $y^2 = 16x$ b) $x^2 = 12y$ c) $y^2 = 4x$

d) $y^2 = -16x$ e) $x^2 = -12y$ f) $x^2 = 24y$

g) $8x^2 - y = 0$ h) $3x^2 + y = 0$

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UNIT V - CALCULUS

Chapter	Title	No. of Teaching hrs.
17.	LIMITS AND CONTINUITY OF A FUNCTION	8 hrs
18.	DIFFERENTIAL CALCULUS	10 hrs
19.	APPLICATION OF DERIVATIVES	08 hrs
20.	INDEFINITE INTEGRALS	08 hrs
21.	DEFINITE INTEGRALS AND APPLICATION TO AREAS	08 hrs
	TOTAL TEACHING HOURS	**42 hrs**

Chapter

17 LIMIT AND CONTINUITY OF A FUNCTION

17.1 Introduction :

Calculus is a latin word which means pebble or a small stone used for calculating. The word calculation is also derived from the same latin word. Sir Issac Newton (1642-1727 A.D) and the German mathematician G.W. Leibnitz (1646 - 1716 A.D) invented and developed the subject independently and almost simultaneously. In this chapter we shall study about different types of functions, definition of limit, standard limits and continuity.

17.2 Variables and Constants:

A quantity which changes in its values is called a **variable.**

If a variable takes real values only then the variable is called a real variable.

For example : $x, y z$.... etc are variables.

A quantity which remains same in its value is called a **constant.**

If the value of a constant is a real number then it is called real constants.

For example: 5,11, e .... are constants.

17.3 Definition of a function

If to each value of real varible *'x'* a unique real number *'y'* is associated by means of rule 'f' then we say the variable y is a real valued function of the real variable x. This is denoted by $y = f(x)$.

The variable x is called **independent** variable

The variable y is called **dependent** variable

17.4 Types of functions

1. **Polynomial function:**

   A function of the form

   $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots\ldots + a_n$

   where $a_0 \neq 0$, n is a non-negative integer and $a_0, a_1, a_2 \ldots\ldots\ldots\ldots a_n$ are real constants.

   Ex : $f(x) = x^4 + 5x^2 + 2x + 1$

2. **Modulus Function**

   The function which associates to each real number x, the number $|x|$ is called the modulus function.

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

Ex: $|3| = 3, |-3| = 3$

3. Exponential function

A function of the form $f(x) = e^x$ where $e > 0$ amd $e \neq 1$ is called an exponential function.

The value of e = 2.71828.... and is an irrational number.

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots\ldots\ldots\ldots + \frac{1}{\lfloor n} + \ldots\ldots\ldots$$

4. Logarithmic function

A function of the form $f(x) = \log_a x$ where $a > 0$ and $a \neq 1$ is called a logarithmic function with base 'a'.

For $a > 0$, $a \neq 1$, $\log_a x = y \Rightarrow a^y = x$

Logarithms with base e are called **natural logarithms.**

5. Rational function

The function defined by $y = \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial functions is called a rational function.

$Ex: \ y = \frac{2x-4}{3x^2+2x+1}$

17.5 Limit of a function

Consider the function $f(x) = \frac{x^2-1}{x-1}$ Clearly, $f(1) = \frac{0}{0}$, which is meaningless.

Thus $f(x)$ is not defined at $x = 1$

Now, $f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = (x+1)$, only when $x \neq 1$

If we give to x, a value not exactly 1 but slightly more than 1 then clearly, the value of $f(x)$ is slightly more than 2. Now if we go on decreasing this value and take it nearer to 1 then clearly the value of $f(x)$ will come nearer to 2, as shown below.

If $x = 1.1$ then $f(x) = 2.1$

$x = 1.01$ then $f(x) = 2.01$

$x = 1.001$ then $f(x) = 2.001$

....

Thus as the value of x approaches 1, the value of $f(x)$ approaches 2

i.e. As $x \to 1$, $f(x) \to 2$

(The symbol '$\to$' stands for 'approaches to' or tends to)

Similarly if we give to x, a value slightly less than 1 then the value of $f(x)$ is slightly less than 2. Now, if we go on increasing this value and take it nearer to 1, the value of $f(x)$ will come nearer to 2, as shown below.

If $x = 0.9$ then $f(x) = 1.9$

$x = 0.99$ then $f(x) = 1.99$

$x = 0.999$ then $f(x) = 1.999$

....

Thus as the value of x approaches 1, the value of $f(x)$ approaches 2.

i.e. As $x \to 1, f(x) \to 2$

We express this fact as $\lim_{x \to 1}\left[\frac{x^2 - 1}{x - 1}\right] = 2$

In general a number 'l' is called the limit of a function $f(x)$ as x tends to *'a'* if the value of $f(x)$ is very nearly equal to 'l', whenever the value of x is very nearly equal to *'a'*. We write this as $\lim_{x \to a}[f(x)] = l$

Definition of Limit

A function $y = f(x)$ is said to tend to limit 'l' as x tends to *'a'* iff the numerical difference **$|f(x) - l|$ can be made as small as we like by taking x, nearer and nearer to *'a'*.**

17.6 Algebra of Limits

(properties of limits without proof)

1. The limit of a constant function is the same constant.

   i.e. If $f(x) = k$, a constant function then $\lim_{x \to a}[f(x)] = k$

2. The limit of a scalar product is equal to the scalar product of the limit.

   i.e. $\lim_{x \to a}[k.f(x)] = k \lim_{x \to a}[f(x)]$

3. The limit of a sum is equal to the sum of the limits.

   i.e $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a}[f(x)] + \lim_{x \to a}[g(x)]$

4. The limit of a difference is equal to the difference of the limits.

   i.e $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a}[f(x)] - \lim_{x \to a}[g(x)]$

5. The limit of a product is equal to the product of the limits.

   i.e. $\lim_{x \to a}[f(x)\, g(x)] = \lim_{x \to a}[f(x)] \lim_{x \to a}[g(x)]$

6. The limit of a quotient is equal to the quotient of the limits provided the limit of the denominator is non–zero

i.e. $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{\lim\limits_{x\to a}[f(x)]}{\lim\limits_{x\to a}[g(x)]}$, provided $\lim\limits_{x\to a}[g(x)]\neq 0$

7. The limit of the nth root is equal to the nth root of the limit provided that the nth root of the limit is a real number.

i.e. $\lim\limits_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim\limits_{x\to a}f(x)}$, provided $\sqrt[n]{\lim\limits_{x\to a}f(x)}$ is a real number.

Note:

1. Limits of polynomials can be found by substitution

$$\lim_{x\to a}\left[p_n x^n+p_{n-1}x^{n-1}+.....+p_1x+p_0\right]$$

$$=\ p_n a^n+p_{n-1}a^{n-1}+.....+p_1a+p_0$$

2. Indeterminate forms:

$\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $0\times\infty$, $\infty-\infty$, 0^o, ∞^o, 1^∞

3. Meaningful forms:

(i) $\dfrac{a}{0}=\infty(a\neq 0)$ (ii) $\dfrac{0}{a}=0(a\neq 0)$ (iii) $\infty\times a=\infty$

(iv) $\infty\times\infty=\infty$ (v) $0\times 0=0$ (vi) $\infty-b=\infty$

(vii) $a-\infty=-\infty$ (viii) $\infty+\infty=\infty$

17.7 Evaluation of limits:

Type A: Direct substitution method

Put $x=a$ in the given function. If *f(a)* is a definite value then $\lim\limits_{x\to a}[f(x)]=f(a)$

Example 1:

Evaluate the following limits

1. $\lim\limits_{x\to 1}\left[x^2+6x+4\right]=\lim\limits_{x\to 1}x^2+\lim\limits_{x\to 1}6x+\lim\limits_{x\to 1}4$

$$=1^2+6(1)+4$$

$$=\ 11$$

2. $\lim\limits_{x\to 4}\left[\dfrac{x^3+4}{1-x}\right] = \dfrac{\lim\limits_{x\to 4}(x^3+4)}{\lim\limits_{x\to 4}(1-x)}$

$= \dfrac{4^3+4}{1-4} = -\dfrac{68}{3}$

3. $\lim\limits_{x\to 2}\sqrt{36-x^2} = \sqrt{\lim\limits_{x\to 2}(36-x^2)}$

$= \sqrt{36-\lim\limits_{x\to 2}x^2} = \sqrt{36-2^2} = \sqrt{36-4} = \sqrt{32}$

Type B : Method of factors:

In this method, *f(x)* and *g(x)* are factorised and the quotient $\dfrac{f(x)}{g(x)}$ is simplified by cancelling the common factors. If the limit of the resulting denominator as $x \to a$ is non-zero, then the quotient rule of limits is applied to get the required limit.

Example 2 :

Evaluate the following limits

1. $\lim\limits_{x\to 2}\dfrac{x^2-3x+2}{x^2-x-2}$

Solution : At $x = 2$, $\dfrac{x^2-3x+2}{x^2-x-2} = \dfrac{0}{0}$ form

$\therefore \lim\limits_{x\to 2}\dfrac{x^2-3x+2}{x^2-x-2} = \lim\limits_{x\to 2}\dfrac{(x-2)(x-1)}{(x-2)(x+1)}$

$= \lim\limits_{x\to 2}\dfrac{x-1}{x+1} = \dfrac{2-1}{2+1} = \dfrac{1}{3}$

2. $\lim\limits_{x\to 3}\left(\dfrac{x^2-9}{x-3}\right)$

Solution: At $x = 3$, $\dfrac{x^2-9}{x-3} = \dfrac{0}{0}$ form

$\therefore \lim\limits_{x\to 3}\left(\dfrac{x^2-9}{x-3}\right) = \lim\limits_{x\to 3}\left[\dfrac{(x+3)(x-3)}{x-3}\right] = \lim\limits_{x\to 3}(x+3)$

$= 3 + 3 = 6$

3. $\lim\limits_{x\to -1}\left[\dfrac{x^3+1}{2x^2+5x+3}\right]$

Solution: At $x=-1$, $\dfrac{x^3+1}{2x^2+5x+3}=\dfrac{0}{0}$ form

$$\therefore \lim_{x\to -1}\left[\frac{x^3+1}{2x^2+5x+3}\right] = \therefore \lim_{x\to -1}\left[\frac{(x+1)(x^2-x+1)}{(x+1)(2x+3)}\right]$$

$$= \therefore \lim_{x\to -1}\left[\frac{x^2-x+1}{2x+3}\right] = \frac{(-1)^2-(-1)+1}{2(-1)+3}$$

$$= \frac{3}{1} = 3$$

4. $\lim\limits_{x\to 2}\left[\dfrac{1}{x-2}-\dfrac{4}{x^3-2x^2}\right]$

Solution: At $x=2$, $\dfrac{1}{x-2}-\dfrac{4}{x^3-2x^2}=\infty-\infty$ form

$$\therefore \lim_{x\to 2}\left[\frac{1}{x-2}-\frac{4}{x^3-2x^2}\right] = \lim_{x\to 2}\left[\frac{1}{x-2}-\frac{4}{x^2(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x^2-4}{x^2(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{(x-2)(x+2)}{x^2(x-2)}\right]$$

$$= \lim_{x\to 2}\left[\frac{x+2}{x^2}\right]$$

$$= \frac{4}{4} = 1$$

Type C: Method of Rationalisation

In this method the expressions involving square roots are rationalised. After the process of rationalisation, if the limit of the resulting denominator as $x\to a$ is non-zero, then the quotient rule of limits is applied to get the required limit.

Example 3:

Evaluate the following limits

1. $\lim_{x\to 0}\dfrac{\sqrt{2-x}-\sqrt{2+x}}{x}$

Solution:

At $x=0$, $\dfrac{\sqrt{2-x}-\sqrt{2+x}}{x}=\dfrac{0}{0}$ form Rationalise in the Numerator

$$\therefore \lim_{x\to 0}\frac{\sqrt{2-x}-\sqrt{2+x}}{x} = \lim_{x\to 0}\left[\frac{\sqrt{2-x}-\sqrt{2+x}}{x}\times\frac{\sqrt{2-x}+\sqrt{2+x}}{\sqrt{2-x}+\sqrt{2+x}}\right]$$

$$= \lim_{x\to 0}\left[\frac{(2-x)-(2+x)}{x\left(\sqrt{2-x}+\sqrt{2+x}\right)}\right]$$

$$= \lim_{x\to 0}\left[\frac{-2x}{x\left(\sqrt{2-x}+\sqrt{2+x}\right)}\right]$$

$$= \lim_{x\to 0}\frac{-2}{\sqrt{2-0}+\sqrt{2+0}}$$

$$= \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

2. $\lim_{x\to 3}\left[\dfrac{x^2-9}{\sqrt{3x+7}-\sqrt{5x+1}}\right]$

Solution: At $x=3$, $\dfrac{x^2-9}{\sqrt{3x+7}-\sqrt{5x+1}}=\dfrac{0}{0}$ form

$$\therefore \lim_{x\to 3}\left[\frac{x^2-9}{\sqrt{3x+7}-\sqrt{5x+1}}\right] = \lim_{x\to 3}\left[\frac{x^2-9}{\sqrt{3x+7}-\sqrt{5x+1}}\times\frac{\sqrt{3x+7}+\sqrt{5x+1}}{\sqrt{3x+7}+\sqrt{5x+1}}\right]$$

$$= \lim_{x\to 3}\left[\frac{(x^2-9)\left(\sqrt{3x+7}+\sqrt{5x+1}\right)}{(3x+7)-(5x+1)}\right]$$

$$= \lim_{x\to 3}\left[\frac{(x^2-9)\left(\sqrt{3x+7}+\sqrt{5x+1}\right)}{-2x+6}\right]$$

$$= \lim_{x\to 3}\left[\frac{(x-3)(x+3)\left(\sqrt{3x+7}+\sqrt{5x+1}\right)}{-2(x-3)}\right]$$

$$= \lim_{x\to 3}\left[\frac{-(x+3)\left(\sqrt{3x+7}+\sqrt{5x+1}\right)}{2}\right]$$

$$= \frac{-(3+3)\left(\sqrt{9+7}+\sqrt{15+1}\right)}{2}$$

$$= \frac{-6\left(\sqrt{16}+\sqrt{16}\right)}{2} = \frac{-6(4+4)}{2}$$

$$= \frac{-6(8)}{2} = -24$$

3. $\lim_{x\to 1}\left[\frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}\right]$

Solution: At $x=1$, $\frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}=\frac{0}{0}$ form

$$\therefore \lim_{x\to 1}\frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1} = \lim_{x\to 1}\left[\frac{\left(\sqrt{3+x}-\sqrt{5-x}\right)\left(\sqrt{3+x}+\sqrt{5-x}\right)}{\left(x^2-1\right)\left(\sqrt{3+x}+\sqrt{5-x}\right)}\right]$$

$$= \lim_{x\to 1}\frac{(3+x)-(5-x)}{(x^2-1)\left(\sqrt{3+x}+\sqrt{5-x}\right)}$$

$$= \lim_{x\to 1}\frac{2x-2}{(x^2-1)\left(\sqrt{3+x}+\sqrt{5-x}\right)}$$

$$= \lim_{x\to 1}\frac{2(x-1)}{(x-1)(x+1)\left(\sqrt{3+x}+\sqrt{5-x}\right)}$$

$$= \lim_{x\to 1} \frac{2}{(x+1)\left(\sqrt{3+x}+\sqrt{5-x}\right)}$$

$$= \frac{2}{(1+1)\left(\sqrt{3+1}+\sqrt{5-1}\right)} = \frac{1}{4}$$

17.8 Evaluation of Standard Limits

Theorem 1 : If *n* is a rational number and a is non-zero real number, then

prove that $\lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = n.a^{n-1}$

Proof :

Case 1: Let *n* be a positive integer.

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}.a + x^{n-3}a^2 + ... + a^{n-1})$$

$\div$ by $(x-a)$ and apply lim $x \to a$ on both sides

$$\therefore \lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = \lim_{x\to a}\left[\frac{(x-a)\left(x^{n-1} + x^{n-2}.a + ... + a^{n-1}\right)}{x-a}\right]$$

$$= \lim_{x\to a}\left[x^{n-1} + x^{n-2}.a + ... + a^{n-1}\right]$$

$$= a^{n-1} + a^{n-2}.a + ... + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + ... + a^{n-1}$$

$$= n.\, a^{n-1}$$

$$\therefore \lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = n.\, a^{n-1}$$

Case 2: Let *n* be a negative integer

put $n = -m, \quad m > 0$

$$\text{consider } \frac{x^n - a^n}{x-a} = \frac{x^{-m} - a^{-m}}{x-a}$$

$$= \left[\frac{\frac{1}{x^m} - \frac{1}{a^m}}{x-a}\right]$$

$$= \frac{a^m - x^m}{x^m a^m (x-a)}$$

$$= \frac{-1}{a^m x^m}\left(\frac{x^m - a^m}{x-a}\right) \text{ apply lim } x \to a \text{ on both sides}$$

$$\therefore \lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = \lim_{x\to a}\frac{-1}{a^m x^m}\left(\frac{x^m - a^m}{x-a}\right)$$

$$= \frac{-1}{a^m a^m} m.a^{m-1} \text{ [By case 1]}$$

$$= (-m)\frac{a^{m-1}}{a^{2m}} = (-m)a^{-m-1}$$

$$= n.\, a^{n-1} \quad [\because -m = n]$$

$$\therefore \lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = n.a^{n-1}$$

Case 3: Let $n = \frac{p}{q}$ where p and q are integers and $q \neq 0$

$$\therefore \lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = \lim_{x\to a}\left[\frac{x^{p/q} - a^{p/q}}{x-a}\right]$$

$$= \lim_{x\to a}\frac{\left(x^{1/q}\right)^p - \left(a^{1/q}\right)^p}{x-a}$$

Let $x^{1/q} = y$ and $a^{1/q} = b$

$\Rightarrow x = y^q$ and $a = b^q$

Also $x \to a$ changes to $y \to b$

$$\therefore \lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = \lim_{y\to b}\left[\frac{y^p - b^p}{y^q - b^q}\right]$$

$$= \lim_{y\to b}\left[\frac{\frac{y^p - b^p}{y-b}}{\frac{y^q - b^q}{y-b}}\right]$$

$$= \frac{p \cdot b^{p-1}}{q \cdot b^{q-1}} \text{ (by case 1)}$$

$$= \frac{p}{q} \cdot b^{p-1-q+1} = \frac{p}{q} \cdot b^{p-q}$$

$$= \frac{p}{q} \cdot \left(a^{1/q}\right)^{p-q} = \frac{p}{q} \cdot a^{p/q-1}$$

$$= n \cdot a^{n-1} \quad [\because p/q = n]$$

$$\because \lim_{x \to a}\left[\frac{x^n - a^n}{n-a}\right] = n \cdot a^{n-1}$$

Hence the result is true for all rational values of n.

Example 4 :

Evaluate the following limits

1. $\lim_{x \to 2}\left[\frac{x^3 - 8}{x-2}\right] = \lim_{x \to 2}\left[\frac{x^3 - 2^3}{x-2}\right]$

$$= 3 \cdot 2^{3-1} = 3(2)^2$$

$$= 3(4) = 12$$

2. $\lim_{x \to -2}\left[\frac{x^5 + 32}{x+2}\right] = \lim_{x \to -2}\left[\frac{x^5 - (-2)^5}{x-(-2)}\right]$

$$= 5(-2)^{5-1} = 5(-2)^4$$

$$= 5(16) = 80$$

3. $\lim_{x \to 2}\frac{x^{10} - 1024}{x^4 - 16} = \lim_{x \to 2}\frac{\frac{x^{10} - 2^{10}}{x-2}}{\frac{x^4 - 2^4}{x-2}} = \frac{\lim_{x \to 2}\left[\frac{x^{10} - 2^{10}}{x-2}\right]}{\lim_{x \to 2}\left[\frac{x^4 - 2^4}{x-2}\right]}$ Divide both Nr and Dr by $(x-2)$

$$= \frac{10(2)^{10-1}}{4(2)^{4-1}} = \frac{10(2)^9}{4(2)^3} = \frac{10(2)^9}{2^2 \times 2^3}$$

$$= 10(2)^{9-5} = 10(2)^4 = 10(16)$$

$$= 160$$

4. $\lim\limits_{x\to 2}\dfrac{x^{5/2}-2^{5/2}}{x^{3/2}-2^{3/2}} = \lim\limits_{x\to 2}\dfrac{\left(\dfrac{x^{5/2}-2^{5/2}}{x-2}\right)}{\left(\dfrac{x^{3/2}-2^{3/2}}{x-2}\right)}$

$$= \frac{\left(\frac{5}{2}\right)2^{\frac{5}{2}-1}}{\left(\frac{3}{2}\right)2^{\frac{3}{2}-1}} = \frac{5}{3}\cdot 2$$

$$= \frac{10}{3}$$

5. $\lim\limits_{x\to 2}\dfrac{x^5\sqrt{x}-32\sqrt{2}}{x^3\sqrt{x}-8\sqrt{2}}$

$$= \lim_{x\to 2}\frac{x^5\cdot x^{1/2}-2^5\cdot 2^{1/2}}{x^3\cdot x^{1/2}-2^3\cdot 2^{1/2}}$$

$$= \lim_{x\to 2}\frac{x^{11/2}-2^{11/2}}{x^{7/2}-2^{7/2}}$$

$$= \lim_{x\to 2}\frac{\dfrac{x^{11/2}-2^{11/2}}{x-2}}{\dfrac{x^{7/2}-2^{7/2}}{x-2}} \quad \div \text{ both Nr and Dr by } (x-2)$$

$$= \frac{\frac{11}{2}\cdot 2^{\frac{11}{2}-1}}{\frac{7}{2}\cdot 2^{\frac{7}{2}-1}}$$

$$= \frac{11}{7}\cdot 2^{\frac{9}{2}-\frac{5}{2}} = \frac{11}{7}\cdot 2^2$$

$$= \frac{11}{7}(4) = \frac{44}{7}$$

6. $\lim\limits_{x\to 0}\dfrac{(x+1)^5-1}{x}$

$= \lim\limits_{x+1\to 1}\dfrac{(x+1)^5-1}{(x+1)-1}$

$= \lim\limits_{y\to 1}\dfrac{y^5-1^5}{y-1}$ where $y = x + 1$

$= 5(1)^{5-1} = 5$

7. $\lim\limits_{x\to 3}\dfrac{(x+2)^{5/3}-5^{5/3}}{x-3}$

Put $x + 2 = y$

As $x \to 3$

$\Rightarrow y \to 5$

$= \lim\limits_{x\to 3}\dfrac{(x+2)^{5/3}-(3+2)^{5/3}}{(x+2)-(3+2)}$

$= \lim\limits_{x\to 5}\dfrac{y^{5/3}-5^{5/3}}{y-5} = \dfrac{5}{3}(5)^{5/3-1}$

$= \dfrac{5}{3}(5)^{2/3}$

$= \dfrac{5^{5/3}}{3}$

8. $\lim\limits_{x\to -2}\dfrac{\dfrac{1}{x}+\dfrac{1}{2}}{x+2}$

$= \lim\limits_{x\to -2}\dfrac{x^{-1}-(-2)^{-1}}{x-(-2)}$

$= (-1)\,(-2)^{-1-1} = (-1)\left(\dfrac{1}{4}\right)$

$= \dfrac{-1}{4}$

EXERCISE 17.1

I. One and two marks questions:

1. Evaluate the following limits

1. $\lim\limits_{x\to 4}\left(\dfrac{4x+3}{x-2}\right)$
2. $\lim\limits_{x\to 1}\left(\dfrac{ax^2+bx+c}{cx^2+bx+a}\right)$
3. $\lim\limits_{x\to -3}\left(\dfrac{x^2+9}{x+3}\right)$
4. $\lim\limits_{x\to 3}\left(\dfrac{x^2-4x}{x-2}\right)$
5. $\lim\limits_{x\to 1}\left(\dfrac{x^3-1}{x-1}\right)$
6. $\lim\limits_{x\to 1/2}\left(\dfrac{4x^2-1}{2x-1}\right)$
7. $\lim\limits_{x\to 2}\dfrac{x-2}{x^{1/3}-2^{1/3}}$
8. $\lim\limits_{x\to -1}\dfrac{x^9+1}{x^5+1}$

II. 3 marks questions:

1. $\lim\limits_{x\to 3}\left[\dfrac{x^2-4x+3}{x^2-2x-3}\right]$
2. $\lim\limits_{x\to 2}\left[\dfrac{2x^2-5x+2}{x^2-3x+2}\right]$
3. $\lim\limits_{x\to 2}\left[\dfrac{3x^2-x-10}{x^2-4}\right]$
4. $\lim\limits_{x\to 3}\dfrac{x^4-81}{2x^2-5x-3}$
5. $\lim\limits_{x\to \frac{1}{2}}\dfrac{2x^2+3x-2}{2x^2-3x+1}$
6. $\lim\limits_{x\to 2}\dfrac{x^2-4}{x\sqrt{x}-2\sqrt{2}}$
7. $\lim\limits_{x\to 0}\left[\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}\right]$
8. $\lim\limits_{x\to 1}\dfrac{\sqrt{x+3}-2}{x-1}$
9. $\lim\limits_{x\to a}\dfrac{x^{3/8}-a^{3/8}}{x^{1/3}-a^{1/3}}$

III. 5 marks questions:

1. $\lim\limits_{x\to 3}\left[\dfrac{1}{x-3}-\dfrac{2}{x^2-4x+3}\right]$
2. $\lim\limits_{x\to 1}\left[\dfrac{1}{x^2+x-2}-\dfrac{x}{x^3-1}\right]$
3. $\lim\limits_{x\to 2}\left[\dfrac{1}{x-2}-\dfrac{2(2x-3)}{x^3-3x^2+2x}\right]$
4. $\lim\limits_{x\to 2}\left[\dfrac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}\right]$

ANSWERS 17.1

I. 1) $\dfrac{19}{2}$ 2) 1 3) –6 4) –3 5) 3 6) 2

7) $3(2)^{\frac{2}{3}}$ 8) $\dfrac{9}{5}$

II. 1) $\frac{1}{2}$ 2) 3 3) $\frac{11}{4}$ 4) $\frac{108}{7}$ 5) –5

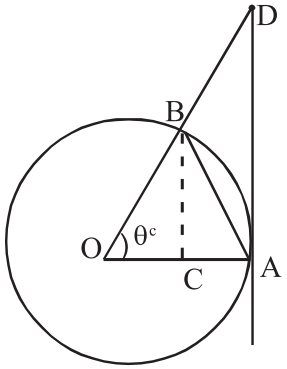
6) $\frac{4\sqrt{2}}{3}$ 7) 1 8) $\frac{1}{4}$ 9) $\frac{9}{8}a^{\frac{1}{24}}$

III. 1) $\frac{1}{2}$ 2) $\frac{-1}{9}$ 3) $\frac{-1}{2}$ 4) –8

Evaluation of Trigonometrical Limits:

Theorem 2 : If angel θ is measured in radians, then prove that $\lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right) = 1$

Proof:



Let O be the centre of a unit circle. Let us first assume that the angle θ is positive in radian.

Let $\angle AOB = \theta^c$. From A draw AD ⊥ OA, meeting OB produced to D.

Let OB = OA = radian.

From the figure, we have Area of ΔOAB < Area of sector OAB < Area of ΔOAD

$$\Rightarrow \frac{1}{2} \cancel{OA} \cdot BC < \frac{1}{2} OA^2 \cdot \theta < \frac{1}{2} \cancel{OA} \cdot DA \quad .....(1)$$

[$\because$ Area of a sector = $\frac{1}{2}$ (radius)^2 × angle in radians]

From the Δ^{le} DOA $\tan\theta = \frac{DA}{OA}$ $\therefore$ $\boxed{DA = r \tan\theta}$

From the ΔBOC, $\sin\theta = \frac{BC}{OB}$ $\therefore$ $\boxed{BC = r \sin\theta}$

(1) becomes,

$\Rightarrow$ BC < θ < DA

$\Rightarrow$ sinθ < θ < tanθ (Dividing by sin θ)

$$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\Rightarrow \cos\theta < \frac{\sin\theta}{\theta} < 1$$

apply $\text{Lt}\ \theta \to 0$

$$\lim_{\theta\to 0} \cos\theta < \lim_{\theta\to 0} \frac{\sin\theta}{\theta} < \lim_{\theta\to 0} 1$$

$$\Rightarrow 1 < \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) < 1$$

$$\Rightarrow \boxed{\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1}$$

$\therefore$ If θ is positive then $\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1$ .....(2)

If angle θ is negative, then $-\theta$ is positive and

$$\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = \lim_{\theta\to 0}\frac{-\sin\theta}{-\theta}$$

$$= \lim_{-\theta\to 0}\frac{\sin(-\theta)}{-\theta} = 1$$

[$\because$ $-\theta$ is +ve and so (2) is applicable]

$\therefore$ For any angle θ, $\boxed{\lim_{\theta\to 0}\left[\frac{\sin\theta}{\theta}\right] = 1}$

Corollary:

1. $\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = 1$

Proof: $\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = \lim_{\theta\to 0}\frac{1}{\frac{\sin\theta}{\theta}}$

$$= \frac{\lim_{\theta\to 0} 1}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}} = \frac{1}{1} = 1$$

2. $\lim_{\theta \to 0}\left(\frac{\tan\theta}{\theta}\right) = 1$

Proof: $\lim_{\theta \to 0}\left(\frac{\tan\theta}{\theta}\right) = \lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta}\right)$

$= \lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta \to 0}\frac{1}{\cos\theta}\right)$

$= (1)\left(\frac{1}{1}\right) = 1$

Example 5 :

Evaluate the following limits:

1. $\lim_{x \to 0}\frac{\sin x^0}{x} = \lim_{x \to 0}\frac{\sin\left(\frac{x\pi}{180}\right)}{x}$ $\left[\because \begin{matrix} 180^o = \pi^c \\ x^o = \left(\frac{x\pi}{180}\right)^c \end{matrix}\right]$

Solution: $\lim_{x \to 0}\frac{\sin\left(\frac{x\pi}{180}\right)}{\frac{x\pi}{180}\cdot\frac{180}{\pi}} = \frac{\pi}{180}\cdot\lim_{x \to 0}\frac{\sin\left(x\pi/180\right)}{\left(x\pi/180\right)}$

$= \frac{\pi}{180}.1 = \frac{\pi}{180}$

2. $\lim_{x \to 0}\frac{\sin ax}{\sin bx}$, $ab \neq 0$

$ab \neq 0 \Rightarrow a \neq 0$ and $b \neq 0$

Solution : $\therefore \lim_{x \to 0}\frac{\sin ax}{\sin bx} = \lim_{x \to 0}\frac{\frac{\sin ax}{ax}\cdot ax}{\frac{\sin bx}{bx}\cdot bx}$

$= \frac{a}{b}\cdot\frac{\lim_{x \to 0}\frac{\sin ax}{ax}}{\lim_{x \to 0}\frac{\sin bx}{bx}} = \frac{a}{b}\cdot\frac{1}{1} = \frac{a}{b}$

3. $\lim_{x\to 0}\dfrac{ax+x\cos x}{b\sin x}$

÷ both Nr and Dr by x

Solution: $\lim_{x\to 0}\dfrac{\frac{1}{x}(ax+x\cos x)}{\frac{b\sin x}{x}}$

$$= \lim_{x\to 0}\frac{a+\cos x}{b\cdot\frac{\sin x}{x}} = \frac{a+\lim_{x\to 0}\cos x}{b\cdot\lim_{x\to 0}\frac{\sin x}{x}} = \frac{a+1}{b}$$

4. $\lim_{x\to 0}\dfrac{\cos^2 x}{1-\sin x}$

Solution: $\lim_{x\to 0}\dfrac{1-\sin^2 x}{1-\sin x} = \dfrac{\lim_{x\to 0}(1+\sin x)\cancel{(1-\sin x)}}{\cancel{(1-\sin x)}} = \lim_{x\to 0}(1+\sin x)$

$= 1 + 0 = 1$

5. $\lim_{\theta\to 0}\dfrac{1-\cos m\theta}{1-\cos n\theta}$

From sub multiple angle

Solution: $\lim_{\theta\to 0}\dfrac{2\sin^2\left(m\theta/2\right)}{2\sin^2\left(n\theta/2\right)} = \lim_{\theta\to 0}\left[\dfrac{\sin\left(m\theta/2\right)}{\sin\left(n\theta/2\right)}\right]^2$ $\left(\cos\theta = 1-2\sin^2\theta/2,\ \therefore 2\sin^2\dfrac{\theta}{2} = 1-\cos\theta\right)$

$$= \lim_{\theta\to 0}\left[\frac{\sin\left(\frac{m\theta}{2}\right)}{\frac{m\theta}{2}}\times\frac{\frac{n\theta}{2}}{\sin\left(\frac{n\theta}{2}\right)}\times\frac{m}{n}\right]^2$$

$$= \frac{m^2}{n^2}\left[\lim_{\theta\to 0}\frac{\sin\left(m\theta/2\right)}{m\theta/2}\times\lim_{\theta\to 0}\frac{n\theta/2}{\sin n\theta/2}\right]^2$$

$$= \frac{m^2}{n^2}(1\times 1)^2 = \frac{m^2}{n^2}$$

6. $\lim_{x\to 0}\dfrac{\sin 5x}{\tan 3x}$

Solution: $\lim_{x\to 0}\left[\dfrac{\sin 5x}{5x}\times\dfrac{3x}{\tan 3x}\times\dfrac{5}{3}\right]$

$= \dfrac{5}{3}\cdot\lim_{x\to 0}\left(\dfrac{\sin 5x}{5x}\right)\cdot\lim_{x\to 0}\left(\dfrac{3x}{\tan 3x}\right)$

$= \dfrac{5}{3}\times 1\times 1 = \dfrac{5}{3}$

7. $\lim_{x\to 0}\dfrac{\cos x}{\pi - x} = \dfrac{\lim_{x\to 0}\cos x}{\lim_{x\to 0}\pi - x}$ $\quad(\because \cos 0^{o} = 1)$

$= \dfrac{1}{\pi - 0} = \dfrac{1}{\pi}$

8. $\lim_{x\to 0}(\operatorname{cosec} x - \cot x) = \lim_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}\right)$

Solution: $= \lim_{x\to 0}\left(\dfrac{1-\cos x}{\sin x}\right)$

$= \lim_{x\to 0}\dfrac{2\sin^{2} x/2}{2\sin x/2\cos x/2}$ [from sub multiple angle]

$= \lim_{x\to 0}\dfrac{\sin x/2}{\cos x/2} = \dfrac{0}{1} = 0$

Example 6:

Evaluate the following limits:

1. $\lim_{\theta\to\pi/2}\dfrac{\cot\theta}{\pi/2 - \theta}$

Solution: Let $\theta = \dfrac{\pi}{2} + h$ $\quad\therefore\ h\to 0$ as $\theta\to\dfrac{\pi}{2}$

$$\therefore \lim_{\theta\to\pi/2}\frac{\cot\theta}{\pi/2-\theta} = \lim_{h\to0}\frac{\cot\left(\pi/2+h\right)}{\pi/2-\left(\pi/2+h\right)}$$

$$= \lim_{h\to0}\frac{-\tan h}{-h} = \lim_{h\to0}\frac{\tan h}{h}$$

$$= 1$$

2. $\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$

Solution: Let $x=\pi+h$ $\therefore\ h\to0$ as $x\to\pi$

$$\therefore \lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)} = \lim_{h\to0}\frac{\sin[\pi-(\pi+h)]}{\pi[\pi-(\pi+h)]}$$

$$= \lim_{h\to0}\frac{\sin(-h)}{\pi(-h)} = \lim_{h\to0}\frac{-\sinh}{-\pi h}$$

$$= \frac{1}{\pi}\cdot\lim_{h\to0}\frac{\sin h}{h} = \frac{1}{\pi}(1)$$

$$= \frac{1}{\pi}$$

3. $\lim_{x\to\pi/2}\frac{\tan 2x}{x-\pi/2}$

Solution : Let $x=\pi/2+h$ $\therefore\ h\to0$ as $x\to\pi/2$

$$\therefore \lim_{x\to\pi/2}\frac{\tan 2x}{x-\pi/2} = \lim_{h\to0}\frac{\tan 2\left(\pi/2+h\right)}{\pi/2+h-\pi/2}$$

$$= \lim_{h\to0}\frac{\tan(\pi+2h)}{h}$$

$$= \lim_{h\to0}\frac{\tan 2h}{h} \qquad \therefore \tan(180^\circ+\theta)=\tan\theta$$

$$= 2.\lim_{h\to0}\frac{\tan 2h}{2h} = 2(1) = 2$$

4. $\lim_{x\to\pi/4}\dfrac{\sin x-\cos x}{\pi/4-x}$

Solution: Let $x=\pi/4+h$ $\quad\therefore\ h\to 0$ as $x\to\dfrac{\pi}{4}$

$$\therefore\ \lim_{x\to\pi/4}\frac{\sin x-\cos x}{\pi/4-x} = \lim_{h\to 0}\frac{\sin\left(\pi/4+h\right)-\cos\left(\pi/4+h\right)}{\pi/4-\left(\pi/4+h\right)}$$

$\because$ from compound angle

$$\begin{bmatrix}\sin(A+B)=\sin A\ \cos B+\cos A\sin B\\ \cos(A+B)=\cos A\ \cos B-\sin A\sin B\end{bmatrix}$$

$$= \lim_{h\to 0}\frac{\left(\frac{1}{\sqrt{2}}\cos h+\frac{1}{\sqrt{2}}\sin h\right)-\left(\frac{1}{\sqrt{2}}\cos h-\frac{1}{\sqrt{2}}\sin h\right)}{-h}$$

$$= -\lim_{h\to 0}\frac{\sqrt{2}\sin h}{h}=-\sqrt{2}\times 1=-\sqrt{2}$$

5. $\lim_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 x}$

$$= \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{\sin^3 x} = \lim_{x\to 0}\frac{\sin x-\sin x\cdot\cos x}{\cos x\cdot\sin^3 x}$$

$$= \lim_{x\to 0}\frac{1-\cos x}{\cos x\cdot\sin^2 x} = \lim_{x\to 0}\frac{1}{\cos x(1+\cos x)}$$

$$= \frac{1}{1\times(1+1)} = \frac{1}{2}$$

Exercise 17.2

I. One and Two marks questions:

Evaluate the following limits:

1. $\lim_{x\to 0}\dfrac{\sin ax}{bx}, a,b\neq 0$
2. $\lim_{x\to 0}\dfrac{\sin x^2}{x}$
3. $\lim_{x\to 0}\dfrac{\sin 4x}{\sin 2x}$

4. $\lim_{\theta\to 0}\frac{\sin 3\theta}{\tan 2\theta}$ 5. $\lim_{x\to 0}\frac{\tan 4x}{\tan 3x}$ 6. $\lim_{x\to 0}\frac{x\cos x+\sin x}{x+\tan x}$

7. $\lim_{x\to 0}\frac{\sin 3x+7x}{4x+\sin 2x}$ 8. $\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}$ 9. $\lim_{x\to 0}\frac{1-\cos x}{x}$

10. $\lim_{x\to 0}\frac{x\tan 4x}{1-\cos 4x}$

II. Three and Five marks questions:

Evaluate the following limits:

1. $\lim_{x\to 0}\frac{1-\cos 2x}{\sin^2 2x}$ 2. $\lim_{\theta\to 0}\frac{1-\cos 4\theta}{1-\cos 6\theta}$ 3. $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

4. $\lim_{x\to 0}\frac{\tan 2x-\sin 2x}{x^3}$ 5. $\lim_{x\to 0}\frac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}$ 6. $\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$

ANSWERS 17.2

I. 1. $\frac{a}{b}$ 2. 0 3. 2 4. $\frac{3}{2}$ 5. $\frac{4}{3}$

6. 1 7. $\frac{5}{3}$ 8. 1 9. 0 10. $\frac{1}{2}$

II. 1. $\frac{1}{2}$ 2. $\frac{4}{9}$ 3. 4 4. 4 5. 4 6. 1

17.9 Statement of some standard limits (without proofs)

1. $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)=1$ **2.** $\lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log_e a$ **5.** $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

3. $\lim_{x\to 0}(1+x)^{1/x}=e$ **4.** $\lim_{x\to 0}\frac{1}{x}\log(1+x)=1$ **6.** $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$

Example 7 :

Evaluate the following limits:

1. $\lim_{x\to 0}\left(\frac{e^{3x}-1}{x}\right) = \lim_{x\to 0}\left(\frac{e^{3x}-1}{3x}\times 3\right)$

$$= \quad 3 \times \lim_{x \to 0}\left(\frac{e^{3x}-1}{3x}\right)$$

$$= \quad 3(1) \quad = \quad 3$$

2. $\lim_{x \to 0}\left(\frac{e^{-x}-1}{x}\right) \quad = \quad \lim_{y \to 0}\left(\frac{e^{y}-1}{-y}\right)$ where $-x = y$

$$= \quad -\lim_{y \to 0}\left(\frac{e^{y}-1}{y}\right) \quad \text{or} \quad -\lim_{x \to 0}\left(\frac{e^{-x}-1}{-x}\right)$$

$$= \quad -1$$

3. $\lim_{x \to 0}\left(\frac{e^{x}-e^{-x}}{x}\right) \quad = \quad \lim_{x \to 0}\left\{\frac{(e^{x}-1)-(e^{-x}-1)}{x}\right\}$

$$= \quad \lim_{x \to 0}\left\{\left(\frac{e^{x}-1}{x}\right)-\left(\frac{e^{-x}-1}{x}\right)\right\}$$

$$= \quad \lim_{x \to 0}\left(\frac{e^{x}-1}{x}\right)-\lim_{x \to 0}\left(\frac{e^{-x}-1}{x}\right)$$

$$= \quad 1-\lim_{y \to 0}\left(\frac{e^{y}-1}{-y}\right), \text{ where } -x = y \left[\because \lim_{x \to 0}\left(\frac{e^{x}-1}{x}\right)=1\right]$$

$$= \quad 1+\lim_{y \to 0}\left(\frac{e^{y}-1}{y}\right)$$

$$= \quad 1 + 1 \quad = \quad 2$$

4. $\lim_{x \to 0}\left[\frac{e^{x}+e^{-x}-2}{x^{2}}\right] \quad = \quad \lim_{x \to 0}\left[\frac{e^{2x}+1-2e^{x}}{x^{2}e^{x}}\right]$

$$= \quad \lim_{x \to 0}\left[\left(\frac{e^{x}-1}{x}\right)^{2} \cdot \frac{1}{e^{x}}\right] \quad = \quad \lim_{x \to 0}\left(\frac{e^{x}-1}{x}\right)^{2} \cdot \lim_{x \to 0}\frac{1}{e^{x}}$$

$$= \quad 1^{2} \times \frac{1}{e^{0}} \quad = \quad 1 \times 1 \quad = \quad 1$$

5. $\lim_{x\to 0}\left(\frac{3^x-2^x}{x}\right)$

$$= \lim_{x\to 0}\left\{\frac{(3^x-1)-(2^x-1)}{x}\right\} = \lim_{x\to 0}\left(\frac{3^x-1}{x}\right)-\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)$$

$$= \log 3 - \log 2$$

$$= \log\left(\frac{3}{2}\right)$$

6. $\lim_{x\to 0}\left(\frac{3^{2x}-1}{2^{3x}-1}\right)$

$$= \lim_{x\to 0}\left\{\frac{\frac{3^{2x}-1}{2x}\cdot 2x}{\frac{2^{3x}-1}{3x}\cdot 3x}\right\}$$

$$= \frac{2}{3}\cdot\frac{\lim_{x\to 0}\left(\frac{3^{2x}-1}{2x}\right)}{\lim_{x\to 0}\left(\frac{2^{3x}-1}{3x}\right)} = \frac{2}{3}\cdot\frac{\log 3}{\log 2}$$

$$= \frac{\log 3^2}{\log 2^2} = \frac{\log 9}{\log 8}$$

7. $\lim_{x\to 1}\frac{\log x}{x-1}$

put $x = y+1$ As $x\to 1$ we have $y\to 0$

$$\therefore \lim_{x\to 1}\frac{\log x}{x-1} = \lim_{y\to 0}\frac{\log(1+y)}{y} = 1 \quad \text{(from std Lt)}$$

8. $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{3n}$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^3 = e^3$$

9. $\lim_{x\to 0}\frac{2}{x}\log(1+x)$

$= 2\cdot\lim_{x\to 0}\frac{1}{x}\log(1+x) = 2\cdot(1) \quad \left(\because \lim_{x\to 0}\frac{1}{x}\log(1+x)=1\right)$

$= 2$

10. $\lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{x+3}$

$= \lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{x+3+1-1}$

$= \lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{x-1} \lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{4}$

$= \lim_{x\to\infty}\left\{\left(1+\frac{4}{x-1}\right)^{\frac{x-1}{4}}\right\}^{4} \times \lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{4}$

$= e^4\times 1 = e^4$

Exercise 17.3

I. One and Two marks questions:

Evaluate the following limits:

1. $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n}$
2. $\lim_{x\to 0}(1+3x)^{\frac{1}{x}}$
3. $lt_{n\to 0}\left(\frac{n+3}{3}\right)^{\frac{2}{n}}$
4. $\lim_{x\to 0}\frac{e^{-3x}-1}{x}$
5. $\lim_{x\to 0}\left(\frac{2^x-1}{3x}\right)$

II. Three marks questions:

Evaluate the following limits:

1. $\lim_{x\to 0}\left[\frac{2^x-1}{\sqrt{1+x}-1}\right]$
2. $\lim_{x\to 0}\frac{9^x+9^{-x}-2}{x^2}$
3. $\lim_{x\to\infty}\left(\frac{x-3}{x+3}\right)^{x+3}$
4. $\lim_{x\to 0}\left(\frac{3^{2x}-2^{3x}}{x}\right)$
5. $\lim_{x\to 0}\left(\frac{3^{2+x}-9}{x}\right)$

ANSWERS 17.3

I. 1. e^2 2. e^3 3. $e^{\frac{2}{3}}$ 4. -3 5. $\frac{1}{3}\log_e^2$

II. 1. $2\log 2$ 2. $(\log 9)^2$ 3. $\frac{1}{e^6}$ 4. $\log\left(\frac{9}{8}\right)$ 5. $9\log 3$

17.10 Limit at infinity and infinite limits:

(i) Meaning of $x \to \infty$:

If x is a variable such that it can take any real value howsoever large, then we say that the variable x approaches $+\infty$ and write this as $x \to +\infty$

(ii) Meaning of $x \to \infty$:

If x is a variable such that it can take any real value howsoever small, then we say that the variable x approaches $-\infty$ and write this as $x \to -\infty$

(iii) Infinite limit of a function:

Let $f(x)$ be a function of x. If the value of $f(x)$ can be made greater than any pre-assigned number by taking x close to a, then we say that the function $f(x)$ becomes positively infinite as x approaches a. We write this as $\lim_{x\to a} f(x) = +\infty$

Similarly if the value of $f(x)$ can be made less than any pre-assigned number by taking x close to a, then we say that the function $f(x)$ becomes negatively infinite as x approaches a. We write this as $\lim_{x\to a} f(x) = -\infty$.

Corollary:

1. $\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$

we have $x \to \infty$

$\therefore$ The variable x can be made as large as we like.

$\therefore$ $\frac{1}{x}$ is positive and grows smaller and smaller as x becomes larger and larger.

$\therefore$ $\frac{1}{x} \to 0$ i.e. $\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$

2. $\lim_{x\to-\infty}\left(\frac{1}{x}\right) = 0$

we have $x \to -\infty$

$\therefore$ The variable x can be made as small as we like.

$\therefore \left(\frac{1}{x}\right)$ is negative and numerically grows smaller and smaller as x becomes smaller and smaller.

$\therefore \frac{1}{x} \to 0$ i.e. $\lim_{x\to-\infty}\left(\frac{1}{x}\right) = 0$

Working rule for evaluating the limit when $x \to \infty$

Rule 1: Write the given expression in the form of rational function i.e. $p(x)/q(x)$ where $p(x), q(x)$ are polynomials and $q(x) \neq 0$.

Rule 2: Divide the numerator and denominator by the highest power of x and simplify and then take the limit.

Example 8 : Evaluate the following limits.

1. $\lim_{x\to\infty}\frac{3x^2-4x+2}{2x^2+5x+7}$

Here the highest power of x in numerator and denominator is 2.

Divide the numerator and denominator by x^2

$$\therefore \lim_{x\to\infty}\left(\frac{3x^2-4x+2}{2x^2+5x+7}\right) = \lim_{x\to\infty}\left(\frac{3-\frac{4}{x}+\frac{2}{x^2}}{2+\frac{5}{x}+\frac{7}{x^2}}\right)$$

$$= \frac{3-0+0}{2-0+0} = \frac{3}{2}$$

2. $\lim_{x\to\infty}\frac{x^7+x^3+1}{x^8-3x^6+5}$

Here, Degree of Nr.(7) < Degree of Dr. (8)

$\therefore$ we divide the numerator and the denominator by x^8.

$$\therefore \lim_{x\to\infty}\frac{x^7+x^3+1}{x^8-3x^6+5} = \lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^5}+\frac{1}{x^8}}{1-\frac{3}{x^2}+\frac{5}{x^8}}$$

$$= \frac{0+0+0}{1-0+0} = 0$$

3. $\lim_{x\to\infty}\frac{3x^2+x+5}{7x-9}$

Here, Degree of Nr.(2) > Degree of Dr. (1)

$$\therefore \lim_{x\to\infty}\frac{3x^2+x+5}{7x-9} = \lim_{x\to\infty}\frac{3x^2\left(1+\frac{1}{3x}+\frac{5}{3x^2}\right)}{7x\left(1-\frac{9}{x}\right)}$$

$$= \lim_{x\to\infty}\left[\frac{3}{7}x\cdot\frac{1+\frac{1}{3x}+\frac{5}{3x^2}}{1-\frac{9}{x}}\right]$$

$$= \infty\times\frac{1+0+0}{1-0} = \infty$$

4. $\lim_{x\to\infty}\frac{(2x-1)^{20}(3x-1)^{30}}{(2x+1)^{50}}$

Here, degree of Nr.(50) = degree of Dr.(50)

$\therefore$ we divide the Nr. and the Dr. by x^{50}

$$\therefore \lim_{x\to\infty}\frac{(2x-1)^{20}(3x-1)^{30}}{(2x+1)^{50}} = \lim_{x\to\infty}\frac{\frac{(2x-1)^{20}}{x^{20}}\cdot\frac{(3x-1)^{30}}{x^{30}}}{\frac{(2x+1)^{50}}{x^{50}}}$$

$$= \lim_{x\to\infty}\frac{\left(\frac{2x-1}{x}\right)^{20}\left(\frac{3x-1}{x}\right)^{30}}{\left(\frac{2x+1}{x}\right)^{50}} = \lim_{x\to\infty}\frac{\left(2-\frac{1}{x}\right)^{20}\left(3-\frac{1}{x}\right)^{30}}{\left(2+\frac{1}{x}\right)^{50}}$$

$$= \frac{(2-0)^{20}(3-0)^{30}}{(2+0)^{50}} = \frac{2^{20}\times3^{30}}{2^{50}}$$

$$= \frac{3^{30}}{2^{30}} = \left(\frac{3}{2}\right)^{30}$$

5. $\lim_{n\to\infty}\frac{1+2+3+.....+n}{(n+5)^2}$

$$= \lim_{n\to\infty}\frac{n(n+1)}{2(n+5)^2} \qquad \left[\because \sum n = 1+2+3+....+n = \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{1\cdot\left(1+1/n\right)}{2\left(1+5/n\right)^2}$$

$$= \frac{1+0}{2(1+0)} = \frac{1}{2}$$

6. $\lim_{x\to\infty} \frac{1^3+2^3+3^3+.....+n^3}{(\sum n)\ (2n^2+3n+1)}$

$$= \left(\because\ 1^3+2^3+....+n^3 = \frac{n^2(n+1)^2}{4} = \sum n^3\right)$$

$$= \lim_{n\to\infty} \frac{(n+1)^2\cdot n^2\cdot 2}{4\cdot n\cdot(n+1)(2n^2+3n+1)}$$

$$= \lim_{n\to\infty} \frac{n(n+1)}{2(2n^2+3n+1)} = \lim_{x\to\infty} \frac{1\left(1+1/n\right)}{2\left(2+3/n+1/n^2\right)}$$

$$= \frac{1+0}{2(2+0+0)} = \frac{1}{4}$$

7. $\lim_{x\to\infty}\left(\sqrt{x+2}-\sqrt{x}\right)$

$$= \lim_{x\to\infty}\left[\sqrt{x+2}-\sqrt{x}\times\frac{\sqrt{x+2}+\sqrt{x}}{\sqrt{x+2}+\sqrt{x}}\right]$$

$$= \lim_{x\to\infty} \frac{(x+2)-x}{\sqrt{x+2}+\sqrt{x}} = \lim_{x\to\infty} \frac{2}{\sqrt{x+2}+\sqrt{x}}$$ ÷ both Nr and Dr by $\sqrt{x}$

$$= \lim_{x\to\infty} \frac{2/\sqrt{x}}{\sqrt{\frac{x+2}{x}}+\sqrt{\frac{x}{x}}} = lt_{x\to\infty} \frac{\frac{2}{\sqrt{x}}}{\sqrt{1+\frac{2}{x}}+1}$$

$$= \frac{0}{\sqrt{1+0}+1} = \frac{0}{2} = 0$$

8. $\lim_{x\to\infty}\left[\sqrt{x}\left(\sqrt{x+5}-\sqrt{x}\right)\right]$

$= \lim_{x\to\infty}\dfrac{\sqrt{x}\left(\sqrt{x+5}-\sqrt{x}\right)\left(\sqrt{x+5}+\sqrt{x}\right)}{\sqrt{x+5}+\sqrt{x}}$

$= \lim_{x\to\infty}\dfrac{\sqrt{x}(x+5-x)}{\sqrt{x+5}+\sqrt{x}} = \lim_{x\to\infty}\dfrac{5\sqrt{x}}{\sqrt{x+5}+\sqrt{x}}$

$= \lim_{x\to\infty}\dfrac{5}{\sqrt{1+5/x}+1} = \dfrac{5}{\sqrt{1+0}+1} = \dfrac{5}{2}$

EXERCISE 17.4

I. Two marks questions:

1. $\lim_{x\to\infty}\dfrac{3x^2-4x+5}{2x^2+5x-1}$
2. $\lim_{x\to\infty}\dfrac{3x^3-4x+7}{2x^4-3x+6}$
3. $\lim_{x\to\infty}\dfrac{(3x+4)(4x+3)}{(2x-7)(x+4)}$
4. $\lim_{x\to\infty}\dfrac{(2x-1)^{30}(3x-1)^{30}}{(2x+4)^{60}}$
5. $\lim_{x\to\infty}\dfrac{x}{\sqrt{4x^2+1}-1}$
6. $\lim_{n\to\infty}\dfrac{1+2+3+.....+n}{n^2+3}$
7. $\lim_{n\to\infty}\dfrac{\sum n^2}{\left(n^2+2n+1\right)(n-1)}$
8. $\lim_{n\to\infty}\dfrac{\sum n^3}{n^2\cdot\sum n}$

II. Three marks questions:

1. $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$
2. $\lim_{x\to\infty}\sqrt{x}\left(\sqrt{x+3}-\sqrt{x}\right)$
3. $\lim_{x\to\infty}x\left(\sqrt{4+x^2}-x\right)$
4. $\lim_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)$
5. $\lim_{x\to\infty}x\left(\sqrt{x^2+1}-x\right)$

ANSWERS 17.4

I. 1) $\frac{3}{2}$ 2) 0 3) 6 4) $\left(\frac{3}{2}\right)^{30}$ 5) $\frac{1}{2}$
6) $\frac{1}{2}$ 7) $\frac{1}{3}$ 8) $\frac{1}{2}$

II. 1) 0 2) $\frac{3}{2}$ 3) 2 4) $\frac{5}{2}$ 5) $\frac{1}{2}$

17.11 Left Hand and Right Hand Limit (one sided limits)

The limit of the function $f(x)$, when x tends to 'a' from the left side of 'a' is called the left hand limit of $f(x)$ as $x \to a$ $\quad \therefore\ LHL = \underset{x \to a-}{lt}\ f(x)$

The limit of the function $f(x)$, when x tends to 'a' from the right side of 'a' is called the right hand limit of $f(x)$ as $x \to a$ $\quad \therefore\ RHL = \lim_{x \to a^+} f(x)$

WORKING RULE

I. To find the left hand limit.

1. Put $x = a - h$ in $f(x)$, where h is a small positive quantity.
2. Take the limit of $f(a-h)$ as $h \to 0$ i.e. $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$

II. To find the right hand limit

1. put $x = a + h$ in $f(x)$, where h is a small positive quantity.
2. Take the limit of $f(a+h)$ as $h \to 0$ i.e. $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$

17.12 Continuity of a Function

A function $y = f(x)$ is said to be continuous at $x = a$ if

(i) $f(a)$ exists (ii) $\lim_{x \to a} f(x) = f(a)$

If a function $f(x)$ is not continuous at $x = a$ then it is said to be discontinuous at $x = a$

Note: The discontinuity of a function $f(x)$ at a point $x = a$ can arise in any one of the following ways

1. The function $f(x)$ is not defined at $x = a$
   i.e. $f(a)$ does not exists.
2. $\lim_{x \to a} f(x)$ exists and $f(a)$ exists

   but $\lim_{x \to a} f(x) \neq f(a)$

Example 9 :

1. Show that the function

Solution : $f(x) = |x|$ is continuous at $x = 0$

$f(x) = |x|$

$$= \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$

LHL $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x) = 0$

$(\because f(x) = -x$ when $x < 0)$

RHL $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x) = 0$

$(\because f(x) = x$ when $x < 0)$

$\therefore$ LHL = RHL = $f(0) = 0$

$\Rightarrow f(x)$ is continuous at $x = 0$

2. Find $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases}$

Solution : LHL $= \lim_{x\to 1^-} f(x) = \lim_{x\to 1}(x^2 - 1) = 1 - 1 = 0$

$\left(\because f(x) = x^2 - 1,\ x < 1\right)$

RHL $= \lim_{x\to 1^+} f(x) = \lim_{x\to 1}(-x^2 - 1)$

$= -1 - 1 = -2$

Thus LHL $\neq$ RHL

$\Rightarrow$ The $f(x)$ is discontinuous at $x = 1$ and the limit does not exist at $x = 1$

3. Prove that

$$f(x) = \begin{cases} x^2 + 1 & when \quad x < 2 \\ 5 & when \quad x = 2 \\ 4x - 3 & when \quad x > 2 \end{cases}$$ is continuous at $x = 2$

Solution : LHL : $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x^2 + 1)$

$= \lim_{h\to 0}(2 - h)^2 + 1 = 5$

RHL : $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(4x - 3)$

$= \lim_{h\to 0}\{4(2 + h) - 3\} = 5$

LHL = RHL $\therefore \lim_{x\to 2} f(x) = 5$

By data $f(2) = 5$

i.e. $\lim_{x\to 2} f(x) = f(2)$

Hence $f(x)$ is continuous at $x = 2$

4. Find k if

$$f(x)=\begin{cases}\dfrac{e^{5x}-1}{2x}, & x\neq 0\\ \dfrac{k+x}{2}, & x=0\end{cases}$$ is continuous at $x=0$

Solution : We have $f(0)=\dfrac{k+0}{2}=\dfrac{k}{2}$

$$\lim_{x\to 0}\frac{e^{5x}-1}{2x}=\frac{1}{2}\cdot\lim_{x\to 0}\frac{e^{5x}-1}{5x}\times 5$$

$$=\frac{5}{2}\cdot\lim_{x\to 0}\left(\frac{e^{5x}-1}{5x}\right)$$

$$=\frac{5}{2}(1)=\frac{5}{2}$$

Since $f(x)$ is continuous at $x=0$

$$\lim_{x\to 0}f(x)=f(0)$$

$$\Rightarrow \frac{5}{2}=\frac{k}{2}\quad\Rightarrow\quad k=5$$

5. If $f(x)=\begin{cases}\dfrac{x^4-256}{x-4} & x\neq 4\\ a & x=4\end{cases}$ is continuous at $x=4$, find a

Solution : $\lim_{x\to 4}f(x)=\lim_{x\to 4}\left(\dfrac{x^4-256}{x-4}\right)$

$$=\lim_{x\to 4}\left(\frac{x^4-4^4}{x-4}\right)$$

$$=4\cdot 4^3=256$$

Since $f(x)$ is continuous at $x=4$

$$\lim_{x\to 4}f(x)=f(4)\Rightarrow 256=a$$

or $a=256$

EXERCISE 17.5

I. 2 and 3 marks question:

1. Show $f(x)$ defined by

$$f(x)=\begin{cases}\dfrac{x^2-25}{x-5} & when \quad x\neq 5\\ 10 & when \quad x=5\end{cases}$$ is continuous at $x=5$

2. Show that the function

$$f(x)\begin{cases}\dfrac{x^2-9}{x-3} & when \quad x\neq 3\\ 4 & when \quad x=3\end{cases}$$ is discontinuous at $x=3$

3. Define $f(0)$ so that $f(x)=\dfrac{x}{1-\sqrt{1-x}}$ becomes continuous at $x=0$

4. Show that the function $f(x)=\begin{cases}(1+3x)^{1/x} & x\neq 0\\ e^3 & x=0\end{cases}$ is continuous at $x=0$

5. Find k for which

$$f(x)=\begin{cases}k+x, & x=1\\ 4x+3, & x\neq 1\end{cases}$$ is continuous at $x=1$

6. If the function

$$f(x)=\begin{cases}(1+2x)^{1/x}, & x\neq 0\\ k \quad , & x=0\end{cases}$$ is continuous at $x=0$, find k

7. Discuss the continuity of $f(x)=\begin{cases}3x^2+1 & if \quad x<1\\ 4 & if \quad x=1\\ 2x+2 & if \quad x>1\end{cases}$ at $x=1$

8. Find the value of k if the function $f(x)=\begin{cases}\dfrac{e^{2x}-1}{x} & x\neq 0\\ k & x=0\end{cases}$ is continuous at $x=0$

ANSWERS 17.5

3) 2 5) 6 6) e^2 7) $f(x)$ is continuous at $x=1$ 8) 2

* * * *

Chapter

18

DIFFERENTIAL CALCULUS

18.1 Introduction:

Calculus is the primary mathematical tool for dealing with change. The concept of 'derivative' is a basic tool in science of calculus. Calculus is essentially concerned with the rate of change of dependent variable with respect to an independent variable.

18.2 Increment:

An increment is any change in a variable

$\therefore$ Increment = Final value – Initial value

Ex: If a variable x changes from

(i) 3.01 to 3.02 then increment = 0.01

(ii) 3.01 to 2.99 then increment = –0.02

Note: 1) The increment of x is denoted by Δx or δx

2) Δx is one symbol and not a product of Δ and x.

3) The ratio $\frac{\Delta y}{\Delta x}$ is called incrementary ratio or average rate of change of y w.r.t. x per unit.

18.3 Derivative of a function:

Let $y = f(x)$ be a given continuous function. Then the value of y depends upon the value of x and it changes with a change in the value of x.

Let Δy be an increment in y corresponding to an increment Δx in x then

$y = f(x)$ and $y + \Delta y = f(x + \Delta x)$ on subtraction,

we get $\Delta y = f(x + \Delta x) - f(x)$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

This $\frac{\Delta y}{\Delta x}$ is called Incremental ratio

$\frac{f(x+\Delta x) - f(x)}{\Delta x}$ is called difference quotient

so, $\lim\limits_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim\limits_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$

The above limit, if it exists finitely is called the **derivative** or **differential coefficient** of $y = f(x)$ with respect to x and it is denoted by

$$\frac{dy}{dx} \text{ or } \frac{d}{dx}\left[f(x)\right] \text{ or } \left[f^1(x)\right] \text{ or } y_1 \text{ or } y^1$$

The process of finding the derivative is known as **differentiation.** The above method is also known as **differentiation from the first principle** or ab initio or by delta method.

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

18.4 Derivative at a point:

The value of $f'(x)$ obtained by putting $x = a$ is called the derivative of $f(x)$ at $x = a$ and is denoted by $f'(a)$ or $\left(\frac{dy}{dx}\right)_{x=a}$ and is defined as

$$\frac{dy}{dx} \text{ or } f'(a) = \lim_{h \to 0}\frac{f(a+h) - f(a)}{h}$$

If $f'(a)$ exists, we say that $f(x)$ is differentiable at $x = a$.

If $f'(x)$ exists for every value of x in the domain of the function, we say that $f(x)$ is differentiable.

Definition (Left hand derivative):

The left hand derivative of a function $y = f(x)$ at $x = a$ is denoted and defined by,

$$L\{f'(a)\} = \lim_{x \to a^-}\frac{f(x) - f(a)}{x - a}, \text{ provided the limit exists.}$$

Definition (Right hand derivative):

The right hand derivative of a function $y = f(x)$ at $x = a$ is denoted and defined by

$$R\{f'(a)\} = \lim_{x \to a^+}\frac{f(x) - f(a)}{x - a}, \text{ provided the limit exists.}$$

18.5 Differentiability:

A function $y = f(x)$ is said to be differentiable at $x = a$ if $L\{f'(a)\}$ and $R\{f'(a)\}$ exist and are equal. This is denoted by $f'(a)$ or $\left(\frac{dy}{dx}\right)_{x=a}$

If for a function $y = f(x)$, any one of $L\{f'(a)\}$ or $R\{f'(a)\}$ (or both) does not exists, or even they exist but not equal then we say $f(x)$ is not differentiable at $x = a$.

18.6 Relation between continuity and differentiability

THEOREM: If a function f is differentiable at a point c, then it is also continuous at that point.

Proof: Since f is differentiable at c, we have

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c)$$

But for $x\neq c$, we have

$$f(x)-f(c)=\frac{f(x)-f(c)}{x-c}\cdot x-c$$

$$\therefore\ \lim_{x\to c}[f(x)-f(c)]=\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\cdot x-c\right]$$

$$\text{or } \lim_{x\to c}[f(x)]-\lim_{x\to c}[f(c)]=\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right]\cdot\lim_{x\to c}(x-c)$$

$$=\ f'(c)\cdot 0\ \ =\ \ 0$$

$$\Rightarrow \lim_{x\to c}f(x)=f(c)$$

Hence f is continuous at $x=c$

Corollary 1 : Every differentiable function is continuous.

We remark that the converse of the above statement is not true. Indeed we have seen that the function defined by $f(x)=|x|$ is a continuous function.

Consider the left hand limit

$$\lim_{h\to 0^-}\frac{f(0-h)-f(0)}{h}=\frac{-h}{h}=-1$$

The right hand limit $\lim_{h\to 0^+}\dfrac{f(0+h)-f(0)}{h}=\dfrac{h}{h}=1$

Since the above left and right hand limits at 0 are not equal, $\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$ does not exist and hence f is not differentiable at 0. Thus f is not a differentiable function.

Note:

Differentiability implies continuity but continuity does not imply differentiability.

18.7 Differentiation of some important standard function from first principles:

1) To differentiate a constant function from first principles.

Let $y = f(x) = k$ where k is constant

$\therefore \quad y + \Delta y = f(x + \Delta x) = k$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{k - k}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 0$$

$$\Rightarrow \frac{dy}{dx} = 0$$

Thus the derivative of a constant function is zero.

i.e. $y = f(x) = k \qquad \Rightarrow \qquad \boxed{\frac{d(k)}{dx} = 0}$

Ex : (i) $\frac{d}{dx}(2) = 0$ (ii) $\frac{d}{dx}\left(\frac{2}{\pi}\right) = 0$ (iii) $\frac{d}{dx}\left(e^{\pi}\right) = 0$

2) To differentiate x^n from the first principles

Let $y = f(x) = x^n$

$\therefore \quad y + \Delta y = f(x + \Delta x) = (x + \Delta x)^n$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(x+\Delta x)^n - x^n}{\Delta x}\right]$$

put $x + \Delta x = z$

$\Delta x = z - x$

As $\Delta x \to o$

$z \to x$

$$= \lim_{\Delta x \to 0}\left[\frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x}\right]$$

$$= \lim_{z \to x}\left[\frac{z^n - x^n}{z - x}\right] \qquad \left(\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = n, a^{n-1}\right)$$

$$\Rightarrow \boxed{\frac{d}{dx}(x^n) = n \cdot x^{n-1}}$$

Ex: (i) $\frac{d}{dx}(x^9) = 9 \cdot x^{9-1} = 9x^8$

(ii) $\frac{d}{dx}\left(\sqrt[3]{x}\right) = \frac{d}{dx}\left(x^{1/3}\right) = \frac{1}{3} \cdot x^{1/3-1} = \frac{1}{3} \cdot x^{-2/3}$

Note : Consider the function $y = \frac{1}{x^n}$, then

$$y = \frac{1}{x^n} \Rightarrow y = x^{-n} \Rightarrow \frac{dy}{dx} = -n \cdot x^{-n-1} \Rightarrow \frac{dy}{dx} = \frac{-n}{x^{n+1}}$$

Thus if $y = \frac{1}{x^n}$ then $\boxed{\frac{dy}{dx} = \frac{-n}{x^{n+1}}}$

Ex: (i) $\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{-3}{x^4}$

(ii) $\frac{d}{dx}\left(\frac{1}{\sqrt[5]{x^4}}\right) = \frac{d}{dx}\left(\frac{1}{x^{4/5}}\right) = \frac{-4}{5x^{9/5}}$

3) To differentiate e^x from the first principes.

Let $y = f(x) = e^x$

$\therefore\ y + \Delta y = f(x + \Delta x) = e^{x+\Delta x}$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{e^{x+\Delta x} - e^x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[e^x\left(\frac{e^{\Delta x} - 1}{\Delta x}\right)\right]$$

$$= e^x \cdot \lim_{\Delta x \to 0}\left(\frac{e^{\Delta x}-1}{\Delta x}\right) = e^x \cdot 1 \qquad \left[\because \lim_{\Delta x \to 0}\frac{e^x-1}{x}=1\right]$$

$$= \boxed{\frac{d}{dx}\left(e^x\right)=e^x}$$

4) To differentiate a^x where ($a > 0$) and $a \neq 1$ from the first principles.

Let $y = f(x) = a^x$

$\therefore \; y+\Delta y = f(x+\Delta x) = a^{x+\Delta x}$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{a^{x+\Delta x}-a^x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[a^x\left(\frac{a^{\Delta x}-1}{\Delta x}\right)\right] \qquad \left[\because \lim_{x \to 0}\frac{a^x-1}{x}=\log a\right]$$

$$= a^x \cdot \lim_{\Delta x \to 0}\left[\frac{a^{\Delta x}-1}{\Delta x}\right]$$

$$\therefore \boxed{\frac{d}{dx}\left(a^x\right)=a^x \log a}$$

5) To differentiate $\log x$ ($x > 0$) from the first principles.

Let $y = f(x) = \log x$

$\therefore \; y+\Delta y = f(x+\Delta x) = \log(x+\Delta x)$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\log(x+\Delta x)-\log x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{1}{\Delta x}\cdot \log\left(\frac{x+\Delta x}{x}\right)\right] \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{1}{\Delta x}\cdot \log\left(1+\frac{\Delta x}{x}\right)\right]$$

put $\frac{\Delta x}{x} = h$

$\Delta x = h \cdot x$

As $\Delta x \to 0$

$h \to 0$

$$= \lim_{h \to 0}\left[\frac{1}{h \cdot x}\cdot \log(1+h)\right]$$

$$= \frac{1}{x}\cdot \lim_{h \to 0}\left[\frac{1}{h}\log(1+h)\right] \qquad \left[\because \lim_{x \to 0}\frac{1}{x}\ \log(1+x) = 1\right]$$

$$= \frac{1}{x}(1)$$

$$\therefore \boxed{\frac{d}{dx}(\log x) = \frac{1}{x}}$$

6) To differentiate '$\sin x$' **from the first principles.**

Let $y = f(x) = \sin x$

$\therefore\ y + \Delta y = f(x + \Delta x) = \sin(x + \Delta x)$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\sin(x+\Delta x) - \sin x}{\Delta x}\right]$$

$$\left[\text{using } \sin C - \sin D = 2\cos\frac{C+D}{2}\cdot \sin\frac{C-D}{2}\right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{2\cos\left(\frac{x+\Delta x+x}{2}\right)\sin\left(\frac{x+\Delta x-x}{2}\right)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[2\cos\left(\frac{2x+\Delta x}{2}\right)\sin\frac{\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\cdot\frac{1}{2}\right] \qquad \left[\because \lim_{x \to 0}\left(\frac{\sin x}{x}\right) = 1\right]$$

$$= 2\cos\left(\frac{2x}{2}\right)\cdot 1\cdot\frac{1}{2}$$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

7) To differentiate cos x from the first principles.

Let $y = f(x) = \cos x$

$\therefore \quad y + \Delta y = f(x + \Delta x) = \cos(x + \Delta x)$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\cos(x+\Delta x) - \cos x}{\Delta x}\right]$$

$$\left[\text{using } \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{-2\sin\left(\frac{x+\Delta x+x}{2}\right)\sin\left(\frac{x+\Delta x-x}{2}\right)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[-2\sin\left(\frac{2x+\Delta x}{2}\right)\cdot\sin\frac{\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\cdot\frac{1}{2}\right] \qquad \left[\because \lim_{x \to 0}\left(\frac{\sin x}{x}\right) = 1\right]$$

$$= -2\sin\left(\frac{2x}{2}\right)\cdot 1\cdot\frac{1}{2}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

8. To differentiate 'tan x' from the first principles.

Let $y = f(x) = \tan x$

$\therefore \quad y + \Delta y = f(x + \Delta x) = \tan(x + \Delta x)$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\tan(x+\Delta x) - \tan x}{\Delta x}\right]$$

$\left[\text{using } \tan A - \tan B = \tan(A-B)\left[1+\tan A\ \tan B\right]\right]$ $\left[\because \begin{matrix} A = x+\Delta x \\ B = x \end{matrix}\right]$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{\tan(x+\Delta x - x)\{1+\tan(x+\Delta x)\tan x\}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\tan \Delta x}{\Delta x}\{1+\tan(x+\Delta x)\cdot \tan x\}\right] \qquad \left[\because \lim_{x\to 0}\left(\frac{\tan x}{x}\right)=1\right]$$

$$= 1\cdot\{1+\tan x\cdot\tan x\} \quad = \quad 1+\tan^2 x$$

$$\therefore \boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

9) To differentiate 'cot *x*' from first principles.

Let $y = f(x) = \cot x$

$\therefore\ y+\Delta y = f(x+\Delta x) = \cot(x+\Delta x)$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\cot(x+\Delta x)-\cot x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left\{\frac{\dfrac{\cos(x+\Delta x)}{\sin(x+\Delta x)} - \dfrac{\cos x}{\sin x}}{\Delta x}\right\}$$

$$= \lim_{\Delta x \to 0}\frac{\sin x\cdot\cos(x+\Delta x)-\cos x\cdot\sin(x+\Delta x)}{\sin(x+\Delta x)\sin x\cdot\Delta x}$$

$\left[\because\ \sin A\cos B - \cos A\sin B = \sin(A-B)\right]$

$$= \frac{-1}{\sin x}\cdot\lim_{\Delta x\to 0}\frac{\sin \Delta x}{\Delta x}\cdot\frac{1}{\lim\limits_{\Delta x\to 0}\sin(x+\Delta x)}$$

$$= \frac{-1}{\sin x}\cdot 1\cdot\frac{1}{\sin x} \quad = \quad -\operatorname{cosec}^2 x$$

$$\therefore \boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$$

10. To differentiate 'cosec x' from first principles.

Let $y = f(x) = \text{cosec}x$

$\therefore \quad y + \Delta y = f(x + \Delta x) = \text{cosec}(x + \Delta x)$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\text{cosec}(x+\Delta x) - \text{cosec}x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{\sin(x+\Delta x)} - \dfrac{1}{\sin x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x - \sin(x+\Delta x)}{\sin(x+\Delta x)\cdot \sin x \cdot \Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2\cos\left(x + \dfrac{\Delta x}{2}\right)\sin\left(\dfrac{\Delta x}{2}\right)}{\sin(x+\Delta x)\cdot \sin x \cdot \Delta x}$$

$$\left[\therefore \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \frac{-1}{\sin x} \cdot \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) \cdot \lim_{\Delta x \to 0} \frac{\sin\left(\dfrac{\Delta x}{2}\right)}{\left(\dfrac{\Delta x}{2}\right)} \cdot \lim_{\Delta x \to 0} \frac{1}{\sin(x+\Delta x)}$$

$$= \frac{-1}{\sin x} \times \cos x \times 1 \times \frac{1}{\sin x} \quad = \quad -\text{cosec}x \cdot \cot x \qquad \left(\therefore \frac{\cos x}{\sin x} = \cot x\right)$$

$\therefore$ $\boxed{\dfrac{d}{dx}(\text{cosec}x) = -\text{cosec}x \cdot \cot x}$

11. To differentiate sec x from first principles.

Let $y = f(x) = \sec x$

$\therefore \quad y + \Delta y = f(x + \Delta x) = \sec(x + \Delta x)$

By definition,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\sec(x+\Delta x) - \sec x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{1}{\cos(x+\Delta x)} - \dfrac{1}{\cos x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x - \cos(x+\Delta x)}{\Delta x \cdot \cos(x+\Delta x) \cdot \cos x}$$

$$= \lim_{\Delta x \to 0} \frac{2\sin\left(x + \dfrac{\Delta x}{2}\right)\sin\left(\dfrac{\Delta x}{2}\right)}{\cos(x+\Delta x) \cdot \cos x \cdot \Delta x}$$

$$\left[\because \cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)\right]$$

$$= \frac{1}{\cos x} \cdot \lim_{\Delta x \to 0} \sin\left(x + \frac{\Delta x}{2}\right) \lim_{\Delta x \to 0} \frac{1}{\cos(x+\Delta x)} \cdot \lim_{\Delta x \to 0} \frac{\sin\left(\dfrac{\Delta x}{2}\right)}{\left(\dfrac{\Delta x}{2}\right)}$$

$$= \frac{1}{\cos x} \times \sin x \times \frac{1}{\cos x} \times 1 \quad = \sec x \cdot \tan x \qquad \left(\because \frac{\sin x}{\cos x} = \tan x\right)$$

$$\therefore \boxed{\frac{d}{dx}(\sec x) = \sec x \cdot \tan x}$$

12. To differentiate $\sqrt{x}$ from first principles

Let $y = f(x) = \sqrt{x}$

$\therefore \quad y + \Delta y = f(x+\Delta x) = \sqrt{x+\Delta x}$

By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^{1/2} - x^{1/2}}{(x+\Delta x) - x}$$

$$= \frac{1}{2} \cdot x^{\frac{1}{2}-1} \qquad \left[\because \lim_{x \to a}\left(\frac{x^n - a^n}{x-a}\right) = n \cdot a^{n-1}\right]$$

$$\therefore \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$$

List of formulae of Derivatives of Basic Functions:

$y = f(x)$ (Function)		$\frac{dy}{dx} = f'(x)$ (Derivative)
1.	$y = k$	$\frac{dy}{dx} = 0$
2.	$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$
3.	$y = \frac{1}{x^n}$	$\frac{dy}{dx} = \frac{-n}{x^{n+1}}$
4.	$\frac{1}{x}$	$\frac{dy}{dx} = \frac{-1}{x^2}$
5.	$\sqrt{x}$	$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$
6.	$\sin x$	$\frac{dy}{dx} = \cos x$
7.	$\cos x$	$\frac{dy}{dx} = -\sin x$
8.	$\tan x$	$\frac{dy}{dx} = \sec^2 x$
9.	$\text{cosec } x$	$\frac{dy}{dx} = -\text{cosec}x \ \cot x$
10.	$\sec x$	$\frac{dy}{dx} = \sec x \cdot \tan x$

11.	$\cot x$	$\frac{dy}{dx} = -\text{cosec}^2 x$
12.	e^x	$\frac{dy}{dx} = e^x$
13.	$\log x$	$\frac{dy}{dx} = \frac{1}{x}$
14.	a^x	$\frac{dy}{dx} = a^x \log_e a$

18.8 Algebra of Derivative of Functions:

Let f and g be two functions such that their derivatives are defined in a common domain. Then

(i) Derivative of sum of two functions is sum of the derivatives of the functions

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

(ii) Derivative of difference of two functions is difference of the derivative of the functions

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

(iii) Derivative of product of two functions is given by the following product rule.

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x)$$

(iv) Derivative of quotient of two functions is given by the following quotient rule (when ever the denominator is non-zero)

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)}{[g(x)]^2}$$

(v) Derivative of scalar multiple of a function is the product of the constant and the derivative of the function

$$\frac{d}{dx}[k.f(x)] = k\cdot\frac{d}{dx}[f(x)]$$

Note 1) If $y = u \pm v \pm w \pm \ldots$ are function of x

then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \ldots\ldots$

2) If $y = uvw.....$ are function of x

then $\frac{dy}{dx} = uv \cdot \frac{dw}{dx} + uw \cdot \frac{dv}{dx} + vw \cdot \frac{du}{dx}$

Example 1: Differentiate the following with respect to x :

1. $x^2 + \frac{4}{x^2} - \frac{2}{3}\tan x + 6e$

Solution: Let $y = x^2 + \frac{4}{x^2} - \frac{2}{3}\tan x + 6e$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) + 4 \cdot \frac{d}{dx}\left(\frac{1}{x^2}\right) - \frac{2}{3}\frac{d}{dx}(\tan x) + 6 \cdot \frac{d}{dx}(e)$$

$$= 2x + 4\left(\frac{-2}{x^3}\right) - \frac{2}{3}\sec^2 x + 6(0)$$

$$= 2x - \frac{8}{x^3} - \frac{2}{3}\sec^2 x$$

2. $x^e + e^x + e^e$

Solution:

Let $y = x^e + e^x + e^e$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^e\right) + \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^e\right)$$

$$= ex^{e-1} + e^x + 0$$

$$= ex^{e-1} + e^x$$

3. $2^x - x^2 - x + 2 - 3\log x - 4\sqrt{x}$

Solution:

Let $y = 2^x - x^2 - x + 2 - 3\log x - 4\sqrt{x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(2^x\right) - \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(x) + \frac{d}{dx}(2) - 3\frac{d}{dx}(\log x) - 4\frac{d}{dx}\sqrt{x}$$

$$= 2^x \log_e 2 - 2x - 1 + 0 - 3\left(\frac{1}{x}\right) - 4\left(\frac{1}{2\sqrt{x}}\right)$$

$$= 2^x \log_e 2 - 2x - 1 - \frac{3}{x} - \frac{2}{\sqrt{x}}$$

4. $\dfrac{3x^2+2x+5}{\sqrt{x}}$

Solution:

Let $y=\dfrac{3x^2+2x+5}{\sqrt{x}}$

$y = 3x^{3/2}+2x^{1/2}+5x^{-1/2}$

[on dividing each term by $\sqrt{x}$)

$$\frac{dy}{dx}=3\cdot\frac{d}{dx}\left(x^{3/2}\right)+2\cdot\frac{d}{dx}\left(x^{1/2}\right)+5\cdot\frac{d}{dx}\left(x^{-1/2}\right)$$

$$= 3\cdot\frac{3}{2}x^{1/2}+2\cdot\frac{1}{2}x^{-1/2}+5\left(\frac{-1}{2}\right)x^{-3/2}$$

$$= \frac{9}{2}\sqrt{x}+\frac{1}{\sqrt{x}}-\frac{5}{2}x^{-3/2}$$

5. $\dfrac{3}{\sqrt[3]{x}}-\dfrac{5}{\cos x}+\dfrac{6}{\sin x}-\dfrac{2\tan x}{\sec x}+7$

Solution:

Let $y=\dfrac{3}{\sqrt[3]{x}}-\dfrac{5}{\cos x}+\dfrac{6}{\sin x}-\dfrac{2\tan x}{\sec x}+7$ $\qquad\left[\tan x=\dfrac{\sin x}{\cos x}\right]$

$y = 3x^{-1/3}-5\sec x+6\ \text{cosec}\ x-2\sin x+7$

$$\frac{dy}{dx}=3\cdot\frac{d}{dx}\left(x^{-1/3}\right)-5\cdot\frac{d}{dx}(\sec x)+6\cdot\frac{d}{dx}(\text{cosec}\ x)-2\cdot\frac{d}{dx}(\sin x)+\frac{d}{dx}(7)$$

$$= 3\left(\frac{-1}{3}\right)x^{-4/3}-5\sec x\tan x-6\text{cosec}\ x\cot x-2\cos x+0$$

$$= \frac{-1}{x^{4/3}}-5\sec x\tan x-6\ \text{cosec}\ x\cot x-2\cos x$$

Example 2: **Differentiate the following w.r.t. x [By Product Rule]**

1. $x^9\cdot 9^x$

Solution: Let $y = x^9\cdot 9^x$

$$\frac{dy}{dx} = x^9 \cdot \frac{d}{dx}\left(9^x\right) + 9^x \cdot \frac{d}{dx}\left(x^9\right)$$

$$= x^9 \cdot 9^x \log 9 + 9^x \cdot 9x^8$$

2. $(x^2 - 2x + 1)(e^x + 4)$

Solution: Let $y = (x^2 - 2x + 1)(e^x + 4)$

$$\frac{dy}{dx} = \left(x^2 - 2x + 1\right)\frac{d}{dx}(e^x + 4) + \left(e^x + 4\right)\frac{d}{dx}\left(x^2 - 2x + 1\right)$$

$$= \left(x^2 - 2x + 1\right)\left(e^x\right) + \left(e^x + 4\right)(2x - 2)$$

3. $\sec x \cdot \tan x$

Solution:

Let $y = \sec x \tan x$

$$\frac{dy}{dx} = \sec x \cdot \frac{d}{dx}(\tan x) + \tan x \cdot \frac{d}{dx}(\sec x)$$

$$= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x$$

$$= \sec x\left(\sec^2 x + \tan^2 x\right)$$

4. $\cot x \cdot \left(\sqrt{3} - 4e^x\right)$

Solution:

Let $y = \cot x \cdot \left(\sqrt{3} - 4e^x\right)$

$$\frac{dy}{dx} = \cot x \cdot \frac{d}{dx}\left(\sqrt{3} - 4e^x\right) + \left(\sqrt{3} - 4e^x\right) \cdot \frac{d}{dx}(\cot x)$$

$$= \cot x\left(0 - 4e^x\right) + \left(\sqrt{3} - 4e^x\right)\left(-\operatorname{cosec}^2 x\right)$$

$$= -4e^x \cot x - \operatorname{cosec}^2 x\left(\sqrt{3} - 4e^x\right)$$

5. $x^2 e^x (\cos x - 4)$

Solution:

Let $y = x^2 e^x (\cos x - 4)$

$$\frac{dy}{dx} = x^2 e^x \cdot \frac{d}{dx}(\cos x - 4) + x^2 (\cos x - 4) \cdot \frac{d}{dx}(e^x) + (\cos x - 4)e^x \cdot \frac{d}{dx}\left(x^2\right)$$

$$= -x^2 e^x \sin x + x^2 e^x (\cos x - 4) + 2xe^x (\cos x - 4)$$

Example 3: **Differentiate the following w.r.t. *x* (By Quotient Rule)**

1. $\dfrac{a^2+x^2}{a^2-x^2}$

Solution:

Let $y=\dfrac{a^2+x^2}{a^2-x^2}$

$$\frac{dy}{dx}=\frac{\left(a^2-x^2\right)\cdot\frac{d}{dx}\left(a^2+x^2\right)-\left(a^2+x^2\right)\frac{d}{dx}\left(a^2-x^2\right)}{\left(a^2-x^2\right)^2}$$

$$=\frac{\left(a^2-x^2\right)(2x)-\left(a^2+x^2\right)(-2x)}{\left(a^2-x^2\right)^2}$$

$$=\frac{2xa^2-2x^3+2xa^2+2x^3}{(a^2-x^2)^2}$$

$$=\frac{4xa^2}{(a^2-x^2)^2}$$

2. $\dfrac{e^x-1}{e^x+1}$

Solution:

Let $y=\dfrac{e^x-1}{e^x+1}$

$$\frac{dy}{dx}=\frac{(e^x+1)\cdot\frac{d}{dx}(e^x-1)-(e^x-1)\cdot\frac{d}{dx}(e^x)}{(e^x+1)^2}$$

$$=\frac{(e^x+1)(e^x)-(e^x-1)(e^x)}{(e^x+1)^2}$$

$$=\frac{e^{2x}+e^x-e^{2x}+e^x}{\left(e^x+1\right)^2}$$

$$=\frac{2e^x}{\left(e^x+1\right)^2}$$

3. $\dfrac{\cos x+\sin x}{\cos x-\sin x}$

Solution:

Let $y = \dfrac{\cos x+\sin x}{\cos x-\sin x}$

$$\frac{dy}{dx}=\frac{(\cos x-\sin x)\frac{d}{dx}(\cos x+\sin x)-(\cos x+\sin x)\frac{d}{dx}(\cos x-\sin x)}{(\cos x-\sin x)^2}$$

$$=\frac{(\cos x-\sin x)(-\sin x+\cos x)-(\cos x+\sin x)(-\sin x-\cos x)}{(\cos x-\sin x)^2}$$

$$=\frac{(\cos x-\sin x)^2+(\cos x+\sin x)^2}{(\cos x-\sin x)^2}=\frac{2(\cos^2 x+\sin^2 x)}{(\cos x-\sin x)^2} \qquad (\because \cos^2 x+\sin^2 x=1)$$

$$=\frac{2}{(\cos x-\sin x)^2}$$

4. $4\left(2^{6\log 2^x}\right)$

Solution:

Let $y = 4\left(2^{\log_2^{x^6}}\right)$ $\qquad \left(\because a^{\log_a^x}=x\right)$

$$y=4\cdot x^6$$

$$\frac{dy}{dx}=24x^5$$

5. $\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

Solution:

Let $y=\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$ $\qquad y=\sqrt{\dfrac{2\sin^2 x}{2\cos^2 x}}$ $\qquad \left[\begin{array}{l}\because \text{From Multiple angle}\\ \cos 2x=2\cos^2 x-1\\ \cos 2x=1-\sin^2 x\end{array}\right]$

$$y=\tan x$$

$$\frac{dy}{dx}=\sec^2 x$$

6. $\dfrac{e^x(x-1)}{x^2+1}$

Solution:

Let $y = \dfrac{e^x(x-1)}{x^2+1}$

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{dx}{dx}\{e^x(x-1)\} - e^x(x-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\{e^x(1)+(x-1)e^x\} - e^x(x-1)(2x)}{(x^2+1)^2}$$

EXERCISE 18.1

One mark questions:

I. Differentiate the following w.r.t. x

1. $5e^x - \log x - 3\sqrt{x}$
2. $\log e^e$
3. $\dfrac{1}{x^{4/3}} - \dfrac{3}{x^{3/2}}$
4. $\dfrac{4x^2-3x}{x}$
5. $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$
6. $\sqrt[3]{x^2} + \dfrac{4}{\sqrt[4]{x^5}} + \dfrac{1}{x^7} + x\sqrt{x}$

Two marks questions:

II. Differentiate the following w.r.t. x

1. $(x-a)(x-b)$
2. $\dfrac{x-a}{x-b}$
3. $(5x^2+3x-1)(x-1)$
4. $x^{-3}(5+3x)$
5. $x^5(3-6x^{-9})$
6. $\sin^2 x$
7. $\dfrac{x+1}{x}$
8. $\dfrac{1}{ax^2+bx+c}$
9. $\dfrac{\cos x}{1+\sin x}$
10. $(x+\cos x)(x-\tan x)$

III. 1. If $f(x) = x^2 - 3x + 10$. Find $f'(50)$ and $f'(11)$

2. If $f(x) = x^n$ and if $f'(1) = 10$. Find the value of n.

3. If $y = x + \dfrac{1}{x}$ show that $x^2\dfrac{dy}{dx} - xy + 2 = 0$

Three marks questions:

IV. Differentiate the following w.r.t. x

1. $\dfrac{x^5-\cos x}{\sin x}$
2. $\dfrac{x^n-a^n}{x-a}$
3. $\dfrac{2}{x+1}-\dfrac{x^2}{3x-1}$
4. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$
5. $\dfrac{2^x \log x}{\sqrt{x}}$

V. 1. Find the derivative of $f(x)=\dfrac{1}{x}$ with respect to x from first principle.

2. If $y=x+\tan x$. Show that $\cos^2 x\cdot\dfrac{dy}{dx}=2-\sin^2 x$

ANSWERS 18.1

One mark questions:

I. 1. $5e^x-\frac{1}{x}-\frac{3}{2}\sqrt{x}$
2. 0
3. $-\frac{4}{3}x^{\frac{7}{3}}+\dfrac{9}{2x^{\frac{5}{2}}}$
4. 4
5. $1-\dfrac{1}{x^2}$
6. $\dfrac{2}{3}x^{-\frac{1}{3}}-\dfrac{5}{x^{\frac{9}{4}}}-\dfrac{7}{x^8}+\dfrac{3x^{\frac{1}{2}}}{2}$

Two marks questions:

II. 1. $2x-a-b$
2. $\dfrac{a-b}{(x-b)^2}$
3. $15x^2-4x-4$
4. $\dfrac{-3}{x^4}(5+2x)$
5. $15x^4+\dfrac{24}{x^5}$
6. $\sin 2x$
7. $\dfrac{-1}{x^2}$
8. $\dfrac{-(2ax+b)}{(ax^2+bx+c)^2}$
9. $\dfrac{-1}{1+\sin x}$
10. $\left(1-\sec^2 x\right)(x+\cos x)+(x-\tan x)(1-\sin x)$

III. 1. 97, 19 2. 10

Three marks questions:

IV. 1. $\dfrac{-x^5\cos x+5x^4\sin x+1}{(\sin x)^2}$
2. $\dfrac{nx^n-anx^{n-1}-x^n+a^n}{(x-a)^2}$

3. $\dfrac{-2}{(x+1)^2}-\dfrac{x(3x-2)}{(3x-1)^2}$

4. $\dfrac{\sqrt{a}}{\sqrt{x}\left(\sqrt{a}-\sqrt{x}\right)^2}$

5. $\dfrac{2x\left[\dfrac{2^x}{x}+2^x\log 2\cdot\log x\right]-2^x\log x}{2x^{\frac{3}{2}}}$

18.9 Composite functions (function of a function)

If $y=f(u)$ and $u=g(x)$ then $y=f[g(x)]$. This function $y=f[g(x)]$ is called a composite function.

Theorem (Chain Rule)

Let f be a real valued function which is composite of two functions u and v, i.e. $f=vou$.

Suppose $y=f(x)$ and $u=g(x)$ $\dfrac{dy}{dx}$ and $\dfrac{du}{dx}$ exists

we have $\therefore \dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$

Note : The chain rule can be extended to the functions which are composition of more than two function.

If $y=f(u),\ u=g(v),\ v=h(x)$

then $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dv}\cdot\dfrac{dv}{dx}$

Example 1 : Differentiate the following w.r.t. x:

1. $\left(3x^2+4x+5\right)^6$

Solution:

Let $y=\left(3x^2+4x+5\right)^6$

put $u=3x^2+4x+5$ then $y=u^6$

$\therefore \dfrac{dy}{du}=6u^5, \dfrac{du}{dx}=6x+4$

$\therefore \dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=6u^5(6x+4)$

$$\Rightarrow \quad \frac{dy}{dx} = 6(3x^2 + 4x + 5)^5 (6x + 4)$$

OR (using direct method) without substitution.

$$\frac{dy}{dx} = 6(3x^2 + 4x + 5) \cdot (6x + 4)$$

2. e^{x^2}

Solution:

Let $y = e^{x^2}$

Put $u = x^2$ then $y = e^u$

$$\frac{dy}{du} = e^u, \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot 2x \quad = \quad 2x \cdot e^{x^2}$$

OR $\frac{dy}{dx} = e^{x^2} \cdot 2x$

3. log ($x^2 - 2$)

Solution:

Let $y = \log(x^2 - 2)$

put $u = x^2 - 2$ then $y = \log u$

$\therefore \quad \frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 2x$

Now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{u} \cdot 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2 - 2}$

OR $\frac{dy}{dx} = \frac{1}{x^2 - 2} \cdot (2x)$

Example 2 : Differentiate the following w.r.t. *x*

1. $\tan^4 x$

Solution: Let $y = \tan^4 x$

$$\frac{dy}{dx} = 4\tan^3 x \frac{d}{dx} \tan x \quad = \quad 4\tan^3 x \cdot \sec^2 x$$

2. sec (sec x)

Solution:

Let $y = \sec(\sec x)$

$$\frac{dy}{dx} = \sec(\sec x)\tan(\sec x)\frac{d}{dx}(\sec x)$$

$$= \sec(\sec x)\tan(\sec x).\sec x\,.\tan x$$

3. $\dfrac{1}{(2-\cos x)^3}$

Solution:

$$\text{Let } y = \frac{1}{(2-\cos x)^3} = (2-\cos x)^{-3}$$

$$\frac{dy}{dx} = \frac{-3}{(2-\cos x)^4} \times (\sin x)$$

4. log(sec x + tan x)

Solution:

Let $y = \log(\sec x + \tan x)$

$$\therefore \frac{dy}{dx} = \frac{1}{\sec x + \tan x}\ \frac{d}{dx}(\sec x + \tan x)$$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x}\quad (\sec x \tan x + \sec^2 x)$$

$$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \sec x$$

Example 3 : Differentiate the following w.r.t. x :

1. $\sqrt{\tan\sqrt{x}}$

Solution:

Let $y = \sqrt{\tan\sqrt{x}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan\sqrt{x}}} \cdot \frac{d}{dx}\left(\tan\sqrt{x}\right)$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}} \sec^2\sqrt{x}\,\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan\sqrt{x}}} \cdot \sec^2\sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$

2. tan [log (sin$x$)]

Solution:

Let $y = \tan\left[\log\left(\sin x\right)\right]$

$$\frac{dy}{dx} = \sec^2\left[\log\left(\sin x\right)\right] \cdot \frac{1}{\sin x} \cdot \cos x$$

$$= \cot x \cdot \sec^2\left[\log\left(\sin x\right)\right] \qquad \left[\because \frac{\cos x}{\sin x} = \cot x\right]$$

3. $\cot\left(x^2 + \frac{1}{x^2}\right)$

Solution:

Let $y = \cot\left(x^2 + \frac{1}{x^2}\right)$

$$\frac{dy}{dx} = -\operatorname{cosec}^2\left(x^2 + \frac{1}{x^2}\right)\left(2x - \frac{2}{x^3}\right)$$

4. $y = \log\left(\frac{1-x^2}{1+x^2}\right)$

Solution:

$$y = \log\left(\frac{1-x^2}{1+x^2}\right) \Rightarrow y = \log\left(1-x^2\right) - \log\left(1+x^2\right) \qquad \left[\because \log\frac{\mathrm{A}}{\mathrm{B}} = \log \mathrm{A} - \log \mathrm{B}\right]$$

$$\frac{dy}{dx} = \frac{1}{1-x^2}(-2x) - \frac{1}{1+x^2}(2x)$$

$$= -2x\left[\frac{1}{1-x^2} + \frac{1}{1+x^2}\right]$$

$$= -2x\left[\frac{\left(1+x^2\right)+\left(1-x^2\right)}{\left(1-x^2\right)+\left(1+x^2\right)}\right]$$

$$= \frac{-4x}{1-x^4}$$

Example 3:

1. **If $y = \log\left[x + \sqrt{1+x^2}\right]$. Prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$**

Solution:

$$y = \log\left[x + \sqrt{1+x^2}\right]$$

$$\frac{dy}{dx} = \frac{1}{\left(x+\sqrt{1+x^2}\right)}\left[\frac{1}{1} + \frac{1}{2\sqrt{1+x^2}} \cdot 2x\right] \text{ take L.C.M}$$

$$= \frac{1}{\left(x+\sqrt{1+x^2}\right)}\left[\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}}\right]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

2. **If $y = \left(x + \sqrt{x^2+1}\right)^n$. Prove that $\left(x^2+1\right)\left(\frac{dy}{dx}\right)^2 = n^2y^2$**

Solution:

$$y = \left(x + \sqrt{x^2+1}\right)^n$$

$$\frac{dy}{dx} = n\left[x + \sqrt{x^2+1}\right]^{n-1} \left[1 + \frac{1}{2\sqrt{x^2+1}} \cdot 2x\right]$$

$$= \frac{n\left[x+\sqrt{x^2+1}\right]^n}{x+\sqrt{x^2+1}} \left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right].$$

$$= \frac{ny}{\sqrt{x^2+1}}$$

$$= \sqrt{x^2+1}\,\frac{dy}{dx} = ny \qquad \because y = \left(x+\sqrt{x^2+1}\right)^n$$

Squaring on both sides

$$\boxed{\left(x^2+1\right)\left(\frac{dy}{dx}\right)^2 = n^2y^2}$$

EXERCISE 18.2

One and Two marks questions:

Differentiate the following w.r.t. x

1. $\left(a^2-x^2\right)^{10}$
2. $\log[\log(\log x)]$
3. $\cos x^3$
4. $\sin^3\sqrt{x}$
5. $\left[\log(\cos x)\right]^2$
6. $\sec\left(x+\frac{1}{x}\right)$
7. $7^{\sin\sqrt{x}}$
8. $\sqrt{\cot\sqrt{x}}$
9. $\log\left(\sin\sqrt{x}\right)$
10. $\log\left[\log\left(\tan x\right)\right]$
11. $\cos 3x \cdot \sin 5x$
12. $\sin x \cdot \sin 2x$
13. $e^{\log e^{\left(x+\sqrt{x^2+a^2}\right)}}$
14. $e^{2x} \cdot \sin 3x$
15. $\cos^5 x \cdot \cos(x^5)$
16. $3^{x^2} \cdot \log x$
17. $\frac{x}{\sqrt{x^2-1}}$
18. $\frac{x}{\sqrt{2x-1}}$
19. $\frac{e^{\sin x}}{\sqrt{\log x}}$
20. $\log\left(\frac{1+\sin x}{1-\sin x}\right)$

Three marks questions:

1. If $y = \left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)$, show that $\frac{dy}{dx} = \sec^2\left(x+\frac{\pi}{4}\right)$

2. If $y = \log\left[\frac{1-\cos x}{1+\cos x}\right]$, prove that $\frac{dy}{dx} = 2 \operatorname{cosec} x$

3. Differentiate e^{2x} w.r.t. x from first principles.

4. Differentiate $\sin 2x$ w.r.t. x from first principles.

5. Differentiate $\tan ax$ w.r.t. x from first principles.

Answers 18.2

One and Two marks questions:

1. $-20x(a^2 - x^2)^9$

2. $\frac{1}{\log(\log x)} \cdot \frac{1}{\log x} \cdot \frac{1}{x}$

3. $-3x^2 \sin x^3$

4. $3\sin^2 \sqrt{x} \cdot \cos \sqrt{x} \cdot \frac{1}{2\sqrt{x}}$

5. $-2 \tan x \cdot \log(\cos x)$

6. $\sec\left(x + \frac{1}{x}\right) \cdot \tan\left(x + \frac{1}{x}\right) \cdot \left(1 - \frac{1}{x^2}\right)$

7. $\frac{(\log 7)\cos\sqrt{x} \cdot 7^{\sin\sqrt{x}}}{2\sqrt{x}}$

8. $\frac{-\operatorname{cosec}^2 \sqrt{x}}{4\sqrt{x}\sqrt{\cot\sqrt{x}}}$

9. $\frac{\cot\sqrt{x}}{2\sqrt{x}}$

10. $\frac{\sec^2 x}{\tan x \cdot \log(\tan x)}$

11. $5 \cos 3x \,.\, \cos 5x - 3 \sin 3x \,.\, \sin 5x$

12. $\frac{1}{2}(-\sin x + 3\sin 3x)$

13. $\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}$

14. $e^{2x}(3\cos 3x + 2\sin 3x)$

15. $-5\cos^4 x\left[x^4 \cos x \cdot \sin x^5 + \sin x \cos x^5\right]$

16. $\frac{3^{x^2}}{x}\left[1 + 2x^2 \log 3 \cdot \log x\right]$

17. $\frac{-1}{(x^2 - 1)^{3/2}}$

18. $\frac{x - 1}{(2x - 1)^{3/2}}$

19. $\frac{(2x \log x \cdot \cos x - 1)e^{\sin x}}{2x(\log x)^{3/2}}$

20. $2 \sec x$

Three marks questions:

3. $2e^{2x}$ 4. $2 \sin x \cos x$ 5. $a \sec^2 ax$

18.10 Differentiation of implicit functions:

If x and y are connected by a relation of the type $y = 2x^4 - 3x^3 + 4x^2 + 7$ then we say that y is defined explicity in terms of x and the function $y = f(x)$ is said to be an explicit function of x.

However if x and y are connected by a relation of the type $x^4 - 2x^2y + y^3 = 2x^2 + 1$ then such an equation is called an **implicit form of a function.** Usually such functions are expressed as $f(x,y) = 0$. Here $f(x,y)$ denotes an expression involving the variables x and y.

WORKING RULE

Step 1 : Differentiate both sides of the equation $f(x,y) = 0$ w.r.t x.

Step 2 : Collect the terms involving $\frac{dy}{dx}$ on one side and the remaining on the other side.

we get $g(x,y)\ \frac{dy}{dx} = h(x,y)$

Step 3 : Solve for $\frac{dy}{dx}$ and write $\frac{dy}{dx} = \frac{h(x,y)}{g(x,y)}$

This process of finding $\frac{dy}{dx}$ by above method is called as implicit differentiation.

Example 1: Find $\frac{dy}{dx}$ if

a) $x^2 + y^2 = a^2$ **b)** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

c) $x^2 + y^2 - 3xy + 2x + 3y - 5 = 0$ **d)** $e^y = \sin(x + y)$

e) $\sin xy + \cos(x + y) = 4$

Solution:

a) $x^2 + y^2 = a^2$ Differentiating w.r.t. x both sides we get

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow\ 2y \cdot \frac{dy}{dx} = -2x \quad \Rightarrow\ \frac{dy}{dx} = -\frac{x}{y}$$

b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Differentiating w.r.t. x both sides we get

$$\frac{1}{a^2}\cdot 2x+\frac{1}{b^2}\cdot 2y\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{1}{b^2}\cdot 2y\cdot\frac{dy}{dx}=\frac{-2x}{a^2}$$

$$\frac{dy}{dx}=\frac{-b^2x}{a^2y}$$

c) $x^2+y^2-3xy+2x+3y-5=0$

Differentiating w.r.t x both sides, we get

$$2x+2y\cdot\frac{dy}{dx}-3\left[x\cdot\frac{dy}{dx}+y\cdot 1\right]+2+3\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left[2y-3x+3\right]=-2x+3y-2$$

$$\therefore \quad \frac{dy}{dx}=\frac{3y-2x-2}{2y-3x+3}$$

d) $e^y=\sin(x+y)$

$$e^y\frac{dy}{dx}=\cos(x+y)\left[1+\frac{dy}{dx}\right]$$

$$e^y\frac{dy}{dx}-\cos(x+y)\frac{dy}{dx}=\cos(x+y)$$

$$\frac{dy}{dx}\left[e^y-\cos(x+y)\right]=\cos(x+y)$$

$$\frac{dy}{dx}=\frac{\cos(x+y)}{e^y-\cos(x+y)}$$

e) $\sin xy+\cos(x+y)=4$

$$\cos xy\left[\frac{d}{dx}(xy)\right]-\sin(x+y)\frac{d}{dx}(x+y)=0$$

apply product rule for xy

$$\cos xy\left[x\cdot\frac{dy}{dx}+y\right]-\sin(x+y)\left[1+\frac{dy}{dx}\right]=0$$

$$\frac{dy}{dx}\left[x\cos xy-\sin(x+y)\right]=\sin(x+y)-y\cos xy$$

$$\therefore\ \frac{dy}{dx}=\frac{\sin(x+y)-y\cos xy}{x\cos xy-\sin(x+y)}$$

Example 2:

If $x\sqrt{1+y}+y\sqrt{1+x}=0$ where $x\neq y$, show that $\frac{dy}{dx}=\frac{-1}{(1+x)^2}$.

Solution:

We have $x\sqrt{1+y}+y\sqrt{1+x}=0$, $\quad x\sqrt{1+y}=-y\sqrt{1+x}$

squaring both sides we get $x^2(1+y)=y^2(1+x)$

$\Rightarrow\ x^2+x^2y-y^2-y^2x=0$

$\Rightarrow\ (x^2-y^2)+xy(x-y)=0$

$\Rightarrow\ (x-y)\,[x+y+xy]=0$

$\therefore\ x+y+xy=0 \qquad [\because\ x\neq y]$

$\Rightarrow\ x+y(1+x)=0$

$$y=\frac{-x}{1+x}\ \text{(By Quotient Rule)}$$

$$\frac{dy}{dx}=\frac{(1+x)(-1)-(-x)1}{(1+x)^2}$$

$$=\frac{-1-x+x}{(1+x)^2}$$

$$\frac{dy}{dx}=\frac{-1}{(1+x)^2}$$

Example 3 :

If $e^x + e^y = e^{x+y}$, show that $\frac{dy}{dx} = -e^{(y-x)}$

Solution:

Consider $e^x + e^y = e^{x+y}$

$$e^x + e^y \cdot \frac{dy}{dx} = e^{x+y}\left[1 + \frac{dy}{dx}\right]$$

$$\Rightarrow \quad e^x + e^y \cdot \frac{dy}{dx} = (e^x + e^y)\left[1 + \frac{dy}{dx}\right] \qquad \left(\because\ e^{x+y} = e^x + e^y\right)$$

$$\Rightarrow \quad \frac{dy}{dx}\left[e^y - e^x - e^y\right] = e^x + e^y - e^x$$

$$\Rightarrow \quad \frac{dy}{dx}\left(-e^x\right) = e^y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-e^y}{e^x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -e^{y-x}$$

Example 4 :

If $\sin y = x\sin(a + y)$. Prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

Solution:

We have $\sin y = x\sin(a + y)$

$\therefore \quad x = \frac{\sin y}{\sin(a+y)}$ Differentiate w.r..t y

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y \cdot \cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin\left[(a+y) - y\right]}{\sin^2(a+y)}$$

$$= \frac{\sin a}{\sin^2(a+y)}$$

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

$$= \frac{\sin^2(a+y)}{\sin a}$$

Example 5:

If $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ then find $\frac{dy}{dx}$ at (a,b)

Solution:

We have $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

Differentiating w.r.t. x

$$n\left(\frac{x}{a}\right)^{n-1} \cdot \frac{1}{a} + n\left(\frac{y}{b}\right)^{n-1} \frac{1}{b} \cdot \frac{dy}{dx} = 0$$

At (a,b)

$$n \cdot (1)^{n-1} \frac{1}{a} + n \cdot (1)^{n-1} \cdot \frac{1}{b} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{n}{a} + \frac{n}{b}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{n}{b}\frac{dy}{dx} = \frac{-n}{a}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-b}{a}$$

EXERCISE 18.3

One and Two marks questions:

Differentiate the following w.r.t. x

1. $3x^2 + 4y^2 = 10$
2. $\sqrt{x} + \sqrt{y} = 3$
3. $y^2 = 4ax$
4. $x^{2/3} + y^{2/3} = a^{2/3}$
5. $x^2 = 4ay$
6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
7. $x^3 + y^3 = 3axy$
8. $x - y = 0$
9. $x^2 - y^2 = a^2$
10. $x + \sqrt{xy} = x^2$

Three marks questions:

Differentiate the following w.r.t. x

1. $\log(xy) = x^2 + y^2$
2. $2^x + 2^y = 2^{x+y}$
3. $x^y = y^x$
4. $\sin xy = \cos(x + y)$
5. $y = 4^{x+y}$

Five marks questions:

1. If $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = a$. Prove that $x \cdot \frac{dy}{dx} = y$
2. If $x^y = e^{y-x}$, show that $\frac{dy}{dx} = \frac{2 - \log x}{(1 - \log x)^2}$
3. If $\cos y = x \cos(a + y)$, show that $\frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin a}$
4. If $e^y = y^x$, show that $\frac{dy}{dx} = \frac{(\log y)^2}{\log y - 1}$
5. If $e^{x+y} = xy$, show that $\frac{dy}{dx} = \frac{y(1 - x)}{x(y - 1)}$
6. If $y^x = x^y$, show that $\frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}$

Answers

One mark questions:

1. $\frac{-3x}{4y}$
2. $-\sqrt{\frac{y}{x}}$
3. $\frac{2a}{y}$
4. $-\left(\frac{y}{x}\right)^{1/3}$
5. $\frac{x}{2a}$
6. $\frac{-b^2x}{a^2y}$
7. $\frac{ay - x^2}{y^2 - ax}$
8. 1
9. $\frac{x}{y}$
10. $-\left[\frac{y + 2\sqrt{xy}}{x + 2\sqrt{xy}}\right]$

Three marks questions:

1. $\frac{y(2x^2-1)}{x(1-2y^2)}$
2. $\frac{2^x(2^y-1)}{2^y(1-2^x)}$
3. $\frac{y[x\log y - y]}{x[y\log x - x]}$
4. $-\left[\frac{\sin(x+y)+y\cos xy}{x\cos xy+\sin(x+y)}\right]$
5. $\frac{y\log 4}{1-y\log 4}$

18.11 Differentiation of Infinite series

When we have to find $\frac{dy}{dx}$ in case y is given as infinite series then we use the fact "If a term is deleted from an infinite series, it remains unaltered".

Example 1:

If $y = x^{x^{x^{x^{\cdots\infty}}}}$. Prove that $\frac{dy}{dx} = \frac{y^2}{x(1-y\log x)}$

Solution:

We have $y = x^{x^{x^{x^{\cdots\infty}}}}$

Apply logarithm on both sides

$$\Rightarrow \quad y = x^y \qquad \Rightarrow \quad \log y = y\log x$$

Diffn. both sides w.r.t. x

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[\frac{1}{y} - \log x\right] = \frac{y}{x} \qquad \Rightarrow \frac{dy}{dx}\left[\frac{1-y\log x}{y}\right] = \frac{y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y^2}{x(1-y\log x)}$$

Example 2:

If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots\ldots\infty}}}$ show that $(2y-1)\frac{dy}{dx} = \frac{1}{x}$

Solution:

We have $y = \sqrt{\log x + \sqrt{\log x + \ldots\ldots\infty}}$

$$y = \sqrt{\log x + y}$$

Squaring on both sides

$$y^2 = \log x + y$$

Diffn. both sides w.r.t. x

$$2y \cdot \frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx} \quad \Rightarrow (2y-1)\frac{dy}{dx} = \frac{1}{x}$$

Example 3:

If $y = \sqrt{\sin x + \sqrt{\sin x + ...\infty}}$. Prove that $\frac{dy}{dx} = \frac{\cos x}{2y-1}$

Solution:

We have $y = \sqrt{\sin x + \sqrt{\sin x + ...\infty}}$

$$y = \sqrt{\sin x + y}$$

Squaring both sides

$$y^2 = \sin x + y$$

Diffn. both sides w.r.t. x

$$2y \cdot \frac{dy}{dx} = \cos x + \frac{dy}{dx} \qquad (2y-1)\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{2y-1}$$

Example 4:

If $y = e^{x+e^{x+e^{x+.....to\ \infty}}}$ show that $\frac{dy}{dx} = \frac{y}{1-y}$

Solution:

We have $y = e^{x+e^{x+.....\infty}}$

$$y = e^{x+y}$$

Apply log on both sides

$$\log y = x + y \qquad \left[\because \log_e^e = 1\right]$$

Diffn. both sides w.r.t. x

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left(\frac{1}{y} - 1\right) = 1$$

$$\frac{dy}{dx} = \frac{y}{y-1}$$

EXERCISE 18.4

1. If $y=\sqrt{x+\sqrt{x+\sqrt{x+.....\infty}}}$ then prove that $\frac{dy}{dx}=\frac{1}{2y-1}$

2. If $y=\sqrt{\tan x+\sqrt{\tan x+.....\infty}}$ then prove that $\frac{dy}{dx}=\frac{\sec^2 x}{2y-1}$

3. If $y=(\sin x)^{(\sin x)^{\cdot^{\cdot^{\cdot^{\infty}}}}}$, show that $\frac{dy}{dx}=\frac{y^2\cot x}{1-y\log(\sin x)}$

4. If $y=(\tan x)^{(\tan x)^{\cdot^{\cdot^{\cdot^{\infty}}}}}$ show that $\frac{dy}{dx}=\frac{2y^2\ \text{cosec}2x}{1-y\log\tan x}$

5. If $y=(e^x)^{(e^x)^{\cdot^{\cdot^{\cdot^{\infty}}}}}$, show that $\frac{dy}{dx}=\frac{y^2}{1-xy}$

18.12 Logarithmic Differentiation:

Many a times, the function whose derivative is required involves products, quotients and powers. In such a case differentiation can be carried out more conveniently if we take logarithms on the two sides and simplify before differentiation. This process is called logarithmic differentiation. This is especially useful when the variable occurs in the exponent.

Example 1:

Differentiate the following w.r.t. x.

1) x^x **2) $(\sin x)^{\tan x}$** **3) $(\log x)^{\cos x}$**

Solution:

1) Let $y=x^x$

Taking log on both sides, we get $\log y=x\log x$

$$\frac{1}{y}\cdot\frac{dy}{dx}=x\cdot\frac{1}{x}+\log x\cdot 1$$

$$\frac{dy}{dx}=y\left[1+\log x\right]$$

$$\therefore\ \frac{dy}{dx}=x^x\left[1+\log x\right]$$

2) Let $y = (\sin x)^{\tan x}$

Taking log on both sides we get $\log y = \tan x \cdot \log(\sin x)$

Differentiating both sides w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \tan x \cdot \frac{1}{\sin x}(\cos x) + \sec^2 x \log \sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left[\tan x \cdot \cot x + \sec^2 x \log \sin x\right] \qquad \left[\because \frac{\cos x}{\sin x} = \cot x\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\tan x}\left[1 + \sec^2 x \cdot \log \sin x\right]$$

3) Let $y = (\log x)^{\cos x}$

Taking log on both sides, we get $\log y = \cos x \cdot \log(\log x)$

Differentiating w.r.t x we get

$$\frac{1}{y}\frac{dy}{dx} = \cos x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + (-\sin x)\log\ (\log x)$$

$$\frac{dy}{dx} = y\left[\frac{\cos x}{x \log x} - \sin x \cdot \log(\log x)\right]$$

$$= (\log x)^{\cos x}\left[\frac{\cos x}{x \log x} - \sin x \cdot \log\ (\log x)\right]$$

Example 2:

Find $\frac{dy}{dx}$ if $y^x + x^y = a^b$

Solution:

Given that $y^x + x^y = a^b$

Putting $u = y^x$, $v = x^y$

we get $u + v = a^b$

$$\therefore \quad \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad .....(1)$$

Now $u = y^x$

Taking logarithm on both sides, we have $\log u = x \log y$

Differentiating both sides w.r.t. x we have

$$\frac{1}{u} \cdot \frac{du}{dx} = x \cdot \frac{d}{dx}(\log y) + \log y \cdot \frac{d}{dx}(x)$$

$$= x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \log y \cdot 1$$

$$\frac{du}{dx} = u\left[\frac{x}{y} \cdot \frac{dy}{dx} + \log y\right]$$

$$= y^x\left[\frac{x}{y} \cdot \frac{dy}{dx} + \log y\right] \quad \text{..... (2)}$$

Also $v = x^y$

Taking logarithm on both sides, we have $\log v = y \log x$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v} \cdot \frac{du}{dx} = y \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{dy}{dx}$$

$$= y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\frac{dv}{dx} = v\left[\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right]$$

$$= x^y\left[\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right] \quad \text{.... (3)}$$

from (1), (2) and (3) we have

$$y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] + x^y\left[\frac{y}{x} + \log x \frac{dy}{dx}\right]$$

$$(x \cdot y^{x-1} + x^y \cdot \log x)\frac{dy}{dx} = -y \cdot x^{y-1} - y^x \log y$$

$$\therefore \quad \frac{dy}{dx} = \frac{-\left[y^x \log y + y \cdot x^{y-1}\right]}{x \cdot y^{x-1} + x^y \log x}$$

Example 3:

Differentiate $\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$ w.r.t. x

Solution:

Let $y = \sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$

Taking logarithm on both sides

$$\log y = \frac{1}{2}\left[\log(x-3) + \log(x^2+4) - \log(3x^2+4x+5)\right]$$

Differentiating both sides w.r.t. x we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

Example 4:

If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Solution:

Given : $x^y = e^{x-y}$

Taking logarithm on both sides

$$\Rightarrow y\log x = (x-y)\log e$$

$$\Rightarrow y\log x = x - y \quad \Rightarrow y(1+\log x) = x$$

i.e. $y = \frac{x}{1+\log x}$

Diffn. w.r.t. x by quotient rule

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot 1 - x\cdot\frac{1}{x}}{(1+\log x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

Example 5:

If $x^m \cdot y^n = (x+y)^{m+n}$. Show that $\frac{dy}{dx} = \frac{y}{x}$

Solution:

We have $x^m \cdot y^n = (x+y)^{m+n}$

Taking log both sides we get $m\log x + n\log y = (m+n)\log(x+y)$

Differentiating both sides w.r.t. x, we get

$$m\cdot\frac{1}{x} + n\cdot\frac{1}{y}\ \frac{dy}{dx} = (m+n)\frac{1}{x+y}\left[1+\frac{dy}{dx}\right]$$

$$\Rightarrow \quad \frac{m}{n} + \frac{n}{y}\cdot\frac{dy}{dx} = \frac{m+n}{x+y}\left[1+\frac{dy}{dx}\right]$$

$$\frac{dy}{dx}\left[\frac{n}{y} - \frac{m+n}{x+y}\right] = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{n(x+y)-y(m+n)}{y(x+y)}\right] = \frac{x(m+n)-m(x+y)}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx}(nx+ny-my-ny) = \frac{y}{x}(mx+nx-mx-my)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cdot y(nx-my)}{x(nx-my)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

Example 6:

If $(xe)^y = e^x$, show that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Solution:

we have $(xe)^y = e^x$

Taking log on both sides $y\log\ (xe) = x\ \log\ e$ $\quad [\because \log xe = \log x + \log e = \log x + 1]$

$y(1+\log x) = x\cdot 1$ $\quad [\because \log e = 1]$

$$\therefore \quad y = \frac{x}{1+\log x}$$

Diffn. w.r.t. x

$$\frac{dy}{dx} = \frac{(1+\log x)\cdot 1 - x\cdot\frac{1}{x}}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2}$$

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

EXERCISE 18.5

Two and Three marks questions:

Differentiate the following w.r.t x:

1. $x^{\sqrt{x}}$
2. $x^{\sin x}$
3. $(\sin x)^x$
4. $x^{5+\log x}$
5. $x^{(\sin x - \cos x)}$

Five marks questions:

Differentiate the following w.r.t x:

1. $x^{\log x} + (\log x)^x$
2. $x^2 \cdot e^{x^2} \cdot \log x$
3. $(x+1)^2(x+2)^3(x+3)^4$
4. $x^{2x} + x^{x^2}$
5. $x^{\left(1+\frac{1}{x}\right)} + \left(1+\frac{1}{x}\right)^x$
6. $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$
7. $x^3 \cdot e^{2x} \cdot \sec^2 x$

ANSWERS 18.5

1. $\frac{x^{\sqrt{x}}}{2\sqrt{x}}(2+\log x)$
2. $x^{\sin x}\left[\frac{\sin x}{x} + \cos x \ \log x\right]$
3. $(\sin x)^x\left[x\cot x + \log\sin x\right]$
4. $x^{5+\log x}\left[\frac{1}{x}(5+2\log x)\right]$
5. $x^{\sin x - \cos x}\left[\frac{\sin x - \cos x}{x} + (\cos x + \sin x)\log x\right]$

Five mark questions:

1. $x^{\log x}\left(\frac{2\log x}{x}\right)+(\log x)^{x}\left[\frac{1}{\log x}+\log(\log x)\right]$

2. $xe^{x^2}\left[2\log x+2x^2\log x+1\right]$

3. $(x+1)^2(x+2)^3(x+3)^4\left[\frac{2}{x+1}+\frac{3}{x+2}+\frac{4}{x+3}\right]$

4. $x^{2x}\left[2+\log x^2\right]+x\cdot x^{x^2}\left[1+\log x^2\right]$

5. $x^{\left(1+\frac{1}{x}\right)}\left[\frac{1}{x}\left(1+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right]+\left(1+\frac{1}{x}\right)^{x}\left[\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}\right]$

6. $\frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$

7. $\left(x^3.e^{2x}\cdot\sec^2 x\right)\left[\frac{3}{x}+2+2\tan x\right]$

18.13 Differentiation of parametric functions:

If the variable x and y of a function are expressed as the functions of another variable 't' or 'θ' then we say the function is defined in parametric form. The variable 't' or 'θ' is called a parameter.

If $x=f(t),\ y=g(t)$

In order to find the derivative of function is such form, we have

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$$

or $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$ $\left(\text{where } \frac{dx}{dt}\neq 0\right)$

Thus $\frac{dy}{dx}=\frac{g'(t)}{f'(t)}$ $\left[\begin{matrix}dy/dt=g'(t)\\ and\ \ dx/dt=f'(t)\end{matrix}\right]$

provided $f'(t)\neq 0$

Example 1: Find $\frac{dy}{dx}$ if

a) $x = a\ \cos\theta,\ y = a\ \sin\theta$ **b)** $x = at^2,\ y = 2at$ **c)** $x = \frac{1-t^2}{1+t^2},\ y = \frac{2t}{1+t^2}$

Solution:

a) We have $x = a\ \cos\theta,\ y = a\ \sin\theta$

$$\therefore\ \frac{dx}{d\theta} = -a\ \sin\theta,\ \frac{dy}{d\theta} = a\ \cos\theta$$

Hence $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{a\ \cos\theta}{-a\ \sin\theta}$$

$$= -\cot\theta$$

b) We have $x = at^2,\ y = 2at$

$$\frac{dx}{dt} = 2at \text{ and } \frac{dy}{dt} = 2a$$

$$\therefore\ \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{2a}{2at}$$

$$= \frac{1}{t}$$

c) We have $x = \frac{1-t^2}{1+t^2},\ y = \frac{2t}{1+t^2}$

$$\frac{dx}{dt} = \frac{\left(1+t^2\right)\left(-2t\right) - \left(1-t^2\right)\left(2t\right)}{\left(1+t^2\right)^2}$$

$$= \frac{-4t}{\left(1+t^2\right)^2}$$

$$\frac{dy}{dt}=\frac{(1+t^2)(2)-2t\cdot 2t}{(1+t^2)^2}=\frac{2(1-t^2)}{(1+t^2)^2}$$

$$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\frac{2(1-t^2)}{(1+t^2)^2}}{\frac{-4t}{(1+t^2)^2}}=\frac{1-t^2}{-2t}$$

$$\frac{dy}{dx}=\frac{t^2-1}{2t}$$

Example 2:

If $x=a\left[\cos t+\log\tan\frac{t}{2}\right]$, $y=a\ \sin\ t$. Show that $\frac{dy}{dx}=\tan t$

Solution:

We have $x=a\left[\cos t+\log\tan\frac{t}{2}\right]$

$$\frac{dx}{dt}=a\left[-\sin t+\frac{1}{\tan\frac{t}{2}}\cdot\sec^2\frac{t}{2}\cdot\frac{1}{2}\right]$$

$$=a\left[-\sin t+\frac{\cos t/2}{2\sin t/2\cos^2 t/2}\right]$$ By Multiple angle $\begin{bmatrix}\sin 2t=2\sin t.\cos t\\ \sin t=2\sin t/2\cos t/2\\ \text{Half angle}\end{bmatrix}$

$$=a\left[-\sin t+\frac{1}{2\sin t/2\cos t/2}\right]=a\left[-\sin t+\frac{1}{\sin t}\right]$$

$$=a\left[\frac{-\sin^2 t+1}{\sin t}\right]=\frac{a\cos^2 t}{\sin t}$$

$$\frac{dx}{dt}=a\cot t\cdot\cos t$$

Again $y = a\sin t \qquad \frac{dy}{dt} = a\cos t$

Now $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{a\cot t \cdot \cos t}$

$\therefore \frac{dy}{dx} = \tan t$

Example 3:

If $x = a\theta$, $y = \frac{a}{\theta}$, prove that $\frac{dy}{dx} + \frac{y}{x} = 0$

Solution:

Take, $xy = a\theta \cdot \frac{a}{\theta}$

$xy = a^2$

Using product rule

$\Rightarrow \quad y \cdot 1 + x \cdot \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{y}{x} + \frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = 0$

Example 4:

Differentiate $\sin^3 x$ w.r.t. $\cos^3 x$

Solution:

Let $u = \sin^3 x$ and $v = \cos^3 x$

$\frac{du}{dx} = 3\sin^2 x \cos x$

$\frac{dv}{dx} = -3\cos^2 x \sin x$

$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{3\sin^2 x\cos x}{-3\cos^2 x\sin x} = -\tan x$

EXERCISE 18.6

Two and Three marks questions:

1. Find $\frac{dy}{dx}$ if

   a) $x = a\left(t - \frac{1}{t}\right),\ y = a\left(t + \frac{1}{t}\right)$

   b) $x = e^{2t},\ y = \log(2t+1)$

   c) $x = \log(1+t),\ y = \frac{1}{1+t}$

   d) $x = \log t,\ y = \frac{1}{t}$

   e) $x = 4t,\ y = \frac{4}{t}$

   f) $x = a\ \sec\theta,\ y = b\ \tan\theta$

   g) $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$

   h) $x = a\ \cos(\log t),\ y = a\ \log(\cos t)$

2. Differentiate $\tan^2 x$ w.r.t. $\cos^2 x$

3. Differentiate $\sin^2 x$ w.r.t. x^2

4. Differentiate $\tan\sqrt{x}$ w.r.t. $\sqrt{x}$

5. Differentiate $\log x$ w.r.t. $1/x$

6. Differentiate $\log \sin x$ w.r.t. $\sqrt{\cos x}$

7. If $x = e^{\log\cos 4\theta},\ y = e^{\log\sin 4\theta}$ show that $\frac{dy}{dx} = \frac{-x}{y}$

8. If $x = a\ \cos^4\theta,\ y = a\ \sin^4\theta$ show that $\frac{dy}{dx} = -\tan^2\theta$

9. If $x = e^t(\cos t + \sin t)\ ,\ y = e^t(\cos t - \sin t)$. Show that $\frac{dy}{dx} = -\tan t$

10. If $x = a\log\sec\theta,\ y = a(\tan\theta - 1)$ show that $\frac{dy}{dx} = 2\ \text{cosec}2\theta$

ANSWERS 18.6

1) a) $\frac{t^2-1}{t^2+1}$ b) $\frac{1}{(2t+1)e^{2t}}$ c) $\frac{-1}{1+t}$

   d) $\frac{-1}{t}$ e) $\frac{-1}{t^2}$ f) $\frac{b}{a}\ \text{cosec}\,\theta$

   g) $\cot\left(\frac{\theta}{2}\right)$ h) $\frac{t\tan t}{\sin(\log t)}$

2) $-\sec^4 x$ 3) $\cos x^2$ 4) $\sec^2 \sqrt{x}$ 5) $-x$

6) $\dfrac{-2(\cos x)^{3/2}}{\sin^2 x}$

18.14 Second order derivative

Let y = *f(x)* then

$$\frac{dy}{dx} = f'(x) = y_1 \quad .....(1)$$

If *f'(x)* is differentiable, we may differentiate (1) again w.r.t *x*. Then the left hand side becomes $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ which is called the second order derivative of *y* w.r.t *x* and is denoted by $\dfrac{d^2y}{dx^2}$. The second order derivative of *f(x)* is denoted by *f''(x)*. It is also denoted by D^2y or y'' or y_2 if $y = f(x)$.

In general the process of differentiating the same function again and again is called successive differentiation, further $\dfrac{d^n y}{dx^n}, y_n$ or $f^{(n)}(x)$ denotes the nth order derivative (if it exists) of the function $y = f(x)$.

1. Find the second order derivative of the following functions w.r.t. ***x*:**

a) $x + x^{-1}$ **b)** $3\cos x + 4\sin x$

c) $x = at^2, y = 2at$ **d)** $x = ct, y = \dfrac{c}{t}$

Solution:

a) we have $y = x + x^{-1} \Rightarrow y = x + \dfrac{1}{x}$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2} \qquad \frac{d^2y}{dx^2} = 0 + \frac{2}{x^3}$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

b) we have $y = 3\cos x + 4\sin x$

$$\frac{dy}{dx} = -3\sin x + 4\cos x$$

$$\frac{d^2y}{dx^2} = -3\cos x - 4\sin x$$

c) we have $x = at^2,\ y = 2at$

$$\therefore \frac{dx}{dt} = 2at,\ \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at}$$

$$\frac{dy}{dx} = \frac{1}{t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-1}{t^2}\cdot\frac{dt}{dx} = \frac{-1}{t^2}\left(\frac{1}{2at}\right) = \frac{-1}{2at^3}$$

d) we have $x = ct,\ y = c/t$

$$\therefore \frac{dx}{dt} = c,\ \frac{dy}{dt} = \frac{-c}{t^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-c/t^2}{c} = \frac{-1}{t^2}$$

$$\frac{d^2y}{dx^2} = -\left(\frac{-2}{t^3}\right)\cdot\frac{dt}{dx} = \frac{2}{t^3}\cdot\frac{1}{c}$$

$$= \frac{2}{ct^3}$$

Example 2 :

If $y^2 + 2y = x^2$. Show that $y_2 = \dfrac{1}{(1+y)^3}$

Solution:

We have $y^2 + 2y = x^2$

Diffn. w.r.t. x

$$2y\cdot y_1 + 2y_1 = 2x \Rightarrow y_1 = \frac{2x}{2y+2} \qquad \left(y_1 = \frac{dy}{dx}\right)$$

$$y_1 = \frac{x}{y+1}$$

Differentiate again w.r.t. x

$$y_2 = \frac{(y+1)\cdot 1 - x\cdot y_1}{(y+1)^2} = \frac{(y+1) - x\cdot\left(\frac{x}{y+1}\right)}{(y+1)^2}$$

$$\Rightarrow \frac{(y+1)^2 - x^2}{(y+1)^3} = \frac{y^2+1+2y-x^2}{(y+1)^3}$$

$$y_2 = \frac{1}{(y+1)^3} \qquad \left(\because\ y^2+2y = x^2\right)$$

Example 3 :

If $y = A\sin x + B\cos x$ then prove that $\frac{d^2y}{dx^2} + y = 0$

Solution:

We have $y = A\sin x + B\cos x$

$$\frac{dy}{dx} = A\cos x - B\sin x$$

$$\frac{d^2y}{dx^2} = -A\sin x - B\cos x \quad \Rightarrow \frac{d^2y}{dx^2} = -y$$

Hence $\frac{d^2y}{dx^2} + y = 0$

Example 4:

If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\cdot\frac{dy}{dx} + 6y = 0$

Solution:

we have $y = 3e^{2x} + 2e^{3x}$

$$\frac{dy}{dx} = 6e^{2x} + 6e^{3x} = 6\left(e^{2x} + e^{3x}\right)$$

Again diffn. w.r.t. x

$$\frac{d^2y}{dx^2} = 6\left(2e^{2x} + 3e^{3x}\right)$$

Hence $\frac{d^2y}{dx^2} - 5 \cdot \frac{dy}{dx} + 6y$

$= 6\left(2e^{2x} + 3e^{3x}\right) - 30\left(e^{2x} + e^{3x}\right) + 6\left(3e^{2x} + 2e^{3x}\right)$

$= 0$

Example 5:

If $y = \left[x + \sqrt{a^2 + x^2}\right]^n$. Show that $\left(a^2 + x^2\right)y_2 + xy_1 - n^2y = 0$

Solution:

we have $y = \left[x + \sqrt{a^2 + x^2}\right]^n$ .....(1)

$$y_1 = n\left[x + \sqrt{a^2 + x^2}\right]^{n-1} \left[1 + \frac{1}{2\sqrt{a^2 + x^2}} \cdot 2x\right]$$

$$= n\left[x + \sqrt{a^2 + x^2}\right]^{n-1} \left[\frac{\sqrt{a^2 + x^2} + x}{\sqrt{a^2 + x^2}}\right] = \frac{n\left(x + \sqrt{a^2 + x^2}\right)^{n-1+1}}{\sqrt{a^2 + x^2}} \text{(same base)}$$

$$\sqrt{a^2 + x^2} \cdot y_1 = n\left(x + \sqrt{a^2 + x^2}\right)^n$$

$$\Rightarrow \sqrt{a^2 + x^2}\ y_1 = ny \qquad [\text{from}(1)]$$

Squaring both sides we get $\left(a^2 + x^2\right)y_1^2 = n^2y^2$

Differentiating both sides $\left(a^2 + x^2\right)2y_1y_2 + 2x \cdot y_1^2 = n^2 \cdot 2y \cdot y_1$

Dividing throughout by $2y_1$, we get $\left(a^2 + x^2\right)y_2 + xy_1 - n^2y = 0$

Example 6:

If $y = e^x \log x$. Show that $xy_2 - (2x - 1)y_1 + (x - 1)y = 0$

Solution:

We have $y = e^x \cdot \log x$

$$\frac{dy}{dx} = e^x \cdot \frac{1}{x} + e^x \cdot \log x$$

$$y_1 = \frac{e^x}{x} + y \qquad \left[\because\ y = e^x \log x\right]$$

$xy_1 = e^x + yx$ .....(1)

Differentiating both sides w.r.t x we get $xy_2 + y_1 \cdot 1 = e^x + (y_1 x + y)$

$\Rightarrow \quad xy_2 + y_1 = xy_1 - yx + xy_1 + y \qquad \left(\because\ e^x = xy_1 - yx \text{ from (1)}\right)$

$\Rightarrow \quad xy_2 + y_1 - 2xy_1 + xy - y = 0$

$\Rightarrow \quad xy_2 - (2x-1)y_1 + (x-1)y = 0$

Example 7:

If $e^y(x+1) = 1$. Show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

Solution:

we have $e^y(x+1) = 1$ applying the product rule

$$e^y \cdot 1 + e^y \cdot \frac{dy}{dx}(x+1) = 0$$

$$e^y(x+1)\frac{dy}{dx} = -e^y$$

$$\frac{dy}{dx} = \frac{-e^y}{e^y(x+1)}$$

$$\frac{dy}{dx} = \frac{-1}{1+x} \qquad \text{.....(1)}$$

Again diffn. w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{1}{(1+x)^2}$$

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \quad \text{From (1)}$$

Example 8 :

If $y = \tan x + \sec x$. Show that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$.

Solution:

we have $y = \tan x + \sec x$

$$\frac{dy}{dx} = \sec^2 x + \sec x \cdot \tan x$$

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \frac{1+\sin x}{\cos^2 x} = \frac{1+\sin x}{1-\sin^2 x} = \frac{1+\sin x}{(1+\sin x)(1-\sin x)}$$

$$= \frac{1}{1-\sin x} \quad \left[\because \cos^2 x = 1-\sin^2 x\right]$$

Now $\frac{dy}{dx} = \frac{1}{1-\sin x}$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{-1}{(1-\sin x)^2}\ (-\cos x)$$

$$\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$$

Example 9:

If $xy + 6y = 2x$. Show that $\frac{d^2y}{dx^2} = \frac{-24}{(x+6)^3}$.

Solution:

Given: $xy + 6y = 2x$

$y(x + 6) = 2x$

$$\therefore \ y = \frac{2x}{x+6}$$

$$\frac{dy}{dx} = \frac{(x+6)(2)-2x(1)}{(x+6)^2} \quad \text{(by quotient rule)}$$

$$= \frac{12}{(x+6)^2}$$

Again diffn. w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{(x+6)^2(0)-12.2(x+6)}{(x+6)^4} = \frac{-24(x+6)}{(x+6)^4}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-24}{(x+6)^3}$$

Example 10:

If $y = \log\left(x + \sqrt{x^2+1}\right)$. Show that $(x^2 + 1)\, y_2 + xy_1 = 0$.

Solution:

Given: $y = \log\left(x + \sqrt{x^2+1}\right)$

$$\frac{dy}{dx} = y_1 = \frac{1}{\left(x + \sqrt{x^2+1}\right)}\left[1 + \frac{1}{2\sqrt{x^2+1}}.2x\right]$$

$$= \frac{1}{\cancel{\left(x + \sqrt{x^2+1}\right)}}\left[\frac{\cancel{\sqrt{x^2+1}} + x}{\sqrt{x^2+1}}\right]$$

$$y_1 = \frac{1}{\sqrt{x^2+1}}$$

$$\therefore \sqrt{x^2+1}.y_1 = 1$$

Again diffn. w.r.t. x

$$\sqrt{x^2+1}.y_2 + y_1\left(\frac{1}{2\sqrt{x^2+1}}.2x\right) = 0$$

$$\sqrt{x^2+1}.y_2 + \frac{xy_1}{\sqrt{x^2+1}} = 0$$

Multiply by $\sqrt{x^2+1}$

$$\therefore \left(x^2+1\right)y_2 + xy_1 = 0$$

Example 11:

If $x^2 + xy + y^2 = a^2$, show that $\dfrac{d^2y}{dx^2} = \dfrac{-6a^2}{(x+2y)^3}$

Solution:

Given : $x^2 + xy + y^2 = a^2$

Diffn. w.r.t. x on both sides we get

$$2x + \left(x\frac{dy}{dx} + y.1\right) + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x+2y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-(2x+y)}{(x+2y)} \qquad \text{.....(1)}$$

Again diffn. w.r.t. x

$$\frac{d^2y}{dx^2} = -\left[\frac{(x+2y)\left(2.1+\frac{dy}{dx}\right)-(2x+y)\left(1+2\frac{dy}{dx}\right)}{(x+2y)^2}\right]$$

$$= -\left[\frac{\cancel{2x} + x\frac{dy}{dx} + 4y + 2y\frac{dy}{dx} - \cancel{2x} - 4x\frac{dy}{dx} - y - 2y\frac{dy}{dx}}{(x+2y)^2}\right]$$

$$= -\left[\frac{\frac{dy}{dx}(x+2y-4x-2y)+(4y-y)}{(x+2y)^2}\right]$$

$$= -\left[\frac{\frac{dy}{dx}(-3x)+3y}{(x+2y)^2}\right]$$

$$= -\left[\frac{-\left(\frac{2x+y}{x+2y}\right)(-3x)+3y}{(x+2y)^2}\right] \text{ from (1)}$$

$$= \frac{\left[\frac{-6x^2-3xy-3y(x+2y)}{(x+2y)}\right]}{(x+2y^2)}$$

$$= \frac{-6x^2-3xy-3xy-6y^2}{(x+2y)^3}$$

$$= \frac{-6(x^2+xy+y^2)}{(x+2y)^3}$$

$$\frac{d^2y}{dx^2} = \frac{-6a^2}{(x+2y)^3} \qquad (\because x^2+xy+y^2 = a^2)$$

Example 12:

If $y = (a^2 + x^2)^6$. **Show that** $(x^2 + a^2)y_2 - 10xy_1 - 12y = 0$

Solution:

$y = (a^2 + x^2)^6$

Diffn. w.r.t. x

$y_1 = 6(a^2 + x^2)^5.2x$

$= 12x.(a^2 + x^2)^5$ .....(1)

Again diffn. w.r.t. x

$y_2 = 12x.5(a^2 + x^2)^4.2x + (a^2 + x^2)^5.(12)$

$= 120x^2(a^2 + x^2)^4 + 12(a^2 + x^2)^5$ .....(2)

LHS $= (x^2 + a^2)y_2 - 10xy_1 - 12y_1$ From (1) and (2)

$= (x^2 + a^2)\ \{120x^2(a^2 + x^2)^4 + 12(a^2 + x^2)^5\} - 10x\ \{12x(a^2 + x^2)^5\} - 12(a^2 + x^2)^6$

$= \cancel{120x^2(a^2 + x^2)^5} + \cancel{12(x^2 + a^2)^6} - \cancel{120x^2(a^2 + x^2)^5} - \cancel{12(a^2 + x^2)^6}$

$= 0 =$ RHS.

Exercise 18.7

One mark questions:

1. Find $\frac{d^2y}{dx^2}$

1) $y = 3x^3 + 4x^2 + 7$ 2) $y = \sqrt{2x+3}$ 3) $y = e^{3x+2}$

4) $y = x^3 \cdot \log x$ 5) $y = \log x + a^x$ 6) $y = e^{-x} \sin 2x$

7) $y = \log(\log x)$ 8) $y = \cos 4x \cos 2x$ 9) $y = \sin 3x \sin 2x$

10) $y = \cos mx \sin nx$

Two and Three marks questions:

1. Find $\frac{d^2y}{dx^2}$ if

1) $x = a\ \cos\theta,\ y = a\ \sin\theta$ 2) $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$

3) $x = a\ \cos^3 t,\ y = -a\ \sin^3 t$

2. If $y = \sin mx$, show that $\frac{d^2y}{dx^2} + m^2y = 0$.

3. If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$.

4. If $y = e^{ax} + e^{-ax}$, show that $y_2 - a^2 y = 0$.
5. If $y = 2 + \log x$, show that $xy_2 + y_1 = 0$

Five marks questions:

1. If $x^2 - xy + y^2 = a^2$, show that $\frac{d^2y}{dx^2} = \frac{6a^2}{(x-2y)^3}$
2. If $x^2 + 2xy + 3y^2 = 1$, show that $y_2 = \frac{-2}{(x+3y)^3}$
3. If $y = a\cos mx + b\sin mx$, show that $\frac{d^2y}{dx^2} + m^2 y = 0$
4. If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$
5. If $y = \log\left(x - \sqrt{x^2+1}\right)$, show that $\left(x^2+1\right)y_2 + xy_1 = 0$
6. If $y = x + \sqrt{x^2-1}$, show that $\left(x^2-1\right)y_2 + xy_1 - y = 0$
7. If $y = \left(x + \sqrt{x^2+1}\right)^m$, show that $\left(x^2+1\right)y_2 + xy_1 - m^2 y = 0$
8. If $y = \sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

ANSWERS

One mark questions:

1. $18x + 8$
2. $\frac{-1}{(2x+3)^{3/2}}$
3. $9y$
4. $x(5 + 6\log x)$
5. $-\frac{1}{x^2} + (\log a)^2 \cdot a^x$
6. $-e^x(3\sin 2x + 4\cos 2x)$
7. $\frac{-1}{x^2} \cdot \frac{1}{\log x}\left[\frac{1}{\log x} + 1\right]$
8. $-2(\cos 2x + 9\cos 6x)$
9. $\frac{1}{2}(-\cos x + 25\cos 5x)$
10. $\frac{1}{2}\left[(m+n)^2 \sin(m+n)x - (m-n)^2 \sin(m-n)x\right]$

Two marks questions:

1. $\frac{1}{a}\operatorname{cosec}^3\theta$
2. $\frac{1}{4a}\sec^4\left(\frac{\theta}{2}\right)$
3. $\frac{1}{3a}\sec^4 t \operatorname{cosec} t$

* * * * *

Chapter

19

APPLICATION OF DERIVATIVES

19.1 Introduction:

Derivative have wide range of application in science engineering, social science and also in the field of commerce. This application information about the behaviour function and hence it in an aid in sketching graph of functions. The derivatives used to define a scope of a tangent to the curve, to find the angle of intersection of two curves. The interpretation of derivate as a rate of change of one variable with respect to another leads to its importance in many field. In chemistry the rate of change of chemical reaction in physics velocity in rectilinear motion in biology the rate of growth of bacteria and in economics it physics concerned with marginal cost, marginal revenue and marginal profit etc.

Even though derivation find many application few application are discussed in this chapter.

19.2 Derivative as a Rate Measure:

If one variable depends on another variable, then the rate at which the first changes with respect to (w.r.t) the scond can be measured by means of derivative.

Note:

$y = f(x)$ be a function of x corresponds to an increment δx of x. Let δy be the increment in y.

$\therefore\ y + \delta y = f(x + \delta x)$

The average change of y w.r.t. x is given by

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

The $\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$ exists and is called the instantaneous rate of change of 'y' w.r.t. 'x'.

hence $\boxed{\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx} = f^{r}(x)}$ represent the rate of change of y w.r.t x

Note:

(Here rate measure w.r.t time (t) change mean = increase or decrease)

(i) If 's' is the displacement of a particle in time 't', v is the velocity of the particle then

v = the rate of change of displacement $\boxed{v = \frac{ds}{dt}}$

Intitial velocity mean t = 0, final velocity or particle at rest means v = 0.

(ii) In v is the velocity of a particle in time 't', 'a' is the acceleration of the particle then

a = the rate of chante of velocity $\therefore a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$

$\therefore \boxed{a = \frac{dv}{dt} \text{ or } \frac{d^2s}{dt^2}}$

(iii) Similarly the rate of change of area 'A' of a circle w.r.t its radius 'r' is given by $\left(\frac{d\text{A}}{dr}\right)$

and w.r.t. time 't' = $\left(\frac{d\text{A}}{dt}\right)$

(iv) The rate of change of volume 'v' of a sphere w.r.t. its radius 'r' = $\left(\frac{d\text{v}}{dr}\right)$ and

w.r.t. time 't' $\left(\frac{d\text{v}}{dt}\right)$

(v) The rate of change of temperature w.r.t. 't' = $\frac{dT}{dt}$

(vi) The rate of change of price 'p' w.r.t. quantity $q = \frac{dp}{dq}$

Important formulae of plane / solid

(1) $\begin{cases} \text{Area of square} = \text{A} = (\text{side})^2 = x^2 \\ \text{Perimeter of square} = 4(\text{side}) = 4x \end{cases}$

(2) $\begin{cases} \text{Area of circle} = \pi \text{r}^2 \\ \text{Perimeter or circumference of circle} = 2\pi r \end{cases}$

(3) $\begin{cases} \text{Surface area of sphere} = 4\pi r^2 \\ \text{Volume of a sphere} = \frac{4}{3}\pi r^3 \end{cases}$

(4) Volume of a cylinder = $\pi r^2 h$

(5) Volume of a cone = $\frac{1}{3}\pi r^2 h$

(r = radius, h = height)

(6) Area of an equilateral triangle $= \frac{\sqrt{3}}{4}$ (side)^2

(7) $\begin{cases} \text{Volume of cube} = (\text{side})^3 = x^3 \\ \text{Surface area of cube} = 6(\text{side})^2 = 6x^2 \end{cases}$

Worked Examples

Example 1 :

A particle moves a distance $S = 6t^2 - t^3 + 5$ (S = mt, t = sec) then find

(i) the velocity and acceleration after 3 second

(ii) the time at which it is rest

Solution :

(i) Formula, $V = \frac{ds}{dt}$

$= 6(2t) - 3t^2 + 0$

$V = 12t - 3t^2$

at $t = 3$ sec, $V = 12(3) - 3(3)^2 = 9$ mt/sec

acceleration $= \frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$

$a = 12(1) - 3(2t)$

$a = 12 - 6t$

at $t = 3$sec $\quad acc = 12 - 6(3) = -6$

$\therefore \quad acc = -6$ mt/sec^2 (retardation)

(ii) At rest $\quad V = 0$

$\therefore \quad \frac{ds}{dt} = 0$

$12t - 3t^2 = 0$

$3t(4 - t) = 0$

$3t = 0 \quad$ or $\quad 4 - t = 0$

$t = 0 \quad$ or $\quad t = 4$ sec

$\therefore$ the particle became to rest at 0 or at 4 sec

Example 2 :

The distance 'S' feet travelled by a particle in time 't' second is given by $s = t^3 - 6t^2 + 15t + 2$. Find when the aceleration is zero.

Solution :

Given acc = 0, t = ?

$$V = \frac{ds}{dt} = 3t^2 - 12t + 15$$

$$\therefore \quad acc = \frac{dv}{dt} = 6t - 12$$

$$0 = 6t - 12 \qquad \therefore 6t = 12$$

$$\boxed{t = 2 \text{ sec}}$$

Example 3 :

According to law of motion of a particle if $s = t^3 - 6t^2 + 9t + 8$, find its initial velocity?

Solution :

$$V = \frac{ds}{dt}$$

$$= 3t^2 - 12t + 9$$

Initial velocity means at t = 0 sec

$\therefore$ Initial velocity = 3(0) – 12(0) + 9 = 9 unit

Example 4 :

If the displacement 's' at any time 't' is given by $s = \sqrt{1-t}$. Show that the velocity is inversely proportional to the displacement.

Solution :

$$V = \frac{ds}{dt}$$

$$= \frac{1}{2\sqrt{1-t}} \frac{d}{dt}(1-t)$$

$$= \frac{1}{2\sqrt{1-t}}(-1) = \frac{1}{2\sqrt{1-t}} = \frac{1}{2} \times \frac{1}{\sqrt{1-t}} = -\frac{1}{2} \times \frac{1}{s}$$

$$\boxed{V \alpha \frac{1}{s}} \quad \left(\because -\frac{1}{2} = a \text{ constant} \right)$$

I. 3 marks question:

Example 5 :

The displacement 's' of a particle at time 't' is given by $S = 2t^3 - 5t^2 + 4t - 3$ find

(i) the time when the acceleration is 14ft sec^2.

(ii) the velocity and displacement at that time.

Solution :

(i) $V = \frac{ds}{dt}$

$= 2(3t^2) - 5(2t) + 4(1)$

$= 6t^2 - 10t + 4$

$\therefore$ acc $= \frac{ds}{dt} = 6(2t) - 10$

acc $= 12t - 10$, given acc $= 14$ ft sec^2

$14 = 12t - 10$

$24 = 12t \quad \therefore \quad \boxed{t = 2 \text{sec}}$

(ii) $\therefore$ at $t = 2$ sec, $V = 6(2)^2 - 10(2) + 4$

$= 24 - 20 - 4 = 8$ ft / sec

at $t = 2$ sec, $S = 2(2)^3 - 5(2)^2 + 4(2) - 3$

$= 16 - 20 + 8 - 3$

$= 24 - 23 = 1$ ft

Example 6 :

When brakes are applied to a moving car, the car travels a distance 's' feet in 't' sec given by $s = 20t - 40t^2$. When and where does the car stop?

Solution :

Given, car stop $v = 0$, $t = ?$ $s = ?$

(i) $v = \frac{ds}{dt}$

$0 = 20(1) - 40(2t)$

$0 = 20 - 80t$

$80t = 20 \quad \therefore \quad t = \frac{20}{80} = \frac{1}{4}$ sec or $\boxed{0.25 \text{sec}}$

$\therefore$ when the brakes are applied the car take 0.25 sec to stop

(ii) $\therefore$ distance travelled during $t = 0.25$ sec

$$s = 20t - 40t^2$$

$$s = 20(0.25) - 40(0.25)^2$$

$$\boxed{s = 2.5 \text{ feet}}$$

Example 7 :

A particle moves according to the laws $s = t^3 + at^2 + bt$. Find 'a' and 'b' if the initial velocity is 5 unit and when $t = 1$ sec it is moving with a velocity which is 4 times its initial velocity.

Solution : $v = \frac{ds}{dt}$

$$= 3t^2 + a(2t) + b(1)$$

$$v = 3t^2 + 2at + b$$

Initial velocity at $t = 0$ sec is 5

$$\therefore 5 = 3(0) + 2a(0) + b$$

$$\boxed{b = 5}$$

And at $t = 1$ sec, velocity = 4 times initial velocity

$$3t^2 + 2at + b = 4 \times 5$$

$$3(1) + 2a(1) + b = 20$$

$$2a + b = 20 - 3$$

$$2a + b = 17 \quad \text{put } b = 5$$

$$2a + 5 = 17$$

$$2a = 17 - 5$$

$$2a = 12$$

$$\therefore \boxed{a = 6}$$

Example 8 :

A particle shot vertically upward rises 's' feet in 't' sec where $s = 40t - 16t^2$. Find the greatest height attained by the particle.

Solution :

The greatest height attained by the particle is $v = 0$

$$v = \frac{ds}{dt}$$

$$0 = 40(1) - 16(2t)$$

$$32t = 40 \quad \therefore\ t = \frac{40}{32} = \frac{5}{4} \text{ or } \boxed{1.25\text{sec}}$$

at t = 1.25sec, the greatest height reached by the particle

$$s = 40\,(1.25) - 16(1.25)^2$$

$$s = 50 - 16\left(\frac{25}{10}\right) = 50 - 25 = \boxed{25 \text{ feet}}$$

Example 9 :

The radius of a circular plate increase at the rate $\frac{8}{3\pi}$ cm/sec. Find the

(i) rate of change in the area when the diameter in 12cm

(ii) and also find the rate of increase of the circumference of circle after 3 sec.

Solution :

Given $\frac{dr}{dt} = \frac{8}{3\pi}$, $\frac{dA}{dt} = ?$ $\frac{dc}{dt} = ?$, r = 12cm

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi.2r\left(\frac{dr}{dt}\right)$$

$$= \pi.2r\left(\frac{8}{3\pi}\right) = 32cm/\sec^2$$

circumference of circle , $C = 2\pi r$ $\quad \therefore \frac{dc}{dt} = 2\pi(1)\frac{dr}{dt}$

$$= 2 \times \frac{22}{7} \times \frac{16r}{3} = \frac{44 \times 16 \times (6)^2}{21}$$

Rate of change of circumference $\frac{dc}{dt} = \frac{88 \times 16}{7} = \frac{1408}{7} = 201.1\ cm/\sec$

Example 10 :

A circular blot of ink increases in area in such a way that the radius at time 't' sec is given by $r = 2t^2 - \frac{t^3}{4}$. What rate the area of the blot increase when t = 4 sec.

Solution :

$A = \pi r^2$

$\therefore \frac{dA}{dt} = \pi 2r \frac{dr}{dt}$ Given, $r = 2t^2 - \frac{t^3}{4}$

$= 2\pi\ 16(4),$

$= 128\ \pi cm^2 / sec$

$[\ \therefore r = 2\ (4)^2 - \frac{4^3}{4} = 32 - 16 = 16]$

$\frac{dr}{dt} = 2(2t) - \frac{3t^2}{4}$

$= 4t - \frac{3t^2}{4}$

at t = 4 sec

$\frac{dr}{dt} = 4(4) - \frac{3(4)}{4} = 16 - 12$

$\therefore \frac{dr}{dt} = 4cm / sec$

Example 11 :

The volume of a sphere is increasing at the rate 4π c.c/sec. Find the rate at which the area of its surface increases when its radius is 10cm.

Solution : Given $\frac{dv}{dt} = 4\pi c.c. / sec$, r = 10cm, $\frac{dA}{dt} = ?$

Volume of a sphere = $V = \frac{4}{3}\pi r^3$

$\frac{dv}{dt} = \frac{4}{3}\pi\left(3r^2 \frac{dr}{dt}\right)$

$4\pi = 4\pi(10)^2 \frac{dr}{dt} \Rightarrow 4\pi = 400\pi \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{4\pi}{400\pi} = \frac{1}{100} cm / sec$

A = surface are of the sphere = $4\pi r^2$

$\frac{dA}{dt} = 4\pi\left(2r \frac{dr}{dt}\right) \Rightarrow 8\pi(10)\left(\frac{1}{100}\right) \therefore \boxed{\frac{dA}{dt} = \frac{4\pi}{5} cm^2 / sec}$

Example 12 :

A spherical balloon is being inflated so that is volume is increasing at the rate of 30c.c/ min. How fast its surface area increasing when its volume is 36π c.c ?

Solution :

Given $\frac{dv}{dt} = 30c.c / \min$, $\frac{dA}{dt} = ?$, $v = 36\pi c.c$

Volume = $V = \frac{4}{3}\pi r^2 \Rightarrow 36\pi = 4\frac{\pi}{3}\left(r^3\right)$

$\frac{36\pi \times 3}{4\pi} = r^3 \quad \therefore r^3 = 27 \quad \therefore \boxed{r = 3cm}$

Again, $V = \frac{4}{3}\pi r^3$

$\therefore \frac{dv}{dt} = \frac{4}{3}\pi\left(3r^2 \frac{dr}{dt}\right)$

$30 = 4\pi(3)^2 \frac{dr}{dt} \Rightarrow 30 = 36\pi \frac{dv}{dt}$

$\therefore \frac{dr}{dt} = \frac{30}{36\pi} = \frac{5}{6\pi} cm / \min$

surface area $A = 4\pi r^2 \Rightarrow \frac{dA}{dt} = 4\pi\ 2r\frac{dr}{dt}$

$= 8\pi(3)\left(\frac{5}{6\pi}\right) = \frac{40}{2}$

$\frac{dA}{dt} = 20\ cm^2 / \min$

Example 13 :

A squareplate in expanding uniformly, the side is increasing the rate of 5cm / sec. What is the rate at which the area and its perimeter is increasing when the side is 20cm long.

Solution : Let the side of the square be 'a' cm

Given, $\frac{da}{dt} = 5cm / \sec$, $\frac{dA}{dt} = ?$, $\frac{dP}{dt} = ?$, a = 20cm

Area of the square $A = a^2$

$$\therefore \frac{dA}{dt} = 2a\left(\frac{da}{dt}\right)$$

$$= 2(20)\,(5) = 200 \text{cm}^2/\text{sec}$$

Again, perimeter $= P = 4a$

$$\therefore \frac{dP}{dt} = 4(1)\frac{da}{dt} \quad = \quad 4(5) \qquad \therefore \boxed{20cm/\text{sec}}$$

Example 14 :

The sides of an equilateral triangel are increasing at the rate 3 cm / sec. How fast is its area increasing when the side is 10 cm.

Solution:

Let us assume that the side equilateral triangle = 'a' cm

$$\frac{da}{dt} = 3cm/\text{sec}, \quad a = 10\text{cm}, \quad \frac{dA}{dt} = ?$$

$$A = \frac{\sqrt{3}}{4}a^2 \qquad \therefore \frac{dA}{dt} = \frac{\sqrt{3}}{4}\left(2a\frac{da}{dt}\right)$$

$$= \frac{\sqrt{3}}{4} \times 2(10)(3) \quad = \quad \frac{60\sqrt{3}}{4} = 15\sqrt{3}$$

$$\therefore \boxed{\frac{dA}{dt} = 15\sqrt{3}\ cm^2/\text{sec}}$$

Example 15 :

Water is flowing out of a vertical cylinderical tank at the rate of 15 cubicft / min. Find how the level water is decreasing if the radius of the tank is 3ft.

Solution :

$$\frac{dv}{dt} = 15\ C\,ft/\text{min}, \qquad \frac{dh}{dt} = ?, \qquad r = 3\,ft$$

Volume of a cylindrical tank, $v = \pi r^2 h$

$$\therefore \frac{dv}{dt} = \pi r^2\left(\frac{dh}{dt}\right) \qquad (r = \text{constant})$$

$$15 = \pi(3)^2\frac{dh}{dt} \qquad \therefore \frac{dh}{dt} \quad = \quad \frac{15}{9\pi} \quad = \quad \frac{5}{3\pi}\text{ft}/\text{min}$$

Example 16 :

The edge of a variable cube is increasing at the rate of 6cm / min. How fast is the volume and its surface increasing when the edge is 10cm long?

Solution :

Let the edge of the cube be 'x' cm.

Given, $\frac{dx}{dt} = 6cm / \min$, $\frac{dv}{dt} = ?$, $\frac{dA}{dt} = ?$, $x = 10\text{cm}$

Volume of the cube = $v = x^3$

$$\therefore \frac{dv}{dt} = 3x^2 \frac{dx}{dt}$$

$$= 3(10)^2 (6) = 3 \times 100 \times 6$$

$$\boxed{\frac{dA}{dt} = 1800cm^3 / \min}$$

surface area of the cube = A = $6x^2$

$$\therefore \frac{dA}{dt} = 6(2x)\frac{dx}{dt} = 12\,(10)\,(6)$$

$$\boxed{\frac{dA}{dt} = 720cm^2 / \min}$$

Example 17 :

The height of a cone is 60cm and it is constant the radius of the base is increasing at the rate of 0.50 cm/sec. Find the rate of increase of volume of the cone when the radius is 10cm?

Solution :

Given, $h = 60$ cm (constant), $\frac{dr}{dt} = 0.50\ cm / \text{sec}$ $\frac{dv}{dt} = ?$, $r = 10$ cm

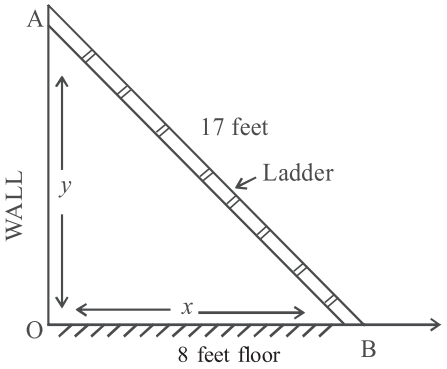
Volume of the cone = $v = \frac{1}{3}\pi r^2 h$

$$\therefore \frac{dv}{dt} = \frac{1}{3}\pi h\left(2r\frac{dr}{dt}\right) \quad (\therefore h = \text{constant})$$

$$= \frac{2}{3}\pi \times 60 \times 10 \times 0.50 \qquad \therefore \boxed{\frac{dv}{dt} = 200\pi\ \text{cm}^2 / \text{sec}}$$

Example 18 : A ladder 17 feet long leans against a smooth vertical wall. If the lower end is moving at the rate of 2 ft min. Find the rate at which the upper end is moving when the lower end is 8 ft from the wall.

Solution :



Given, $\frac{dy}{dt}$ = Rate at which the upper end moves = 2ft min.

$\frac{dx}{dt}$ = Rate at which the lower end moves = ?

ΔABC is a right angle triangle

$\therefore$ from pythogoras theorem

$$OB^2 + OA^2 = AB^2$$

$$x^2 + y^2 = 17^2 \qquad .....(1)$$

$$8^2 + y^2 = 17^2 \Rightarrow 64 + y^2 = 289$$

$$\therefore y^2 = 289 - 64 = 225 \qquad \therefore y = 15 \text{ feet}$$

Again differentiate (1) w.r.t. t

$$\therefore 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\div 2, \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \qquad \therefore 8\frac{dx}{dt} + 15(2) = 0$$

$$8\frac{dx}{dt} = -30 \qquad \therefore \frac{dx}{dt} = \frac{-30}{8}$$

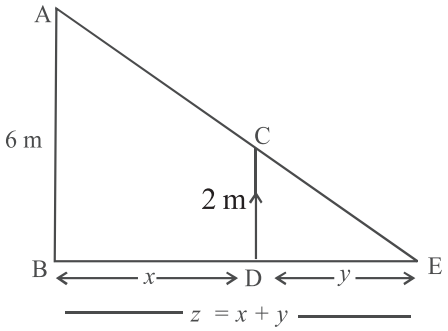
$$\therefore \boxed{\frac{dx}{dt} = -3.75 \text{ ft / min}}$$

(negative sign shows that the ladder in moving away from the wall)

Example 19:

A man 2mt height walks at a uniform speed of 6km / hour away from the lamp post 6m high. Find the rate at which the (i) length of his shadow increase (ii) the rate at which the tip of the shadow is moving.

Solution :



Given the rate of man walking $\frac{dx}{dt} = 6km / hr$

Rate of shadow length increase $= \frac{dy}{dt} = ?$

Rate tip of shadow nowing $\frac{dz}{dt} = ?$

ΔABE and ΔCDE are similar

$$\therefore \quad \frac{AB}{CD} = \frac{BE}{DE} \quad \Rightarrow \quad \frac{6}{2} = \frac{x+y}{y} \quad \Rightarrow \quad \frac{3}{1} = \frac{x+y}{y}$$

$$3y = x + y \quad \therefore \quad \boxed{2y = x} \text{ Diff. w.r.t. t}$$

$$\therefore \quad 2\frac{dy}{dt} = \frac{dx}{dt} = 6$$

$\therefore$ Rate of increase of length of shadow $= \boxed{\frac{dy}{dt} = 3km / hr}$

Again, $z = x + y$ diff. w.r.t.

$$\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt} \quad \Rightarrow \quad 6 + 3 = 9 \text{ km/hr}$$

$\therefore$ Rate at which tip of shadow is moving = 9km / hr

Example 20 :

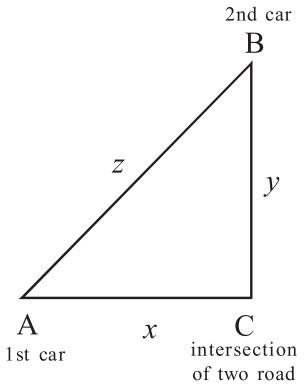
Two car, one going due east the rate 90 mt/min and the other going due south at the rate 60 mt/min all travelling towards an intersection of the two roads. Show that the two cars are approaching each other at the rate 108 mt/min. At the instant when the 1st car is 200 mt and the 2nd car is 150 mt from the intersection.

Solution :

Position of 1st car is at A, 2nd car is at B

Let x = distnace of 1st car from the point of intersection of two road (at C) = 200mt

y = 2nd car = 150 mt



$\frac{dx}{dt} = -90 \text{ mt / min}$, $\frac{dy}{dt} = -60 \text{ mt / min}$

(negative sign show both are approaching towards the point of Intersection)

z = distance between two car

$\therefore$ ABC is a right angled triangle

$$z^2 = x^2 + y^2 \text{.....(1)}$$

$$= 200^2 + 150^2 = 40000 + 22500$$

$z^2 = 62500 \qquad \therefore z = \sqrt{62500} \qquad \boxed{z = 250\text{mt}}$

Diff (1) w.r.t 't'

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dz}{dt}$$

$\div 2, \quad z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dz}{dt}$

$$250\left(\frac{dz}{dt}\right) = 200(-90) + 150(-60) = -18000 - 9000$$

$$250\left(\frac{dz}{dt}\right) = -27000$$

$$\therefore \frac{dz}{dt} = \frac{-27000}{250} = \boxed{-180 \text{ mt / min}}$$

$\therefore$ The two car approach at the rate 180 mt/min

EXERCISE 19.1

I. 2 marks questions:

1. The displacement 's' of a particle at time 't' is given by $S = 4t^3 - 6t^2 + t - 7$. Find the velocity and acceleration when t = 2 sec.
2. If $S = 5t^2 + 4t - 8$. Find the initial velocity and acceleration. (s = displacement, t = time)
3. A stone thrown vertically upward rises 's' ft. in 't' sec. where $s = 80t - 16t^2$. What its velocity after 2 sec? Find the acceleration?
4. A body is thrown vertically upwards its distance S feet is 't' sec is given by $S = 5 + 12t - t^2$. Find the greatest height reached by the body.
5. If $v = \sqrt{s^2 + 1}$ prove that acceleration is 'S'. (V = velocity, S = displacement)
6. If $S = at^3 + bt$. Find a and b given that when t = 3 velocity is '0' and the acceleration is 14 unit. (S = displacement, t = time).
7. When the brakes are applied to a moving car, the car travels a distance 's' ft in 't' see given by $s = 8t - 6t^2$ when does the car stop?

II. 3 marks questions:

1. The radius of a sphere is increasing at the rate of 0.5 mt/ sec. Find the rate of increase of its surface area and volume after 3 sec?
2. The surface area of a spherical bubble is increasing at the rate of a 0.8 cm^2 / sec find at what rate is its volume increasing when r = 2.5cm [r = radius of the sphere).
3. A spherical balloon is being inflated at the rate 35 cc / sec. Find the rate at which the surface area of the balloon increases when its diameter is 14 cm.
4. The radius of a circular plate is increasing at the rate of $\frac{2}{3\pi}$ cm/sec. Find the rate of change of its area when the radius is 6cm.
5. A circular patch of oil spreads on water the area growing at the rate of 16cm^2 / min. How fast are radius and the circumference increasing when the diameter is 12 cm.
6. A stone is dropped into a pond waved in the form of circles are generated and the radius of the outer most ripple increases at the rate 2 inches / sec. How fast is the area increasing when the a) radius is 5 inches b) after 5 sec ?
7. The side of an equilateral triangle is increasing at the rate $\sqrt{3}$ cm/sec. Find the rate at which its area is increasing when its side is 2 meters.

8. Water is being poured at the rate of 30 mt^3 /min into a cylindrical vessel whose base is a circle of radius 3 mt. Find the rate at which the level of water is rising ?
9. Sand is being dropped at the rate of 10 mt^3 /sec into a conical pile. If the heihgt of the pile twice the radius of the base, at what rate is the height to the pile is increasing when the sand in the pile is 8mt high.
10. A ladder of 15ft long leans against a smooth vertical wall. If the top slides downwards at the rate of 2 ft / sec. Find how fast the lower and is moving when the lower end is 12 ft from the wall.
11. An edge of a variable cube is increasing at the rate of 10cm/sec. How fast in the volume and also its surface area is increasing when the edge is 5cm long.
12. A man 6ft tall is moving directly away from a lamp post of height 10 ft above the ground. If he is moving at the rate 3 ft / sec. Find the rate at whcih the length of his shadow is increasing and also the tip of his shadow is moving?
13. The height of circular cone is 30 cm and it is constant. The radius of the base is increasing at the rate of 0.25 cm/ sec. Find the rate of increase of volume of the cone when the radius of base is 10cm.
14. The volume of a spherical ball in increasing at the rate 4πcc /sec. Find the rate of increase of the radius of the ball when the volume is 288 πcc.
15. A drop of ink spreads over a blotting paper so that the circumferenes of the blot is 4πcm and it changes 3cm/sec. Find the rate of increase of its radius and also find the rate of increase of its area?
16. A circular plate of metal is heated so that its radius increases at the rat of 0.1 mm / min. At what rate is the plate's area increasing when the radius is 25cm [1cm = 10mm]
17. The surface area of a spherical soap bubble increasing at the rate of 0.6cm^2 /sec. Find the rate at which its volume is increasing when its radius is 3cm.
18. A rod 13 feet long slides with it end A and B as two straight lines at right angles which meet at 'O' . If A is moved away from O with a uniform speed at 4ft / sec, find the speed of the end B move when A is 5 feet from O.
19. A street lamp is hung 12 feet above a straight horizontal floor on which a man of 5 feet is walking how fast his shadow lengthening when he is walking away from the lamp post at the rate of 175 ft/min.
20. Find a point on the parabola $y^2 = 4x$ at which the ordinate increases at twice the rate of the abscissa [ordinate = y, absissa = x].

ANSWERS 19.1

I. 1) V = 25 unit, acc = 36 unit
2) I.V = 4 unit, acc = 10 unit
3) V = 10ft / sec, acc = –32ft/sec^2
4) S = 41 feet
6) $a = \frac{7}{9}$, $b = -21$
7) $t = \frac{+2}{3}$ sec

II. 1) $6\pi m^3$/sec, $4.5\pi m^3$/sec
2) 1.cc/sec
3) 10cm^2/sec
4) 8cm^2/sec
5) $\frac{4}{3\pi}$cm / min, $\frac{8}{3}$cm/min
6) 20π inch^2/sec, 40π inch^2/sec
7) 300 cm^2 /sec
8) $\frac{10}{3\pi}$ mt/min
9) $\frac{5}{8\pi}$ mt/sec
10) $\frac{3}{2}$ft/sec
11) 750 cm^2 /sec, 600 cm^2 /sec
12) 4.5ft/ sec, 7.5 ft/sec
13) 50π cm^2 /sec
14) $\frac{1}{36}$ cm/sec
15) $\frac{3}{2\pi}$ cm/sec, 6cm^2 /sec
16) 50π mm^2 /min
17) 0.9 cc/sec
18) $\frac{5}{3}$ ft /sec
19) 125 ft/min
20) $(x, y) = \left(\frac{1}{4}, 1\right)$

19.3 Increasing and Decreasing function

Definition : A function $y = f(x)$ be a continuous function defined on an Interval I. Let x_1 and x_2 are any two points in the interval. Then $f(x)$ is said to

(i) Increasing on the interval if $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

(ii) Strictly increasing on the interval if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

(iii) Decreasing on the interval if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

(iv) Strictly decreasing on the interval if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Graphical Representation:

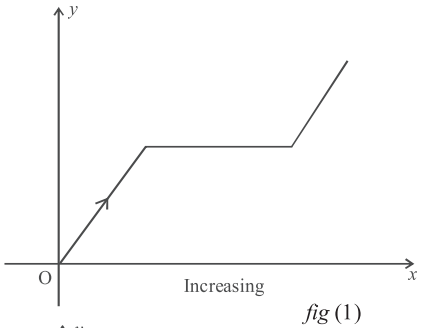


fig (1)

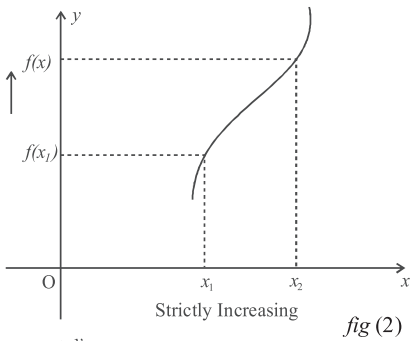


fig (2)

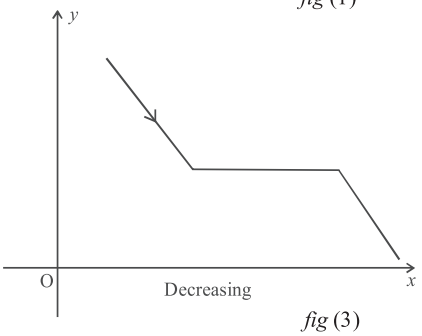


fig (3)

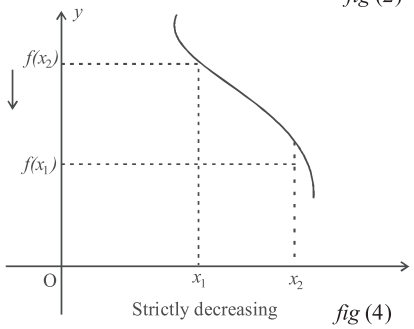


fig (4)

Note :(i) When the function $y = f(x)$ is increasing in an Interval then $\frac{dy}{dx} = f^{1}(x) > 0$ (positive)

(ii) When the function $y = f(x)$ is decreasing in an Interval then $\frac{dy}{dx} = f^{1}(x) < 0$ (negative)

(iii) If $y = f(x)$ is said to be **stationary at a point *f(x)* is neither increases nor decreases then** $\frac{dy}{dx} = 0$ i.e. the tangent is parallel x – axis .....fig (5)

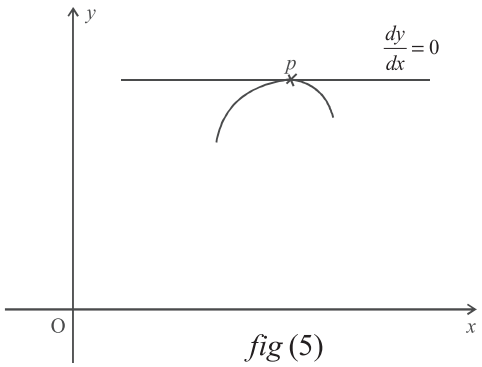


fig (5)

(iv) The value of x for which the function $y = f'(x)$ is stationary are more frequently called **Critial value** for the function.

Example 1 :

Find the interval in which $f(x) = 5 + 36x + 3x^2 - 2x^3$ is

(i) increasing (ii) decreasing

Solution :

Let $y = f(x) = 5 + 36x + 3x^2 - 2x^3$

$\therefore \quad \frac{dy}{dx} = f^1(x) = 36(1) + 3(2x) - 2(3x^2)$

(i) for increasing $36 + 6x - 6x^2 > 0$

$\div\ 6,\ 6 + x - x^2 > 0$

$-(x^2 - x - 6) > 0$

multiplying by (–) sign $(x^2 - x - 6) < 0$

$(x - 3)(x + 2) < 0$

$\Rightarrow\ x - 3 < 0$ or $x + 2 > 0 \qquad \Rightarrow\ x < 3$ or $x > -2$

$\Rightarrow\ \boxed{-2 < x < 3}$

$\therefore\ f(x)$ is increasing in the interval (–2, 3)

(ii) for Decreasing $f^1(x) < 0$

$\therefore 36 + 6x - 6x^2 < 0$

$\div\ 6,\ 6 + x - x^2 < 0$

$-\left(x^2 - x - 6\right) < 0$

multiply by (–) sign $x^2 - x - 6 > 0$

$(x - 3)(x + 2) > 0 \quad \Rightarrow \quad x - 3 > 0$ or $x + 2 < 0$

$x > 3$ or $x < -2 \qquad \therefore f(x)$ is decreasing in the Interval $(-\infty, -2) \cup (3, \infty)$

Example 2 :

Find whether the function are increasing or decreasing or neither increasing nor decreasing.

Solution :

(i) $f(x) = 2x^2 + 3x - 7$ at $x = 2$

$f^1(x) = 4x + 3$

$f^1(2) = 4(2) + 3 = 11 > 0(+)$

$\therefore$ *f(x)* is increasing at $x = 2$

(ii) $f(x) = x^3 - 6x^2 + 12x - 1$

$\therefore f^1(x) = 3x^2 - 6(2x) + 12(1)$

$= 3x^2 - 12x + 12 = 3(x^2 - 4x + 4)$

$3(x-2)^2 > 0$ for all the value of 'x'

$\therefore$ *f(x)* is increasing at all points

(iii) $f(x) = x^2 - 6x + 5$

$f^1(x) = 2x - 6(1) = 2x - 6 = 2(x-3)$

$f^1(x) > 0 \Rightarrow x - 3 > 0 \Rightarrow x > 3$

$f^1(x) < 0 \Rightarrow x - 3 < 0 \Rightarrow x < 3$

$\therefore$ *f(x)* is increasing at all $x > 3$

decreasing for all $x < 3$

(iv) $f(x) = x^3 - 4x^2 + 5x + 1$ at $x = 1$

$f^1(x) = 3x^2 - 4(2x) + 5(1) = 3x^2 - 8x + 5$

$\therefore f^1(1) = 3(1)^2 - 8(1) + 5 = 3 - 8 + 5 = 0$

$\therefore f(x)$ is neither increasing or decreasing at $x = 1$

Example 3 :

Find the critical (stationary) points of the function $f(x) = 2x^3 - 9x^2 + 12x + 6$

Solution : $f(x) = 2(3x^2) - 9(2x) + 12(1)$

$= 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$

for critical point $f^1(x) = 0$

$\therefore 6(x^2 - 3x + 2) = 0 \Rightarrow (x-1)(x-2) = \frac{0}{6} = 0$

$\therefore x = 1$ and $x = 2$ $\therefore$ the critical points are 1 and 2

Example 4 :

Find the stationary points and the corresponding value of the function $f(x) = x^3 - 3x^2 - 9x + 5$

Solution :

$f^1(x) = 3x^2 - 3(2x) - 9(1) = 3x^2 - 6x - 9$

At stationary point $f^1(x) = 0$

$\therefore 0 = 3(x^2 - 2x - 3) \quad \therefore 0 = 3(x+1)(x-3)$

$\Rightarrow x + 1 = 0$ or $x - 3 = 0$

$x = -1$ or $x = 3$

the stationary point are –1 and 3.

the values are at $x = -1$

$f(x) = (-1) - 3(-1)^2 - 9(-1) + 5$

$= -1 - 3 + 9 + 5 = \boxed{10}$

at $x = 3$ $f(3) = (3)^3 - 3(3)^2 - 9(3) + 5$

$= 27 - 27 - 27 + 5 = \boxed{-22}$

$\therefore$ the stationary values are 10 and –22

Example 5 :

Find the domain of increasing and decreasing of $f(x) = x^2 - 4x + 3$

Solution :

$f(x) = x^2 - 4x + 3 \quad \Rightarrow \quad f^1(x) = 2x - 4(1)$

for increasing $\quad f^1(x) > 0 \quad \therefore 2x - 4 > 0$

$\therefore 2x > 4 \qquad \boxed{x > 2}$

for decreasing function $f^1(x) < 0 \quad \Rightarrow \quad 2x - 4 < 0$

$2x - 4 \qquad \therefore \quad \boxed{x < 2}$

$\therefore$ the given function increasing in $(2, \infty)$

given function decreasing in $(-\infty, 2)$

Example 6 :

Find whether the following are increasing or decreasing or neither. And also find the critical point if they exist.

(i) $f(x) = 2x^3 - 21x^2 + 36x - 20$ **at** $x = 0, -1$

Solution :

$$f^1(x) = 2(3x^2) - 21(2x) + 36(1) = 6x^2 - 42x + 36$$

at $x = 0$, $\quad f^1(0) = 6(0) - 42(0) + 36 = 36 > 0$

at $x = -1$, $\quad f^1(1) = 6(1)^2 - 42(1) + 36 = 6 + 42 + 36 = 84 > 0$

$\therefore$ $f(x)$ is increasing at both $x = 0$ and $x = -1$

for critical point

$f^1(x) = 0$

$6x^2 - 42x + 36 = 0 \Rightarrow 6(x^2 - 7x + 6) = 0$

$x^2 - 7x + 6 = 0 \Rightarrow (x - 1)(x - 6) = 0$

$x - 1 = 0$ and $x - 6 = 0$

$\therefore$ $x = 1,\ 6$ all critical point

(ii) $f(x) = x + \dfrac{1}{x}$ **at** $x = 2, 3$

Solution : $f^1(x) = (1) + (-1x^{-1-1}) = 1 - \dfrac{1}{x^2}$

at $x = 2$, $\quad f^1(2) = 1 - \dfrac{1}{(2)^2} = 1 - \dfrac{1}{4} = \dfrac{3}{4} > 0$

$x = 3$, $\quad f^1(3) = 1 - \dfrac{1}{3^2} = 1 - \dfrac{1}{9} = \dfrac{8}{9} > 0$

$\therefore$ $f(x)$ is increasing at $x = 2$ and 3

for critical value, $f^1(x) = 0 \quad \therefore 1 - \dfrac{1}{x^2} = 0$

$\dfrac{1}{x^2} = 1 \qquad \therefore x^2 = 1$

$\boxed{x = \pm 1}$ are the critical points

(iii) $f(x)=\dfrac{x^2-7x+6}{x-10}$ **at** $x=1,-1$

Solution : $f^1(x)=\dfrac{(x-10)(2x-7)-(x^2-7x+6)(1)}{(x-10)^2}$

at $x=1$ $\quad f^1(1)=\dfrac{(1-10)(2-7)-(1-7+6)}{(1-10)^2}$

$$=\frac{(-9)\ (-5)-(0)}{(-9)^2}=\frac{45}{81}>0$$

at $x=-1$ $\quad f^1(-1)=\dfrac{(-1-10)(-2-7)-\left((-1)^2-7(-1)+6\right)}{(-1-10)^2}$

$$=\frac{(-11)(-9)-(1+7+6)}{(-11)^2}=\frac{99-14}{121}=\frac{85}{121}>0$$

$\therefore f(x)$ is increasing at $x=1$ and -1

for critical value $f^1(x)=0$

$$\frac{(x-10)(2x-7)-(x^2-7x+6)}{(x-10)^2}=0$$

$$(x-10)(2x-7)-(x^2-7x+6)=0$$

$$2x^2-\cancel{7x}-20x+70-x^2+\cancel{7x}-6=0$$

$$x^2-20x+64=0 \Rightarrow (x-16)(x-4)=0$$

$\therefore$ $\boxed{x=16,\ 4}$ are critical value

(iv) $f(x)=e^x+1$ **at** $x=0$

$f^1(x)=e^x$

at $x=0$ $\quad f^1(0)=e^0=1>0$ $\quad \therefore f(x)$ is increasing at $x=0$

for critical point $f^1(x)=0$ $\quad \therefore e^x=0$ but e^x can never be '0' for any value of 'x'

$\therefore f(x)$ has no critical point

EXERCISE 19.2

2 and 3 mark

I. Find whether the following functions are increasing or decreasing or neither.

(i) $f(x)=x^4-8x^3+22x^2-24x+5$ at $x=0,-2$

(ii) $f(x)=4x^3-15x^2+12x-2$ at $x=1,-1$

(iii) $f(x)=(x-1)(x-2)^2$ at $x=1,\ 3$

II. Find the value of x (Interval) for which the function is increasing or decreasing.

(i) $f(x)=2x^3-15x^2-84x+7$

(ii) $f(x)=x^4-2x^3+1$

(iii) $f(x)=x^3-3x^2+3x-100$

(iv) $f(x)=2x^2-96x+5$

(v) $f(x)=10-6x-2x^2$

(vi) $f(x)=2x^3+9x^2+12x+20$

ANSWER 19.2

I. (i) Decreasing at $x=0$, Increasing at $x=-2$

(ii) Decreasing at $x=1$, Increasing at $x=-1$

(iii) Increasing at $x=1$, $x=3$

II. (i) Increasing : $x<-2$ and $x>7$, Interval $(-\infty,-2)\cup(7,\infty)$

(ii) Increasing : $x>\frac{3}{2}$, Interval $\left(\frac{3}{2},\infty\right)$

(iii) Increasing for all x

(iv) Decreasing $-4<x<4$

(v) Decreasing $x>-\frac{3}{2}$, Increasing $x<-\frac{3}{2}$

(vi) Increasing $(-\infty,-2)\cup(-1,\infty)$,
Decreasing $(-2,-1)$

19.4 Maxima and Minima

Open Interval & Closed Interval : If $x\in R$ (R = set of finite real number) 'a' and 'b' the open interval with end point a and b is denoted by (a,b) is the set $\{x\in R, a<x<b\}$ for closed Interval $[a,b]$ is the set $\{x\in R, a\le x\le b\}$

Local Maxima (Relative Maxima)

In the fig(1) A is called a point of Local Maxima at $x = a$ because the value of the function at A is greater than at any point in the neighbourhood M and N ie., at $x = a - h$ and $x = a + h$

$\therefore f(x)$ is said to have a local maxima at $x = a$ there exist an interval $(a-h, b+h)$ neighbourhood of 'a' such that $f(a) > f(a+h)$ and $f(a) > f(a-h)$.

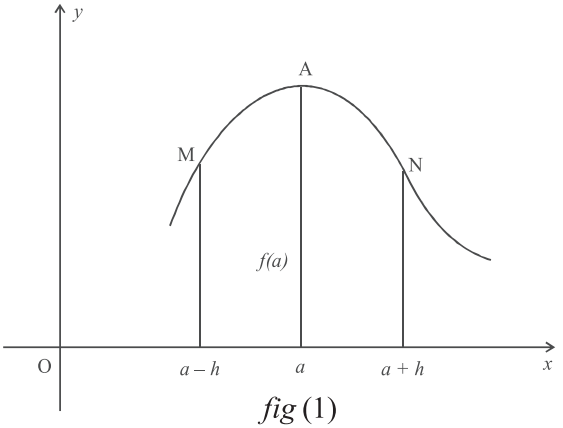


fig (1)

Local Minimum (Relative Minima)

In the *fig*(2) B is called a point of Local Minima at $x = b$ because the value of the function at B is small than at any point in the neighbourhood M and N ie., at $x = b - h$ and $x = b + h$

$\therefore f(x)$ is said to have a local minima at $x = b$ there exist an interval $(b-h, b+h)$ neighbourhood of b such that

$$f(b) < f(b-h)$$
$$f(b) < f(b+h)$$

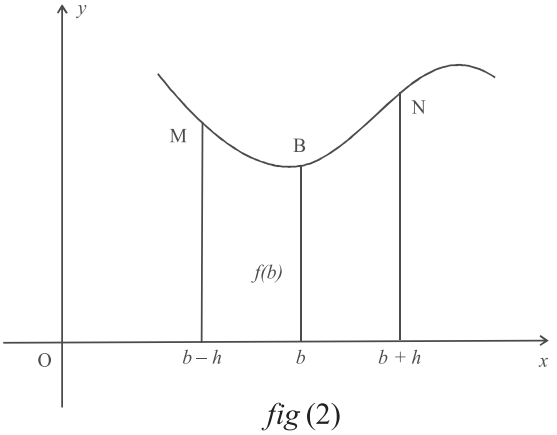


fig (2)

Note:

(i) A function may have many local mixima and many local minimal values. In *fig*(3) *f(x)* has attained maxima at A, C, E, G and minima at B, D, F, H.

(ii) Points of local maxima and local minima of a continuous function always occur alternately.

(iii) A maximal value in a neighbourhood can even be small than a minimal value in another neighbourhood ie., in the fig(3) D and G.

Maxima

Definition : A function $y = f(x)$ in said to have a maximum at $x = a$ is there is an open interval $(a-h,\ a+h)$ such that $f(x) \le f(a)$ for all $x \in (a-h, a+h)$ and *f(x)* is maximum value of the given function.

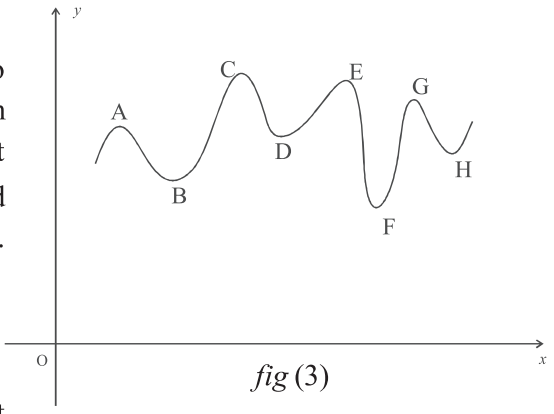


fig (3)

From the fig(4) it is clear at A. The tangent is parallel to x - axis

$\therefore$ slope = 0

$\therefore \quad \dfrac{dy}{dx} = f(x) = 0$

at $x = a - h$, $\quad f^1(x) > 0$

at $x = a + h$, $\quad f^1(x) > 0$

Hence is *f(x)* is maximum at $x = a$ the $f^1(x)$ changes sign from +ve to –ve

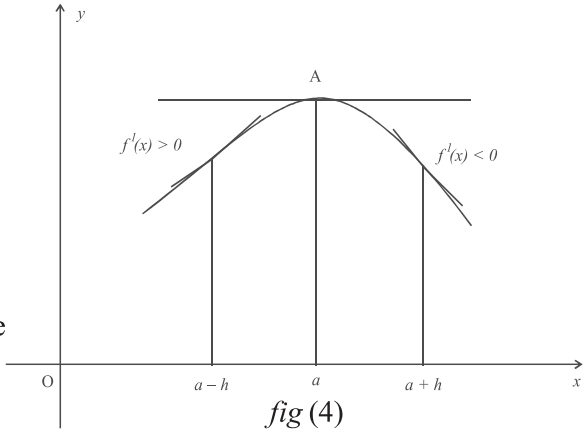


fig (4)

Minima :

A function $y = f(x)$ is said to have a minimum at $x = b$ is there exist an open intervel $(b-h,\ b+h)$ such that $f(x) \ge f(a)$ for all $x \in (b-h, b+h)$ and *f(b)* is the minimum value of the given function.

From fig(5) it is clear at B the tangent is parallel to x -axis

$\therefore$ slope = 0

$\therefore \quad \dfrac{dy}{dx} = f(x) = 0$

at $x = b - h$, $\quad f^1(x) > 0$

at $x = b + h$, $\quad f^1(x) > 0$

Hence *f(x)* is minimum at $x = b$ then *f(x)* changes sign from –ve to +ve

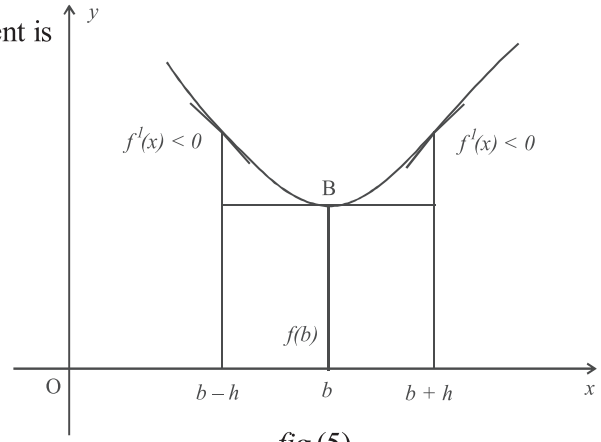


fig (5)

Working method for finding Maxima or Minima

Step (1) : find $\frac{dy}{dx} = f^1(x)$

Step (2) : put $\frac{dy}{dx} = 0$. Solve the equation. Let $x = a,\ x = b$ etc. be the value of x

Step (3) : Find $\frac{d^2y}{dx^2}$ (second derivativetent) = $f^1(x)$

for $x = a$, If $\frac{d^2y}{dx^2}$ = +ve then $f(x)$ is minimum

for $x = a$, If $\frac{d^2y}{dx^2}$ = –ve then $f(x)$ is miximum

for $x = a$, If $\frac{d^2y}{dx^2} = 0$ then $f(x)$ is neither maximum nor minimum then $x = a$ is said to be a **point of Inflexion**

Step (4) : To find the maximum or minimum value the corresponding value of $x = ab$ etc. is substitute in $y = f(x)$ to get y_{max} or y_{min}.

Example 1 :

Find the minimum value of $x^2 + \frac{250}{x}$

Solution :

Let $y = f(x) = x^2 + \frac{250}{x}$ .....(1)

$$\therefore \frac{dy}{dx} = 2x + 250\left(\frac{-1}{x^2}\right) \Rightarrow \frac{dy}{dx} = 2x - \frac{250}{x^2}$$

put $\frac{dy}{dx} = 0$, $\therefore\ 0 = 2x - \frac{250}{x^2}$

$$\frac{250}{x^2} = 2x \ = \ 2x^3 = 250 \Rightarrow x^3 = 125 = (5)^3 \quad \therefore \boxed{x = 5}$$

Again, $\frac{d^2y}{dx^2} = 2(1) - 250\left(\frac{-2}{x^3}\right) = 2 + \frac{500}{x^3}$

at $x = 5$, $\frac{d^2y}{dx^2} = 2 + \frac{500}{53} = \frac{2+500}{125} = 6(+)$ $\therefore f(x)$ is minimum

for minimum value sub $x = 5$ in (1)

$\therefore y \min = (5)^2 + \frac{250}{2} = 25 + 50 = 75$

Example 2 :

Find the maximum and minimum value of $x^3 - 9x^2 + 15x - 1$

Solution :

Let $x^3 - 9x^2 + 15x - 1$ ..... (1)

$\frac{dy}{dx} = 3x^2 - 9(2x) + 15(1) = 3x^2 - 18x + 15$

put $\frac{dy}{dx} = 0$ $\therefore 0 = 3x^2 - 18x + 15$

$0 = 3(x^2 - 6x + 5) \Rightarrow \frac{0}{3} = x^2 - 6x + 5$

$x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0 \Rightarrow x = 1, 5$

Now, $\frac{d^2y}{dx^2} = 3(2x) - 18(1) = 6x - 18$

at $x = 1$, $\frac{d^2y}{dx^2} = 6(1) - 18 = -12$ (–ve) < 0 . Hence the function attain maxima

at $x = 5$, $\frac{d^2y}{dx^2} = 6(5) - 18 = 30 - 18 = 12$ (+ve) > 0 $\therefore$ the function attain minima

for maximum value sub $x = 1$ in (1)

$\therefore y \max = (1)^3 - 9(1)^2 + 15(1) - 1$

$= 1 - 9 + 15 - 1 = \boxed{6}$

for minimum value substitute $x = 5$ in (1)

$\therefore y \min = (5)^3 - 91(5)^2 + 15(5) - 1$

$= 125 - 225 + 75 - 1 = \boxed{-26}$

Example 3 :

Find the maximum value of $\frac{\log x}{x}$ (x in real)

Solution :

Let $y = \frac{\log x}{x}$ .....(1) by quotient rule

$$\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \log x(1)}{x^2} \quad \Rightarrow \quad 0 = \frac{1 - \log x}{x^2}$$

$$0 = 1 - \log x$$

$$\log_e x = 1 \qquad \therefore \quad \boxed{x = e}$$

Now, $\frac{d^2y}{dx^2} = \frac{(x^2)\left(-\frac{1}{x}\right) - (1 - \log x)(2x)}{(x^2)^2}$

$$= \frac{-x - 2x + 2x \log x}{x^4} = \frac{-3x + 2x \log x}{x^4}$$

$$= \frac{x(-3 + 2\log x)}{x^4} \qquad \therefore \quad \boxed{\frac{d^2y}{dx^2} = \frac{-3 + 2\log x}{x^3}}$$

at $x = e$, $\frac{d^2y}{dx^2} = \frac{-3 + 2\log_e^e}{e^3} \quad \Rightarrow \quad e = \frac{-3 + 2(1)}{e^3} \qquad \left(\because \log^e = 1\right)$

$$= -\frac{1}{e^3} < 0 \quad (\because \text{ e is +ve})$$

$\therefore$ the function is maximum at $x = e$

for maximum value put $x = e$ in (1)

$$\therefore y \max = \frac{\log_e^e}{e} = \boxed{\frac{1}{e}}$$

Example 4 : Show that $\frac{1}{x^x}$ is maximum at $x = e$

Solution : $y = \frac{1}{x^x}$ .....(1)

apply log on both side

$$\log y = \log \frac{1}{x^x} \quad \Rightarrow \quad \log y = \frac{1}{x} \log x$$

diff. w.r.t. x by produce rule

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\left(\frac{1}{x}\right) + \log x\left(\frac{-1}{x^2}\right)$$

put $\frac{dy}{dx} = 0, \ \frac{1}{y}(0) = \frac{1}{x^2} - \frac{\log x}{x^2}$

$$0 = \frac{1 - \log x}{x^2} \Rightarrow 0 = 1 - \log x$$

$$\therefore \ \log_e^x = 1 \qquad \therefore \ \boxed{x = e}$$

Again, $\frac{dy}{dx} = y\left[\frac{1 - \log x}{x^2}\right]$ apply quotient rule

$$\frac{d^2y}{dx^2} = \frac{x^2\left(y\left(\frac{-1}{x}\right) + (1 - \log x)\frac{dy}{dx}\right) - y(1 - \log x)2x}{\left(x^2\right)^2}$$

at $x = e$, $\frac{d^2y}{dx^2} = \frac{-e.e^{1/e}}{e^4} = \frac{-e^{1/e}}{e^3} < 0$

$\therefore$ the function has a maxima at $x = e$

Example 5 :

Show that x^x is minimum at $x = \frac{1}{e}$

Solution :

Let $y = x^x$ .....(1)

apply 'log' on both sides

$\therefore \log y = \log x^x \quad \Rightarrow \quad \log y = x \log x$ Diff. w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \log x(1)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \log x$$

put $\frac{dy}{dx}=0$ $\therefore \frac{1}{y}(0)=1+\log x \Rightarrow 0=1+\log x$

$\Rightarrow \log_e^x=-1 \quad \therefore x=e^{-1} \quad \therefore \boxed{x=\frac{1}{e}}$

Now $\boxed{\frac{dy}{dx}=y(1+\log x)}$

Diffn. w.r.t. x

$$\frac{d^2y}{dx^2}=y\left(\frac{1}{x}\right)+(1+\log x)\frac{dy}{dx}$$

$$= \frac{y}{x}+(1+\log x)y(1+\log x) = y\left[\frac{1}{x}+(1+\log x)^2\right]$$

$$= x^x\left(\frac{1}{x}+(1+\log x)^2\right)$$

$$\left(\frac{d^2y}{dx^2}\right) \text{ at } x=\frac{1}{e}=\left(\frac{1}{e}\right)^{1/e}\left[e+\left(1+\log_e^{1/e}\right)^2\right] > 0 (\because e>0)$$

$\therefore$ the function has a minima at $x=\frac{1}{e}$

Example 6 :

Find the maxima and minima of the function $f(x)=3x^3-9x^2-27x+30$

Solution :

Given $f(x)=y=3x^3-9x^2-27x+30$ .....(1)

$\therefore \frac{dy}{dx}=3(3x^2)-9(2x)-27(1)$

$\frac{dy}{dx}=9x^2-18x-27$

$0=9(x^2-2x-3) \qquad x^2-2x-3=0/9=0$

$(x-3)(x+1)=0 \qquad \therefore x=3,\ -1$

Now, $\frac{d^2y}{dx^2} = 9(2x) - 18(1) = 18x - 18$

at $x = 3$, $\frac{d^2y}{dx^2} = 18(3) - 18 = 54 - 18 = 36 > 0$ $\therefore$ $f(x)$ is maximum at $x = -1$

at $x = -1$, $\frac{d^2y}{dx^2} = 18(-1) - 18 = 36 < 0$ $\therefore$ $f(x)$ is maximum at $x = -1$

$\therefore$ for minimum value put $x = 3$ in (1)

$$y_{min} = 3(3)^3 - 9(3)^2 - 27(3) + 30$$
$$= 81 - 81 - 81 + 30 = -51$$

for maximum value put $x = -1$ in (1)

$$y_{max} = 3(-1)^3 - 9(-1)^2 - 27(-1) + 30$$
$$= -3 - 9 + 27 + 30 = 45$$

Example 7 :

Find the maximum and minimum value of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution :

Let $y = x^5 - 5x^4 + 5x^3 - 1$ .....(1)

$$\frac{dy}{dx} = 5x^4 - 5(4x^3) + 5(3x^2) = 5x^4 - 20x^3 + 15x^2$$

put $\frac{dy}{dx} = 0$, $5x^4 - 20x^3 + 15x^2 = 0$

$$5x^2(x^2 - 4x + 3) = 0 \Rightarrow 5x^2 = 0,\ x^2 - 4x + 3 = 0$$

$$x^2 = \frac{0}{5},\ (x-3)(x-1) = 0$$

$$\boxed{x = 0,\ x = 3,\ x = 1}$$

Now $\frac{d^2y}{dx} = 5(4x^3) - 20(3x^2) + 15(2x) = 20x^3 - 60x^2 + 30x$

at $x = 0$, $\frac{d^2y}{dx^2} = 20(0)^3 - 60(0)^2 + 30(0) = 0$

$\therefore$ *f(x)* is neither maxima nor minima at $x = 0$

at $x=1$, $\frac{d^2y}{dx^2} = 20(1)^3 - 60(1)^2 + 30(1) = 20 - 60 + 30 = -10 < 0$

$\therefore$ $f(x)$ is maximum at $x = 1$

at $x=3$, $\frac{d^2y}{dx^2} = 20(3)^3 - 60(3)^2 + 30(3) = 540 - 540 + 90 = 90 > 0$

$\therefore$ $f(x)$ is minimum at $x = 3$

$\therefore$ for maximum value put $x = 1$ at (1)

$\therefore$ y max $= (1)^5 - 5(1)^4 + 5(1)^4 - 1 = 1 - 5 + 5 - 1 = 0$

for minimum value put $x = 3$ at (1)

$\therefore$ y min $= (3)^5 - 5(3)^4 + 5(3)^3 - 1 = 243 - 405 + 135 - 1 = 28$

Example 8 :

Show that $x^3 - 6x^2 + 12x - 3$ has neither a maximum nor a minimum at $x = 2$.

Solution : Let $y = f(x) = x^3 - 6x^2 + 12 - 3$

$$\frac{dy}{dx} = 3x^2 - 6(2x) + 12(1)$$

$$0 = 3x^2 - 12x + 12 \quad \Rightarrow \quad 0 = 3(x^2 - 4x + 4)$$

$$x^2 - 4x + 4 = \frac{0}{3} = 0 \quad \Rightarrow \quad (x-2)^2 = 0 \quad \therefore \quad \boxed{x = 2}$$

$$\frac{d^2y}{dx^2} = 3(2x) - 12(1) = 6x - 12$$

at $\frac{d^2y}{dx^2} = 3(2x) - 12(1) = 6x - 12$

$\therefore$ $f(x)$ has neither maxima nor minima at $x = 2$

Example 9 :

Divide the number 40 into two parts suh that their product is maximum.

Solution : Let the number be x and y

$\therefore$ $x + y = 40$

$y = 40 - x$ .....(1)

product , $p = xy$

$p = x(40 - x)$

$p = 40x - x^2$

for the product to be maximum diffn.

$\frac{dp}{dx} = 40(1) - 2x$ $\qquad$ $\frac{d^2p}{dx^2} = -2 < 0$

$0 = 40 - 2x \Rightarrow 2x = 40$ $\quad \therefore \boxed{x = 20}$ $\quad \therefore$ product is maximum at $x = 20, y = 20$

sub in (1) $\quad \therefore y = 40 - 20$ $\quad \boxed{y = 20}$

$\therefore$ the numbers are 20 and 20

Example 10 :

The product of two natural number is 144, find the numbers if their sum is minimum.

Solution :

Let the numbers are x and y

Given $\quad xy = 144$

$$\therefore y = \frac{144}{x} \quad .....(1)$$

sum of numbers = $S = x + y$

$$S = x + \frac{144}{x} \quad \text{from (1)}$$

for the sum to be minimum differentiate w.r.t. x

$$\frac{ds}{dx} = 1 + 144\left(-\frac{1}{x^2}\right)$$

$$0 = 1 - \frac{144}{x^2}$$

$$\frac{144}{x^2} = 1 \quad \Rightarrow \quad x^2 = 144 \quad \therefore \boxed{x = 12}$$

Second derivative test for minimum

$$\frac{d^2y}{dx^2} = -144\left(-2/x^3\right)$$

at $x = 12$, $\frac{d^2s}{dx^2} = \frac{1}{6}$ (+ ve)

sub in (1) $\quad y = \frac{144}{2} \quad \therefore y = 12$

$\therefore$ the two numbers are 12 and 12

Example 11 :

Show that $f(x) = 4x^5 - 25x^4 + 40x^3 - 3$ has a point of Inflexion at $x = 0$

Solution : Let $f(x) = 4x^5 - 25x^4 + 40x^3 - 3$

$$\frac{dy}{dx} = 4(5x^4) - 25(4x^3) + 40(3x^2) = 20x^4 - 100x^3 + 120x^2$$

$$\therefore \frac{dy}{dx} = 20x^2(x^2 - 5x + 6) \quad \Rightarrow \quad 0 = 20x^2(x^2 - 5x + 6)$$

$$\therefore\ 20x^2 = 0,\ x^2 - 5x + 6 = 0$$

$$x^2 = \frac{0}{20},\ (x-2)(x-3) = 0$$

$$\boxed{x = 0,\ x = 2,3}$$

Now, $\frac{d^2y}{dx^2} = 20(4x^3) - 100(3x^2) + 120(2x) = 80x^3 - 300x^2 + 240x$

at $x = 0$, $\frac{d^2y}{dx^2} = 80(0) - 300(0) + 240(0) = 0$

$\therefore$ *f(x)* in neither maxima nor minima

$\therefore$ *f(x)* has a point of inflexion at $x = 0$

Example 12 :

Divide 64 into two pars such that the sum of the cubes of two parts is minimum

Solution :

Let the two parts are x and y

Given $x + y = 64$

$\therefore\ y = 64 - x$ .....(1)

sum of their cubes, $s = x^3 + y^3$

$s = x^3 + (64 - x)^3$ from (1)

for the sum to be minimum diffn. w.r.t. x

$$\frac{ds}{dx} = 3x^2 + 3(64 - x)^2(-1)$$

$$0 = 3x^2 - 3(64 - x)^2$$

$$\therefore\ 3x^2 = 3(64 - x)^2$$

$\Rightarrow x^2 = (64-x)^2 \Rightarrow x = 64 - x$

$\therefore\ 2x = 64, x = 32$

Second derivative test for minima

$$\frac{d^2s}{dx^2} = 3(2x) - 3.2(64-x)(-1) = 6x + 6\ (64-x)$$

at $x = 32$, $\frac{d^2s}{dx^2} = 6(32) + 6\ (64-32) = 192 + 192 = 384$ (+ve)

$\therefore$ the sum is minimum

To find 2nd part sub x = 12 in (1)

$\therefore y = 64 - 32 \quad \therefore y = 32$

$\therefore$ the two parts are 32 and 32

Example 13 :

Find the maximum and minimum of the function $f(x) = x^3 - 12x^2 + 36x - 4$

Solution :

Let $y = f(x) = x^3 - 12x^2 + 36x - 4$ ....(1)

$$\frac{dy}{dx} = 3x^2 - 12(2x) + 36(1)$$

$0 = 3x^2 - 24x + 36 \Rightarrow 0 = 3(x^2 - 8x + 12)$

$\therefore\ x^2 - 8x + 12 = \frac{0}{3} = 0 \Rightarrow (x - 6(x-2) = 0 \quad \therefore x = 2$ and 6

Now $\frac{d^2y}{dx^2} = 3(2x) - 24(1) = 6x - 24$

at $x = 2$, $\frac{d^2y}{dx^2} = 6(2) - 24 = 12 - 24 = -12(-ve)$

$\therefore$ *f(x)* is maximum at $x = 2$

at $x = 6$, $\frac{d^2y}{dx^2} = 6(6) - 24 = 36 - 24 = +12(+ve)$

$\therefore$ *f(x)* is manimum at $x = 6$

$\therefore$ for maximum value put $x = 2$ in (1)

$$y_{max} = (2)^3 - 12(2)^2 + 36(2) - 4 = 8 - 48 + 72 - 4 = 80 - 52 = 28$$

for minimum value put $x = 6$ in (1)

$$y_{min} = (6)^3 - 12(6)^2 + 36(6) - 4 = 216 - 432 + 214 - 4 = -6$$

Exercise 19.3

2 and 3 mark questions:

Find the maximum and minimum value of the following function.

1. (i) $f(x) = x^3 - 3x$ (ii) $f(x) = x^3 - 6x^2 + 9x + 15,\ (0 \le x \le 6)$

   (iii) $f(x) = x^4 - 62x^2 + 120x + 9$ (iv) $f(x) = 2x^3 - 3x^2 - 12x + 12$

   (v) $f(x) = 2x^3 - 3x^2 - 36x + 10$ (vi) $f(x) = 9x^2 + 12x + 2$

   (vii) $f(x) = 2x^3 - 15x^2 + 36x + 10$ (viii) $f(x) = 2x^3 - 21x^2 + 36x - 20$

   (ix) $f(x) = 12x^5 - 45x^4 + 40x^3 + 6$

2) The sum of two natural numbers is 48. find the numbers when their product is maximum.
3) Find two positive numbers whose sum is 14 and the sum of whose square is minimum.
4) Find two positive numbers whose sum is 30 and the sum of their cubes is minimum.
5) The product of two natural numbers is 64. Find the numbers is their sum is minimum.

Answers 19.3

1. (i) Min at $x = 1$ is -2, Max at $x = 1$ is 2
   (ii) Min at $x = 1$ is 19, Max at $x = 3$ is 15
   (iii) Min at $x = -6$ is -1647, Max at $x = 1$ is 8 and Min at $x = 5$ is -316
   (iv) Min at $x = 2$ is -8, Max at $x = -1$ is 19
   (v) Min at $x = 3$ is -17, Max at $x = -2$ is 54
   (vi) Min at $x = \frac{-2}{3}$ is -2
   (vii) Min at $x = 3$ is 37, Max at $x = 2$ is 38
   (viii) Min at $x = 6$ is -128, Max at $x = 1$ is -3
   (ix) at $x = 0$, neither Max nor Min
   Min at $x = 1$ is 13, Mx at $x = 2$ is -10
2. 24, 24 3. 7, 7 4. 15 and 15 5. 32 and 32

19.5 Total cost, Average cost and Marginal cost

Total cost (cost function) : The outflow usually raw material, rent, pay & salaries from total cost.

It is the sum total of all costs

Total cost = variable cost + fixed cost

$T\ C = ax + b$ [x = output / quantity produced]

Average cost : Average cost is the cost per unit of the output

$$A.C = \frac{\text{Total cost}}{\text{Quantity}} = \frac{T.C}{x}$$

Margtinal cost : Marginal cost is the additional cost incurred as a result of producing and selling one more unit of the product ie., Instantaneous rate of change of total cost with respect to output

$$\therefore M.C = \frac{d(T.C)}{dx} \quad [x = \text{output}]$$

19.6 Total Revenue, Average Revenue and Marginal Revenue

Total Revenue (Revenue function) : The money which flow into an organisation from either giving service or selling product is called as Revenue

$\therefore$ Total Revenue = price / unit × quantity sold

$T\ R = p \times x$ or $p \times q$

Average Revenue : Average Revenue refers to Revenue / unit Quantity sold

$$\therefore AR = \frac{T.C}{x}$$

Marginal Revenue : Change in total revenue bought about by infinitesimal change in quantity sold

$$\therefore M.R = \frac{d(TR)}{dx}$$

Profit function : It is the difference between total Revenue and Total cost

$\therefore$ profit = TR – TC

Profit Maximization : For many production situation the Marginal Revenue exceeds the marginal cost. As the level of output increases the amount by which M.R. exceeds M.C. become smaller.

$\therefore$ for profit maximization **Marginal Revenue = Marginal cost** at which the level of output can be indentified.

Total profit (TP) : Total profit is the difference between

Total Revenue and Total Cost

TP = TR – TC

REMEMBER

(1) Total Cost (TC) $= \begin{cases} \text{(i) variable cost} + \text{Fixed Cost} \\ \text{(ii) A.C} \times \text{Quantity} \end{cases}$

(2) Marginal Cost (MC) $= \frac{d}{dx}(\text{T.C})$

(3) Average Cost (AC) $= \frac{\text{T.C}}{\text{Quantity}}$

(4) Total Revenue (TR) $= \begin{cases} \text{(i) Price} \times \text{Quantity} \\ \text{(ii) A.R} \times \text{Quantity} \end{cases}$

(Revenue function)

(5) Average Revenue = AR $= \frac{\text{TR}}{\text{Quantity}}$

(Demand function)

(6) Marginal Revenue = MR $= \frac{d}{dx}(\text{TR})$

(7) Total Profit = TP = TR – TC

(8) Profit Maximization : MR = MC

(9) Fixed cost = F.C = T.C (at $x = 0$)

Example 1:

If the cost function of a firm is given by $c(x) = x^3 - 3x + 7$. Find Average cost and Marginal cost when the output $x = 6$ unit.

Solution : $\text{A.C} = \frac{\text{T.C}}{x}$

$$= \frac{x^3 - 3x + 7}{x} = x^2 - 3 + \frac{7}{x}$$

$$\text{Marginal cost (MC)} = \frac{d}{dx}(T.C) = \frac{d}{dx}(x^3 - 3x + 7)$$

$$= 3x^2 - 3$$

$$\text{at } x = 6,\ \text{MC} = 3\,(6)^2 - 3$$

$$= 108 - 3 = 105$$

Example 2:

The total cost function is given by

$C = f(q) = q^3 - 3q^2 + 15q + 27$ (q = output). Find the average cost and the marginal cost. What is the fixed cost in the cost function $f(q)$

Solution :

$$\text{Average cost} = \text{A.C} = \frac{T.C}{q}$$

$$= \frac{q^3 - 3q^2 + 15q + 27}{q} = q^2 - 3q + 15 + \frac{27}{q}$$

$$\text{Marginal cost} = \text{M.C} \ \frac{d}{dq}(T.C)$$

$$= 3q^2 - 3(2q) + 15\ (1) = 3q^2 - 6q + 15$$

Fixed cost function is when $q = 0$ in $f(0)$

then $f(q) = 0 - 3\ (0) + 15\ (0) + 27.$

$$\therefore \boxed{f(0) = 27 = \text{fixed cost}}$$

Example 3:

If the total cost function $C = 9q - 3q^2 + \frac{q^3}{3}$, find the level of output at which Average cost is minimized.

Solution : $\text{A.C} = \frac{T.C}{q}$

$$= \frac{9q - 3q^2 + \frac{q^3}{3}}{q} \quad \Rightarrow \quad \text{AC} = 9 - 3q\frac{q^2}{3}$$

or A.C to be minimise differentiate w.r.t. q

$$\frac{d}{dq}(A.C) = -3(1) + \frac{2q}{3}$$

$$0 = -3 + \frac{2q}{3} = 3 = \frac{2q}{3}$$

$$\therefore \boxed{q = 9/2} \text{ unit.}$$

Example 4:

The Total Revenue function is given by $R = 400x - 2x^2$, and the total cost function is given by $C = 2x^2 + 40x + 4000$. Find

(i) the Marginal Revenue and Marginal cost functions

(ii) the Average Revenue and Average cost

(iii) the output at which Marginal Revenue = marginal cost.

Solution :

(i) $\text{MR} = \frac{d}{dx}(\text{TR})$

$= \frac{d}{dx}(400x - 2x^2)$

$= 400\,(1) - 2\,(2x) = 400 - 4x$

$\text{MC} = \frac{d}{dx}(\text{T.C})$

$= \frac{d}{dx}(2x^2 + 40x + 4000)$

$= 2(2x) + 40(1)$

$= 4x + 40$

(ii) $\text{AR} = \frac{\text{T.R}}{x} = \frac{400x - 2x^2}{x} = 400 - 2x$

$\text{AC} = \frac{\text{T.R}}{x} = \frac{2x^2 + 40x + 4000}{x}$

$= 2x + 40 + \frac{4000}{x}$

(iii) Given MR = MC

$400 - 4x = 4x + 40$

$400 - 40 = 4x + 4x$

$360 = 8x$

$x = \frac{360}{8}$

$\therefore \boxed{x = 45 \text{ unit}}$

Example 5:

If the demand function is given by $p = 50 - 2q$ (p = price, q = output). Find the value of 'p' and 'q' at which the revenue is maximum and the revenue corresponding to these values.

Solution :

Given $p = 50 - 2q$ ..... (1)

Total Revenue = TR = price × quantity

$= p \times q$

$= (50 - 2q) \times q$

$= 50q - 2q^2$

$\therefore$ for Maximum Revenue diffn. TR w.r.t. q

$$\frac{dR}{dq} = 50(1) - 2(2q)$$

$$0 = 50 - 4q$$

$$4q = 50 \qquad \therefore q = \frac{50}{4} = \boxed{\frac{25}{2}} \text{ units}$$

$$\frac{d^2R}{dq^2} = -4 < 0 \qquad \therefore \text{ Revenue is Maximum}$$

sub $q = \frac{25}{2} in (1)$ $\qquad \therefore p = 50 - 2\left(\frac{25}{2}\right)$

$p = 50 - 25$ $\qquad \boxed{p = ₹25}$

$\therefore$ TR = $p \times q$ = $\qquad 25 \times {}^{25}\!/_{2} = ₹\frac{625}{2} = ₹312.0$

Example 6:

The total cost of commodity is given by $C = x^2 - 7x + 2$ where 'x' is the number of units and the price / unit is ₹ 5.00. Find the profit function.

Solution :

$p(x)$ = profit function = $R(x) - C(x)$ .....(1)

T.R (x) = price × quantity

$= 5 \, . \, x$

T.C $(x) = x^2 - 7x + 2$ sub in (1)

$$p(x) = 5x - (x^2 - 7x + 2)$$
$$= 5x - x^2 + 7x - 2$$

$$\boxed{p(x) = -x^2 + 12x - 2}$$

Example 7:

The cost 'C' of manufacturing an article is given by $C = 5 + \frac{48}{q} + 3q^2$ **(q = number of article). find the number of article produced at the minimum cost and also find the minimum cost.**

Solution :

Given $C = 5 + \frac{48}{q} + 3q^2$ ..... (1)

for minimum cost dittn. w.r.t. q

$$\therefore \frac{dc}{dq} = 0 + 48\left(\frac{-1}{q^2}\right) + 3(2q)$$

$$0 = \frac{-48}{q^2} + 6q \qquad 6q = \frac{48}{q^2} \qquad \therefore 6q^3 = 48$$

$$q^3 = 8 \qquad \therefore \boxed{q = 2}$$

$$\frac{d^2C}{dq^2} = -48\left(\frac{-2}{q^3}\right) + 6(1)$$

$$= \frac{96}{q^3} + 6$$

at $q = 2$, $\frac{d^2C}{dq^2} = \frac{96}{2^3} + 6 = \frac{96}{8} + 6 = 12 + 6 = 18(+)$

$\therefore$ C is minimum

for minimum cost sub $q = 2$ in (1)

$$C = 5 + \frac{48}{2} + 3(2)^2 = 5 + 24 + 12$$

$$\boxed{C = ₹41}$$

Example 8:

Let the demand function of an article be $p = 75 - 2x$ and the cost function be $C(x) = 350 + 12x + \frac{x^2}{4}$. Find the number of units and the price at which the total profit is maximum [p = price, x = output]

Solution :

Total profit = T.R – T.C or

$= R(x) - C(x)$ ..... (1)

$R(x)$ = price × quantity

$= (75 - 2x)x$

$= 75x - 2x^2$

$C(x) = 350 + 12x + \frac{x^2}{4}$ sub in (1)

$\therefore p(x) = \left[75x - 2x^2\right] - \left[350 + 12x + \frac{x^2}{4}\right]$ ..... (2)

$= 75x - 2x^2 - 350 - 12x - \frac{x^2}{4}$

$p(x) = -2x^2 - \frac{x^2}{4} + 63x - 350$

for maximum profit dittn. w.r.t. x

$\frac{d\left[p(x)\right]}{dx} = -2(2x) - \frac{2x}{4} + 63$

$0 = -4x - \frac{2x}{4} + 63 \quad \Rightarrow \quad 0 = -4x - \frac{x}{2} + 63$

$\frac{x}{2} = -4x + 63$

$x = -8x + 126 \qquad \therefore 9x = +126$

$x = \frac{+126}{9} = 14(+)$

for total profit put $x = 14$ in p = $75 - 2x$

$\therefore p = 75 - 2(14) = 75 - 28$

$\therefore p = ₹ 47.$

Example 9:

A manufacture gives the total cost $C = \frac{q^3}{3} - 7q^2 + 111q + 50$ and $q = 100p$. (q = output, p = price) find the level of output at which profit is maximised?

Solution :

$$\text{T.C} = \frac{q^3}{3} - 7q^2 + 111q + 50$$

$$\therefore \text{M.C} = \frac{dc}{dq} = \frac{3q^2}{3} - 7(2q) + 111(1)$$

$$= q^2 - 14q + 1111$$

T.R = price × quantity

$$= (100 - q)\, q \qquad \begin{pmatrix} q = 100 - p \\ \therefore\ p = 100 - q \end{pmatrix}$$

$$= 100\, q - q^2$$

$$\therefore \text{MR} = \frac{d}{dq}(\text{TR}) = 100(1) - 2q \quad = 100 - 2q$$

for profit maximese

MR = MC

$$100 - 2q = q^2 - 14q + 1111$$

$$q^2 - 14q + 2q + 111 - 100$$

$$q^2 - 12q + 11 = 0 \qquad \Rightarrow (q - 1)\,(q - 11) = 0$$

$$\therefore \boxed{q = 1 \text{ or } q = 11}$$

To find for which output the profit maximise

$$\therefore \text{Take} \frac{d}{dq}(\text{MK}) < \frac{d}{dq}(\text{MC})$$

$$\frac{d}{dq}(100 - 2q) < \frac{d}{dq} \qquad (q^2 - 14q + 111)$$

$$-2 < 2q - 14 \quad = \quad -2 + 14 < 2q$$

$$= \quad 12\ (2q)$$

at $q = 1$ $\quad 12 < 2\ (1)$

$12 < 2$ does not satisfied

$\therefore q = 11 \quad 12 < 2(11)$

$12 < 22$ (satified)

$\therefore \boxed{q = 11}$

Example 10:

The demond function of a firm is $p = 500 - 0.2q$ and the total cost $C = 25q + 10000$ (p = price, q = output). Find the output at which the profit of the firm is maximised. What is the price charged.

Solution :

Given T.C = $25q = 10000$

T.R = price × quantity

$= (500 - 0.2q)q$

$= 500\ q - 0.2q^2$

for profit maximise : MR = MC ..... (1)

$$MR = \frac{d}{dq}(TR) = \frac{d}{dx}\left(500q - 0.2q^2\right)$$

$= 500\ (1) - 0.2(2q) = 500 - 0.4q$

$$MC = \frac{d}{dx}(TC)$$

$= 25(1) = 25$

sub is (1)

MR = MC

$500 - 0.4q = 25$

$0.4q = 500 - 25$

$0.4q = 475$

$$\therefore q = \frac{475}{0.4} = \frac{4750}{4} = 1187.5 \text{ unit}$$

sub is $p = 500 - 0.2\ (1187.5)$

$\boxed{p = ₹\ 262.50}$

Exercise 19.4

2 and 3 mark questions:

1) Find the Average cost and Marginal cost if the total cost function of an article is given by $C = 5x^2 + 2x + 3$ (x = quantity)

2) The total cost fo a commodity is given by $C = -x^2 + 5x + 7$ (x = number of unit) and the price per unit is ₹ 12. Find the profit function.

3) For the demond function $2x - 5y = 7$ (x = number of unit and y is the price / unit). Find the Total Revenue, Marginal Revenue and Average Revenue.

4) The total cost of C of output Q is given by $C = 300Q - 10Q^2 + \frac{Q^3}{3}$. find the output level at which the marginal cost and the Average cost attain their respective minimum.

5) The total cost of the production of a firm in given by the following function $C = 0.7x + 18$ (x = output) Find

   (i) the Total cost for an output 10 unit

   (ii) the Average cost for an output 9 unit

   (iii) the Marginal cost for an output of 6 unit

6) If $R = 250x + 45x^2 - x^3$, (R = total Revenue, x = no. of unit) what will be the Marginal Revenue if $x = 25$ unit and the Average Revenue if $x = 10$ unit.

7) The total cost function of a manufacturer is $C = 5x^2 + 500x + 50000$. Find the output (x) when AC = MC.

8) If $R = x\left(15 - \frac{x}{30}\right)$. What is the Marginal Revenue function and what will be the Marginal Revenue if 100 unit were produced?

9) The Total Revenue (R) and the total cost (C) function of a company are given by $\boldsymbol{R(q) = 300q - q^2}$ and $C(q) = 20 + 4q$ (q = output). Find the equilibrium output (Hint : equilbriums MR = MC)

Answers 19.4

1) $AC = 5x + 2 + \frac{3}{x}$, $M.C. = 10x + 2$

2) $p(x) = x^2 + 7x - 7$

3) Price = $y = \frac{2x-7}{5}, \text{TR} = \frac{2x^2-7x}{5}, \text{MR} = \frac{4x-7}{5}, \text{AR} = \frac{2x-7}{5}$

4) MC = ₹ 10, AC = ₹ 15

5) TC = ₹ 25, AC = 2.7, MC = 0.7

6) MR = ₹ 625, AR = ₹ 600

7) $x = 100$ units

8) $\text{MR}(x) = 15 - \frac{x}{15}$ and MR = ₹ 8.33

9) $q = 13$ unit.

*****

Chapter

20

INDEFINITE INTEGRALS

20.1 Introduction

Students are already familiar with differentiation and have to find out the differential coefficient of a given function. In this chapter we have to find out the function whenever its derivative is given. The required function is known as **anti derivative** (or primitive) of the given derivative. The anti derivatives are called the **Indefinite integral.**

Integration is the 'Inverse' or 'Reverse' or 'opposite' process of differentiation.

There are two forms of integrals (i) Indefinite and (ii) definite integral. Definite integral is used to find out the area bounded by the graph of a function under certain conditions. Definite integral has wide range of application in the field of engineering, science, economics, finance, commerce etc. Definite integral will be discussed later. In this chapter we shall study indefinite integral.

20.2 Primitive or Anti derivative

Definition : A function F(x) is called a primitive or an anti derivative or an integral of a function f(x) if $\frac{d}{dx}\,F(x) = f(x)$

***Eg* :** (i) $\frac{x^2}{2}$ is primitive of x, because $\frac{d}{dx}\left(\frac{x^2}{2}\right) = x$

(ii) $\sin x$ is primitive of $\cos x$, because $\frac{d}{dx}\,(\sin x) = \cos x$

(iii) $\log x$ is primitive of $\left(\frac{1}{x}\right)$, because $\frac{d}{dx}\,(\log x) = \frac{1}{x}$

Note: that $\frac{x^2}{2} + c$ is also primitive of x, because $\frac{d}{dx}\left(\frac{x^2}{2} + c\right) = x$

$(\sin x + c)$ is also primitive of $\cos x$, because $\frac{d}{dx}\,(\sin x + c) = \cos x$

$(\log x + c)$ is also primitive of $\frac{1}{x}$, because $\frac{d}{dx}\,(\log x + c) = \frac{1}{x}$

From the above examples, we can observe that the primitives or anti derivatives or integrals of functions are not unique. There are infinitely many antiderivatives of each of these functions and can be find out by choosing C arbitrarly from the set of real numbers.

In general, let F*(x)* be primitive of *f(x)* and let C be arbitrary constant. Then

$$\frac{d}{dx}[F(x)+c]=f(x)$$

$\therefore$ F*(x)* + *c*, where C$\in$ R denotes family of antiderivatives of *f*.

20.3 Indefinite Integral

Definition : Let *f(x)* be a function. Then the family of all its primitives (or antiderivatives) is called the indefinite integral of *f(x)* and is denoted by $\int f(x)dx$

$$\therefore \frac{d}{dx}[F(x)+c]=f(x) \qquad \int f(x)dx = F(x)+c$$

where F*(x)* is primitive of *f(x)* and c is an arbitrary constant called the constant of integration.

Symbols and their meanings

Symbols / terms	Meaning
$\int f(x)\ dx$	Integral of f with respect to x
$f(x)$ in $\int f(x)\ dx$	Integrand
x in $\int f(x)\ dx$	variable of integration
Integrate	Find the integral
An integral of f or primitive of f or antiderivative of f	A function F such that $F'(x) = f(x)$
Integration	The process of finding integral
Constant of integration	Any real number C, considered as constant function.

20.4 Standard integrals

Integrals (Anti derivatives)	Derivatives
(i) $\int x^n dx = \frac{x^{n+1}}{n+1}+C,\ n\neq -1$	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$
Note : $\int dx = x+c$	$\frac{d}{dx}(x)=1$
(ii) $\int \frac{1}{x}dx = \log x + c$	$\frac{d}{dx}(\log x)=\frac{1}{x}$

(iii) $\int e^x dx = e^x + c$	$\frac{d}{dx}(e^x) = e^x$
(iv) $\int a^x dx = \frac{a^x}{\log a} + c$	$\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$
(v) $\int \sin x \ dx = -\cos x + c$	$\frac{d}{dx}(-\cos x) = \sin x$
(vi) $\int \cos x \ dx = \sin x + c$	$\frac{d}{dx}(\sin x) = \cos x$
(vii) $\int \sec^2 x \ dx = \tan x + c$	$\frac{d}{dx}(\tan x) = \sec^2 x$
(viii) $\int \text{cosec}^2 x \ dx = -\cot x + c$	$\frac{d}{dx}(-\cot x) = \text{cosce}^2 x$
(ix) $\int \sec x \tan dx = \sec x + c$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
(x) $\int \text{cosec } x \cot x \ dx = -\text{cosec } x + c$	$\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

20.5 Properties of the Idefinite integral

Property (i)

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

and $\int f^1(x)\,dx = f(x) + c$, where c is any arbitrary constant

Property (ii)

If $\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\int g(x)dx$

Then $\int f(x)dx + c_1 = \int g(x)dx + c_2$

Property (iii)

If $\int [f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx$

Property (iv)

$\int k\,f(x)dx = k\int f(x)dx$, where k is any real number

Property (v)

$$\int [k_1 f_1(x) + k_2 f_2(x) + k_3 f_3(x) + - - + k_n f_n(x)]\, dx =$$

$$k_1 \int f_1(x)dx + k_2 \int f_2(x)dx + k_3 \int f_3(x)dx + - - k_n \int f_n(x)dx$$

WORKED EXAMPLES

Example 1 :

Evaluate the following integrals:

(a) $\int x^5 dx$ **(b)** $\int 6\sqrt{x}\ dx$ **(c)** $\int \frac{1}{\sqrt[3]{x^5}} dx$ **(d)** $\int \frac{1}{5e^{-x}} dx$ **(e)** $\int 7.5^x dx$

(f) $\int \frac{3}{x} dx$ **(g)** $\int \frac{8}{\text{cosec}\, x} dx$ **(h)** $\int \frac{9}{\sin^2 x}\, dx$ **(i)** $\int 4\sec^2 x\ dx$

Solution :

(a) $\int x^5 dx = \frac{x^{5+1}}{5+1} + c$

$= \frac{x^6}{6} + c$

(b) $\int 6\sqrt{x}\ dx = 6\int x^{\frac{1}{2}} dx$

$= 6 \times \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = 6 \times \frac{2}{3} x^{\frac{3}{2}} + c$

$= 4x^{\frac{3}{2}} + c$

(c) $\int \frac{1}{\sqrt[3]{x^5}} dx = \int x^{-\frac{5}{3}} dx$

$= \frac{x^{\frac{-5}{3}+1}}{-\frac{5}{3}+1} + c = \frac{x^{-\frac{2}{3}}}{-\frac{2}{3}} + c$

$= \frac{-3}{2} x^{-\frac{2}{3}} + c$

(d) $\int \frac{1}{5e^{-x}} dx = \frac{1}{5}\int e^x dx = \frac{1}{5}e^x + c$

(e) $\int 7.5^x dx = 7\int 5^x dx = 7\frac{5^x}{\log 5} + c$

(f) $\int 3\frac{1}{x} dx = 3\int \frac{1}{x} dx = 3\log x + c$

(g) $\int \frac{8}{\text{cosec } x} dx = 8\int \sin x \, dx = -8 \cos x + c$

(h) $\int \frac{9}{\sin^2 x} dx = 9\int \text{cosec}^2 x dx = -9 \cot x + c$

(i) $\int 4\sec^2 x dx = 4\int \sec^2 x dx = 4 \tan x + c$

Example 2 :

Evaluate the following integrals:

(a) $\int \left(2x^2 - \frac{3}{x} + 2e^x\right) dx$

(b) $\int \left(\frac{x^4 + 3x^2 - 5x}{x^2}\right) dx$

(c) $\int \left(x + \frac{1}{x}\right)^3 dx$

(d) $\int \sqrt{x}\left(1 - \frac{1}{x}\right) dx$

(e) $\int \left(e^{5 \log x} + 5^{3 \log_5 x}\right) dx$

(f) $\int \left(\frac{7^x - 6.8^x}{5^x}\right) dx$

(g) $\int \frac{\left(3^x - 2^x\right)^2}{3^x 2^x} dx$

(h) $\int \left(3e^x + 5a^x - e^{\log a}\right) dx$

Solution:

(a) $\int \left(2x^2 - \frac{3}{x} + 2e^x\right) dx$

$= 2\int x^2 dx - 3\int \frac{1}{x} dx + 2\int e^x dx$

$= 2\frac{x^3}{3} - 3 \log x + 2e^x + c$

(b) $\int\left(\frac{x^4+3x^2-5x}{x^2}\right)dx$

$= \int\frac{x^4}{x^2}dx+3\int\frac{x^2}{x^2}dx-5\int\frac{x}{x^2}dx$

$= \int x^2dx+3\int 1dx-5\int\frac{1}{x}dx$

$= \frac{x^3}{3}+3x-5\ \log x+c$

(c) $\int\left(x+\frac{1}{x}\right)^3dx$ $\quad\because (a+b)^3=a^3+b^3+3ab\ (a+b)$

$= \int\left(x^3+\frac{1}{x^3}+3x\frac{1}{x}\left(x+\frac{1}{x}\right)\right)dx$

$= \int\left(x^3+\frac{1}{x^3}+3x+\frac{3}{x}\right)dx$

$= \int x^3dx+\int x^{-3}dx+3\int xdx+3\int\frac{1}{x}dx$

$= \frac{x^4}{4}-\frac{x^{-2}}{2}+\frac{3x^2}{2}+3\ \log x+c$

(d) $\int\sqrt{x}\left(1-\frac{1}{x}\right)dx$

$= \int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx = \int\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)dx$

$= \int x^{\frac{1}{2}}dx-\int x^{\frac{-1}{2}}dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}-\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$

$= \frac{2}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}+c$

(e) $\int \left(e^{5 \log x} + 5^{3 \log_5 x}\right) dx$

$= \int \left(e^{\log x^5} + 5^{\log_5 x^3}\right) dx \qquad \left[\because m \log a = \log a^n\right]$

$= \int \left(x^5 + x^3\right) dx \qquad \left[\because a^{\log \frac{x}{a}} = x\right]$

$= \int x^5 dx + \int x^3 dx$

$= \frac{x^6}{6} + \frac{x^4}{4} + c$

(f) $\int \left(\frac{7^x - 6.8^x}{5^x}\right) dx$

$= \int \left(\frac{7}{5}\right)^x dx - 6\int \left(\frac{8}{5}\right)^x dx$

$= \frac{\left(\frac{7}{5}\right)^x}{\log\left(\frac{7}{5}\right)} - 6\frac{\left(\frac{8}{5}\right)^x}{\log\left(\frac{8}{5}\right)} + c$

(g) $\int \frac{(3^x - 2^x)^2}{3^x 2^x} dx$

$= \int \frac{(3^{2x} + 2^{2x} - 2.3^x.2^x)}{3^x 2^x}$

$= \int \left(\frac{3^x}{2^x} + \frac{2^x}{3^x} - 2\right) dx$

$= \int \left(\frac{3}{2}\right)^x dx + \int \left(\frac{2}{3}\right)^x - \int 2dx$

$= \frac{\left(\frac{3}{2}\right)^x}{\log \frac{3}{2}} + \frac{\left(\frac{2}{3}\right)^x}{\log \frac{2}{3}} - 2x + c$

(h) $\int \left(3e^x + 5a^x - e^{\log a}\right) dx$ $\quad [\because e^{\log_e a} = a]$

$= 3\int e^x + 5\int a^x - \int a\ dx$

$= 3e^x + 5\dfrac{a^x}{\log a} - ax + c$

Example 3 :

Integrate the following w.r.t.x.

(a) $\tan^2 x$ **(b)** $\dfrac{5}{3}\dfrac{\sin x}{\cos^2 x}$ **(c)** $\dfrac{1}{1+\cos x}$ **(d)** $\sqrt{1+\cos 2x}$

(e) $\sqrt{1-\sin 2x}$ **(f)** $\dfrac{\sin x}{1+\sin x}$ **(g)** $\dfrac{\cos^2 x}{1+\sin x}$ **(h)** $\sec x\sqrt{\sec^2 x - 1}$

(i) $\operatorname{cosec} x\,[\operatorname{cosec} x + \cot x]$ **(j)** $5x^3 - 3\operatorname{cosec}^2 x + 6^x$ **(k)** $\dfrac{5+7\cos x}{\sin^2 x}$

Solution:

(a) $\int \tan^2 x\ dx$

$= \int (\sec^2 x - 1) dx \qquad \because 1 + \tan^2 x = \sec^2 x$

$= \tan x - x + c$

(b) $\int \dfrac{5\ \sin x}{3\ \cos^2 x} dx$

$= \dfrac{5}{3}\int \dfrac{\sin x}{\cos^2 x} dx = \dfrac{5}{3}\int \sec x \tan x\ dx$

$= \dfrac{5}{3}\sec x + c$

(c) $\int \dfrac{1}{1+\cos x} dx$

$= \int \dfrac{1}{1+\cos x} \times \dfrac{1-\cos x}{1-\cos x} dx$ (Rationalise the Denominator)

$= \int \dfrac{1-\cos x}{1-\cos^2 x} dx = \int \dfrac{1-\cos x}{\sin^2 x} dx$

$$= \int \frac{1}{\sin^2 x} dx - \int \frac{\cos x}{\sin^2 x} dx$$

$$= \int \text{cosec}^2 x \, dx - \int \text{cosec}\, x \, \cot x \, dx$$

$$= -\cot x + \text{cosec} x + c$$

(d) $\int \sqrt{1+\cos 2x} \, dx$

$$= \int \sqrt{2\cos^2 x} \, dx \qquad \because \cos 2x = 2\cos^2 x - 1$$

Multiple angle

$$= \sqrt{2} \int \cos x \, dx$$

$$= \sqrt{2} \sin x + c$$

(e) $\int \sqrt{1-\sin 2x} \, dx$

$$= \int \sqrt{\left(\cos^2 x + \sin^2 x - 2\sin x \cos x\right)} \, dx \qquad [\because \cos^2 x + \sin^2 x = 1]$$

$$= \int \sqrt{(\cos x - \sin x)^2} \, dx$$

$$= \int (\cos x - \sin x) dx$$

$$= \int \cos x \, dx - \int \sin x \, dx$$

$$= \sin x + \cos x + c$$

(f) $\int \frac{\sin x}{1+\sin x} dx$

$$= \int \left(\frac{\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} \right) dx$$

$$= \int \frac{\sin x - \sin^2 x}{1-\sin^2 x} dx$$

$$= \int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$$

$$= \int \frac{\sin x}{\cos^2 x} dx - \int \frac{\sin^2 x}{\cos^2 x} \, dx$$

$$= \int \frac{1}{\cos x} \times \frac{\sin x}{\cos x} dx - \int \tan^2 x \, dx$$

$$= \int \sec x \tan x \, dx - \int (\sec^2 x - 1) dx$$

$$= \sec x - \int \sec^2 x dx + \int dx$$

$$= \sec x - \tan x + x + c$$

(g) $\int \frac{\cos^2 x}{1+\sin x} dx$

$$= \int \frac{\cos^2 x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx$$

$$= \int \frac{\cos^2 x(1-\sin x)}{1-\sin^2 x} dx$$

$$= \int \frac{\cos^2 x(1-\sin x)}{\cos^2 x} dx = \int (1-\sin x) dx$$

$$= x + \cos x + c$$

(h) $\int \sec x \sqrt{\sec^2 x - 1} \, dx$

$$= \int \sec x \sqrt{\tan^2 x} \, dx$$

$$= \int \sec x \tan x \, dx = \sec x + c$$

(i) $\int \text{cosec } x \, (\text{cosec } x + \cot x) dx$

$$= \int (\text{cosec}^2 x + \text{cosec} x \cot x) dx$$

$$= \int \text{cosec}^2 x \, dx + \int \text{cosec} x \, \cot x \, dx$$

$$= -\cot x - \text{cosec } x + c$$

(j) $\int (5x^3 - 3\text{cosec}^2 x + 6^x) dx$

$$= 5\int x^3 dx - 3\int \text{cosec}^2 x \, dx + \int 6^x dx$$

$$= 5\frac{x^4}{4} + 3 \cot x + \frac{6^x}{\log_e 6} + c$$

(k) $\int\left(\frac{5+7\cos x}{\sin^2 x}\right)dx$

$= 5\int\frac{1}{\sin^2 x}dx+7\int\frac{\cos x}{\sin^2 x}dx$

$= 5\int \operatorname{cosec}^2 x\,dx+7\int \operatorname{cosec} x\cot x\,dx$

$= -5\cot x-7\operatorname{cosec} x+c$

Exercise 20.1

I One Mark Questions:

(a) $x^2-\frac{6}{x}+5e^x$ (b) $7^x-10.9^x$ (c) $x^e+e^x-\log a$ (d) $7^{6\log_7 x}$ (e) $\cot^2 x$

(f) $\frac{9}{5}\frac{\cos x}{\sin^2 x}$ (g) $7x^2-4\sec^2 x$

II Two Marks Questions:

(a) $\frac{x^5+5x^2-7x}{\sqrt{x}}$ (b) $\left(x-\frac{1}{x}\right)^3$ (c) $\sqrt{x}\left(1-\frac{1}{\sqrt{x}}\right)$ (d) $x^{\frac{3}{2}}\left(x-\frac{1}{x^2}\right)$ (e) $\frac{6^x-3^x}{5^x}$

(f) $3^{3\log_3 x}-7^{7\log_7 x}$ (g) $\sqrt{1-\cos 2x}$ (h) $\frac{1+\sin x}{\cos^2 x}$ (i) $\sec x(\sec x-\tan x)$

III Three Marks Questions:

(a) $\frac{(a^x+b^x)^2}{a^x b^x}$ (b) $\operatorname{cosec} x\sqrt{\operatorname{cosec}^2 x-1}$ (c) $\sqrt{1+\sin 2x}$ (d) $\frac{1-\cos 2x}{1+\cos 2x}$

IV Five Marks Questions:

(a) $\frac{1}{1+\sin x}$ (b) $\frac{\sec x}{1+\sec x}$ (c) $\frac{\cos x}{1+\cos x}$ (d) $\frac{\sin^2 x}{1+\cos x}$

Answers 20.1

I (a) $\frac{x^3}{3}-6\log x+5e^x+c$ (b) $\frac{7^x}{\log 7}-10\frac{9^x}{\log 9}+c$ (c) $\frac{x^{e+1}}{e+1}+e^x-x\log a+c$

(d) $\frac{x^7}{7}+c$ (e) $-\cot x-x+c$ (f) $-\frac{9}{5}\operatorname{cosec} x+c$ (g) $\frac{7x^3}{3}-4\tan x+c$

II (a) $\frac{2}{11}x^{\frac{11}{2}} + 2x^{\frac{5}{2}} - \frac{14}{3}x^{\frac{3}{2}} + c$ (b) $\frac{x^4}{4} + \frac{1}{2x^2} - \frac{3x^2}{2} + 3\log x + c$

(c) $\frac{2}{3}x^{\frac{3}{2}} - x + c$ (d) $\frac{2}{7}x^{\frac{7}{2}} - 2x^{\frac{1}{2}} + c$ (e) $\frac{\left(\frac{6}{5}\right)^x}{\log\left(\frac{6}{5}\right)} - \frac{\left(\frac{3}{5}\right)^x}{\log\left(\frac{3}{5}\right)} + c$

(f) $\frac{x^4}{4} - \frac{x^8}{8} + c$ (g) $-\sqrt{2}\cos x + c$ (h) $\tan x + \sec x + c$

(i) $\tan x - \sec x + c$

III (a) $\frac{\left(\frac{a}{b}\right)^x}{\log\left(\frac{a}{b}\right)} + \frac{\left(\frac{b}{a}\right)^x}{\log\left(\frac{b}{a}\right)} + 2x + c$ (b) $-\operatorname{cosec} x + c$ (c) $\sin x - \cos x + c$

(d) $\tan x - x + c$

IV (a) $\tan x - \sec x + c$ (b) $-\cot x + \operatorname{cosec} x \cot x + c$

(c) $-\operatorname{cosec} x + \cot x + x + c$ (d) $x - \sin x + c$

20.6 Integration by Substitution:

In the previous topic we have evaluated the integrals by using the standard integrals directly. While the integrals of some functions cannot be obtained directly, but those functions can be reduced to standard integrals by using proper method of integration.

Important methods of integration are:

1. **Integration by substitution** 2. **Integration by partial fractions**
3. **Integration by parts**

In this topic, we will discuss the method of integration by substitution.

The method of evaluating an integral by reducing it to standard form by proper substitution is called Integration by substitution.

Some special forms of substitution

(i) If $\int f(x)dx = \phi(x) + c$, then $\int f(ax+b)dx = \frac{1}{a}\phi(ax+b) + c$

(ii) $\int \frac{f^1(x)}{f(x)}\,dx = \log[f(x)] + c \quad \left[\frac{d}{dx}f(x) = f'(x)\right]$

(iii) $\int [f(x)]^n f^1(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c$

STANDARD THIFOREMS

(i) $\int f(x)\,dx = \phi(x) + c,\ \int f(ax+b)dx = \frac{1}{a}\phi(ax+b) + c,\ a \neq 0$

Proof: Put $ax + b = t$

$$\Rightarrow adx = dt \Rightarrow dx = \frac{1}{a}dt$$

$$\therefore \int f(ax+b)dx = \int f(t)\frac{1}{a}dt$$

$$= \frac{1}{a}\int f(t)dt \quad = \frac{1}{a}\ \phi(t) + c$$

$$= \frac{1}{a}\phi(ax+b) + c,\ (a \neq 0)$$

(ii) $\int \frac{f^1(x)}{f(x)}\,dx = \log[f(x)] + c, f(x) \neq 0$

Proof: put $f(x) = t$

$$\Rightarrow f^1(x)dx = dt \quad \therefore \int \frac{f^1(x)}{f(x)}\,dx = \int \frac{1}{t}dt$$

$$= \log\ t + c$$

$$= \log\ [f(x)] + c$$

(iii) $\int [f(x)]^n f^1(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1$

Proof: put $f(x) = t$

$$\Rightarrow f^1(x)\,dx = dt$$

$$\therefore \int [f(x)]^n\ f^1(x)\,dx = \int t^n dt = \frac{t^{n+1}}{n+1} + c = \frac{[f(x)]^{n+1}}{n+1} + c$$

Standard results:

(i) $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c,\ n \neq -1 \qquad a \neq 0$

(ii) $\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax+b| + c \qquad a \neq 0$

(iii) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$ $a \neq 0$

(iv) $\int a^{bx+c}dx = \frac{1}{b}\frac{a^{bx+c}}{\log a} + c, a > 0$ and $a \neq 1$ $a \neq 0$

(v) $\int \sin(ax+b)dx = \frac{-1}{a}\cos(ax+b) + c$ $a \neq 0$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$ $a \neq 0$

(vii) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$ $a \neq 0$

(viii) $\int \text{cosec}^2(ax+b)dx = \frac{-1}{a}\cot(ax+b) + c$ $a \neq 0$

(ix) $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + c$ $a \neq 0$

(x) $\int \text{cosec}(ax+b)\cot(ax+b)dx = \frac{-1}{a}\text{cosec}(ax+b) + c$ $a \neq 0$

(xi) $\int \tan(ax+b)\ dx = \frac{1}{a}\log\ \sec\ |ax+b| + c$ $a \neq 0$

(xii) $\int \cot(ax+b)dx = \frac{1}{a}\log|\sin(ax+b)| + c$ $a \neq 0$

(xiii) $\int \sec(ax+b)dx = \frac{1}{a}\log|\sec(ax+b) + \tan(ax+b)| + c$ $a \neq 0$

(xiv) $\int \text{cosec}(ax+b)dx = \frac{1}{a}\log|\text{cosec}(ax+b) - \cot(ax+b)| + c$ $a \neq 0$

Some other results:

(i) $\int \tan x\ dx = \log|\sec x|$ (ii) $\int \cot x\ dx = \log|\sin x|$

(iii) $\int \sec x\ dx = \log|(\sec x + \tan x)| = \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|$

(iv) $\int \text{cosec}\, x\ dx = \log|\text{cosec}\, x - \cot(x)| = \log\left|\tan\frac{x}{2}\right|$

Type I : $\boxed{\int f(ax+b)dx = \frac{1}{a}\phi(ax+b) + c}$

Worked Examples

Example 4:

Integrate the following functions w.r.t 'x'

(a) $(3x+5)^2$ **(b)** $\sqrt{2x-3}$ **(c)** $\frac{1}{7x+8}$ **(d)** e^{5x+7} **(e)** 7^{3x+4}

(f) $\text{cosec}^2(5x-3)$ **(g)** $\sec(2x+3)\tan(2x+3)$

Solution :

(a) $\int (3x+5)^2\,dx$

$$= \frac{(3x+5)^{2+1}}{3(2+1)} + c = \frac{(3x+5)^3}{9} + c$$

(b) $\int \sqrt{2x-3}\,dx$

$$= \int (2x-3)^{\frac{1}{2}}\,dx = \frac{1}{2}\frac{(2x-3)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)} + c$$

$$= \frac{1}{2}\times\frac{2}{3}(2x-3)^{\frac{3}{2}} + c$$

$$= \frac{1}{3}(2x-3)^{\frac{3}{2}} + c$$

(c) $\int \frac{1}{7x+8}\,dx = \frac{\log\,(7x+8)}{7} + c$

(d) $\int e^{5x+7}dx = \frac{e^{5x+7}}{5} + c$

(e) $\int 7^{3x+4}dx = \frac{7^{3x+4}}{3\log 7} + c$

(f) $\int \text{cosec}^2(5x-3)dx = \frac{-\cot\,(5x-3)}{5} + c$

(g) $\int \sec(2x+3)\tan(2x+3)dx = \frac{\sec(2x+3)}{2} + c$

Example 5:

Integrate the following functions w.r.t. 'x'

(a) $\sin^2 x$ **(b)** $\tan^2(3-2x)$ **(c)** $\sin^3 x$

(d) $\sin 3x \cos 2x$ **(e)** $\cos 5x \cos 3x$ **(f)** $\sin 5x \sin 2x$

Solution :

(a) $\int \sin^2 x dx$

$$= \int \frac{1-\cos 2x}{2} dx$$

$$= \frac{1}{2}x - \frac{\sin 2x}{4} + c$$

$$\left[\begin{array}{l}\text{From multiple angle} \\ \cos 2x = 2\sin^2 x - 1 \\ \therefore \sin^2 x = \dfrac{1-\cos 2x}{2}\end{array}\right]$$

(b) $\int \tan^2(3-2x) dx$

$$= \int \left[\sec^2(3-2x) - 1\right] dx$$

$$= \frac{\tan(3-2x)}{-2} - x + c$$

$$= \frac{-1}{2} \tan(3-2x) - x + c$$

(c) $\int \sin^3 x dx$

$$= \int \frac{(3\sin x - \sin 3x)}{4} dx \qquad [\because \sin 3x = 3\sin x - 4\sin^3 x]$$

$$= \frac{1}{4}\left[\int 3\sin x dx - \int \sin 3x \, dx\right]$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + c$$

$$= \frac{-3}{4}\cos x + \frac{1}{12}\cos 3x + c$$

(d) $\int \sin 3x \cos 2x \, dx$

$$= \frac{1}{2}\int [\sin(3x+2x)+\sin(3x-2x)]dx \quad \left[\begin{array}{l}\text{From Transformation formula} \\ \because \sin A \cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))\end{array}\right]$$

$$= \frac{1}{2}\int (\sin 5x + \sin x)dx$$

$$= \frac{1}{2}\left[\frac{-\cos 5x}{5} - \cos x\right] + c$$

$$= -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$$

(e) $\int \cos 5x \cos 3x \, dx$

$$= \frac{1}{2}\int [(\cos(5x+3x)+\cos(5x-3x)]dx \quad [\because \cos A \ \ \cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$

$$= \frac{1}{2}\left[\int \cos 8x dx + \int \cos 2x dx\right]$$

$$= \frac{1}{2}\left[\frac{+\sin 8x}{8} + \frac{\sin 2x}{2}\right] + c$$

$$= \frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + c$$

(f) $\int \sin 5x \sin 2x \, dx$

$$= \frac{1}{2}\int [\cos(5x-2x)-\cos(5x+2x)]dx \quad \left[\because \sin A \sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)\right]$$

$$= \frac{1}{2}\int (\cos 3x - \cos 7x)dx$$

$$= \frac{1}{2}\left[\frac{\sin 3x}{3} - \frac{\sin 7x}{7}\right] + c$$

$$= \frac{\sin 3x}{6} - \frac{\sin 7x}{14} + c$$

Example 6 :

Evaluate the following integrals:

(a) $\int \sqrt{1-\sin x}\, dx$ **(b)** $\int \frac{1+\cos x}{1-\cos x} dx$ **(c)** $\int \frac{\sin x}{1+\cos x} dx$ **(d)** $\int \frac{1}{1+\cos x} dx$

Solution :

(a) $\int \sqrt{1-\sin x}\, dx$

$= \int \sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - 2\sin \frac{x}{2}\cos \frac{x}{2}}\, dx \qquad \left[\because \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1\right]$

$= \int \sqrt{\left(\sin \frac{x}{2} - \cos \frac{x}{2}\right)^2}\, dx$

$= \int \left(\sin \frac{x}{2} - \cos \frac{x}{2}\right) dx$

$= \frac{-\cos \frac{x}{2}}{\frac{1}{2}} - \frac{\sin \frac{x}{2}}{\frac{1}{2}} + c$

$= -2\cos \frac{x}{2} - 2\sin \frac{x}{2} + c$

(b) $\int \frac{1+\cos x}{1-\cos x} dx \qquad \left[\begin{array}{l} \cos 2x = 2\cos^2 x - 1 \\ \therefore \cos x = 2\cos^2 \frac{x}{2} - 1 \end{array}\right]$

$= \int \frac{2\cos^2 \frac{x}{2}}{2\sin^2 \frac{x}{2}} dx = \int \cot^2 \frac{x}{2} dx$

$= \int \left(\text{cosec}^2 \frac{x}{2} - 1\right) dx \qquad \left(\therefore 1 + \cot^2 x = \text{cosec}^2 x\right)$

$= \frac{-\cot \frac{x}{2}}{\frac{1}{2}} - x + c = -2\cot \frac{x}{2} - x + c$

(c) $\int \frac{1}{1+\cos x} dx$

$= \int \frac{1}{2\cos^2 \frac{x}{2}} dx = \frac{1}{2}\int \frac{1}{\cos^2 \frac{x}{2}} dx$

$= \frac{1}{2}\int \sec^2 \frac{x}{2} dx = \frac{1}{2} \frac{\tan \frac{x}{2}}{\frac{1}{2}} dx$

$= \tan \frac{x}{2} + c$

Example 7:

Integrate the following functions w.r.t. x

(a) $\int \frac{1}{\sqrt{x}+\sqrt{1+x}} dx$ **(b)** $\frac{1}{\sqrt{3x+5}-\sqrt{3x+2}}$ **(c)** $\frac{3}{\sqrt{2x}+\sqrt{2x+3}}$

(d) $\frac{x}{\sqrt{x+9}}$ **(e)** $\frac{3x+2}{2x-5}$

Solution: (a) $\int \frac{1}{\sqrt{x}+\sqrt{1+x}} dx$

$= \int \frac{1}{\sqrt{x}+\sqrt{1+x}} \times \frac{\sqrt{x}-\sqrt{1+x}}{\sqrt{x}-\sqrt{1+x}} dx = \int \frac{\sqrt{x}-\sqrt{1+x}}{\left(\sqrt{x}\right)^2 - \left(\sqrt{1+x}\right)^2} dx$

$= \int \frac{\sqrt{x}-\sqrt{1+x}}{x-1-x} dx$

$= \int \frac{\sqrt{x}-\sqrt{1+x}}{-1} dx$

$= \int \left(\sqrt{1+x}-\sqrt{x}\right) dx$

$= \frac{2}{3}\left[(1+x)^{\frac{3}{2}} - x^{\frac{3}{2}}\right] + c$

(b) $\int \frac{1}{\sqrt{3x+5}-\sqrt{3x+2}}dx$

$= \int \frac{1}{\sqrt{3x+5}-\sqrt{3x+2}} \times \frac{\sqrt{3x+5}+\sqrt{3x+2}}{\sqrt{3x+5}+\sqrt{3x+2}}dx$

$= \int \frac{\sqrt{3x+5}+\sqrt{3x+2}}{3x+5-3x-2}dx$

$= \frac{1}{3}\int\left(\sqrt{3x+5}+\sqrt{3x+2}\right)dx$

$= \frac{1}{3}\left[\frac{2}{3}\frac{(3x+5)^{\frac{3}{2}}}{3}+\frac{2}{3}\frac{(3x+2)^{\frac{3}{2}}}{3}\right]+c$

$= \frac{2}{27}(3x+5)^{\frac{3}{2}}+\frac{2}{27}(3x+2)^{\frac{3}{2}}+c$

(c) $\int \frac{3}{\sqrt{2x}+\sqrt{2x+3}}dx = \int \frac{3}{\sqrt{2x}+\sqrt{2x+3}} \times \frac{\sqrt{2x}-\sqrt{2x+3}}{\sqrt{2x}-\sqrt{2x+3}}dx$

$= \int \frac{3\left(\sqrt{2x}-\sqrt{2x+3}\right)}{2x-2x-3}dx$

$= \int \frac{3\left(\sqrt{2x}-\sqrt{2x+3}\right)}{-3}dx$

$= \int \left(\sqrt{2x+3}-\sqrt{2x}\right)dx$

$= \frac{2}{3}\frac{(2x+3)^{\frac{3}{2}}}{2}-\frac{2}{3}\frac{(2x)^{\frac{3}{2}}}{2}+c$

$= \frac{1}{3}(2x+3)^{\frac{3}{2}}-\frac{1}{3}(2x)^{\frac{3}{2}}+c$

(d) $\int \frac{x}{\sqrt{x+9}} dx$

$$= \int \frac{(x+9-9)}{\sqrt{x+9}} dx = \int \frac{(x+9)}{\sqrt{x+9}} dx - \int \frac{9}{\sqrt{x+9}} dx$$

$$= \int \sqrt{x+9}\, dx - \int \frac{9}{\sqrt{x+9}}\, dx$$

$$= \frac{2}{3}(x+9)^{\frac{3}{2}} - \frac{9(x+9)^{\frac{-1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{2}{3}(x+9)^{\frac{3}{2}} - 18(x+9)^{\frac{1}{2}} + c$$

(e) $\int \frac{3x+2}{2x-5} dx$

Multiply and divide by 3

$$= 3\int \frac{x+\frac{2}{3}}{2x-5} dx$$

Multiply and divide by 2

$$= \frac{3}{2}\int \frac{2x+\frac{4}{3}}{2x-5} dx \quad = \quad \frac{3}{2}\int \frac{2x-5+5+\frac{4}{3}}{2x-5} dx$$

$$= \frac{3}{2}\int \frac{(2x-5)+\frac{19}{3}}{2x-5} dx$$

$$= \frac{3}{2}\left[\int \frac{2x-5}{2x-5} dx + \frac{19}{3}\int \frac{1}{2x-5} dx\right]$$

$$= \frac{3}{2}\left[\int 1 dx + \frac{19}{3}\frac{\log(2x-5)}{2}\right] + c$$

$$= \frac{3}{2}\left[x + \frac{19}{6}\log(2x-5)\right] + c$$

EXERCISE 20.2

Integrate the following functions w.r.t. x

I Three and Five mark questions:

(a) $(7x-3)^4$ (b) $(2x+5)^{\frac{3}{2}}$ (c) $\frac{1}{10x+3}$ (d) e^{3-4x} (e) 3^{5x-3}

(f) $\sec^2(x-5)$ (g) $\operatorname{cosec}(3-5x)\ \cot(3-5x)$

II Two marks questions:

(a) $\cos^2 x$ (b) $\cot^2(5x+3)$ (c) $\cos^3 x$ (d) $(3x+4)^3+5^{7-3x}$

III Three marks questions:

(a) $\sin 2x \cos 3x$ (b) $\cos 9x \sin 4x$ (c) $\cos 7x \cos 6x$ (d) $\sin 11x \sin 7x$

(e) $\int \frac{x}{\sqrt{x-5}}$ (f) $\frac{2x}{2x+3}$ (g) $\frac{3x}{5x-1}$ (h) $\frac{2x+5}{3x+4}$

IV Four marks questions:

(a) $\sqrt{1+\sin x}$ (b) $\int \frac{1}{1-\cos x}dx$ (c) $\frac{1}{\sqrt{x}-\sqrt{2+x}}$

(d) $\frac{4}{\sqrt{x+1}+\sqrt{x+2}}$ (e) $\frac{5}{\sqrt{3x+1}-\sqrt{3x+4}}$

ANSWERS 20.2

I (a) $\frac{(7x-3)^5}{35}+c$ (b) $\frac{1}{5}(2x+5)^{\frac{5}{2}}+c$ (c) $\frac{\log\ (10x+3)}{10}+c$

(d) $\frac{-e^{3-4x}}{4}+c$ (e) $\frac{3^{5x-3}}{5\ \log\ 3}+c$ (f) $\tan(x-5)+c$

(g) $\frac{\operatorname{cosec}\ (3-5x)}{5}+c$

II (a) $\frac{1}{2}x+\frac{1}{4}\sin 2x+c$ (b) $\frac{-\cot(5x+3)}{5}-x+c$ (c) $\frac{\sin 3x}{12}+\frac{3\sin x}{4}+c$

(d) $\frac{(3x+4)^4}{12}-\frac{5^{7-3x}}{3\log 5}+c$

III (a) $\frac{-1}{10}\cos 5x+\frac{1}{2}\cos x+c$ (b) $\frac{-1}{26}\cos 13x+\frac{1}{10}\cos 5x+c$

(c) $\frac{-1}{26}\sin 3x-\frac{1}{2}\sin x+c$ (d) $\frac{-\sin 4x}{8}+\frac{\sin 18x}{36}+c$

(e) $\frac{2}{3}(x-5)^{\frac{3}{2}}+10(x-5)^{\frac{1}{2}}+c$ (f) $x-\frac{3}{2}\log(2x+3)+c$

(g) $\frac{3}{5}x+\frac{3}{25}\log(5x-1)+c$ (h) $\frac{2}{3}x+\frac{14}{18}\log(3x+4)+c$

IV (a) $2\sin\frac{x}{2}-2\cos\frac{x}{2}+c$ (b) $-\cot\frac{x}{2}+c$

(c) $\frac{-1}{3}x^{\frac{3}{2}}-\frac{1}{3}(2+x)^{\frac{3}{2}}+c$ (d) $\frac{-8}{3}(x+1)^{\frac{3}{2}}+\frac{8}{3}(x+2)^{\frac{3}{2}}+c$

(e) $\frac{-10}{27}(3x+1)^{\frac{3}{2}}-\frac{10}{27}(3x+4)^{\frac{3}{2}}+c$

TYPE II $\boxed{\int\frac{f^1(x)}{f(x)}dx=\log[f(x)]+c}$

Worked Examples

Example 8:

Evaluate the following:

(a) $\int\frac{2x}{1+x^2}\,dx$ **(b)** $\int\frac{e^x}{e^x+1}\,dx$ **(c)** $\int\frac{1}{x(3+\log x)}\,dx$

(d) $\int\frac{2x+5}{x^2+5x+3}\,dx$ **(e)** $\int\frac{7x^6+7^x\log 7}{x^7+7^x}\,dx$

Solution: (a) Let $I=\int\frac{2x}{1+x^2}dx$

put $1+x^2=t$

$= 2x\,dx = dt \quad \therefore I=\int\frac{dt}{t}$

$= \log t + c \quad = \quad \log(1+x^2)+c$

(b) Let $I = \int \frac{e^x}{e^x + 1} dx$

put $e^x + 1 = t \quad \Rightarrow e^x dx = dt$

$\therefore I = \int \frac{dt}{t}$

$= \log t + c = \log (e^x + 1) + c$

(c) Let $I = \int \frac{1}{x(3 + \log x)} dx$

put $3 + \log x = t \quad \Rightarrow \frac{1}{x} dx = dt$

$\therefore I = \int \frac{1}{t} dt$

$= \log t + c = \log (3 + \log x) + c$

(d) Let $I = \int \frac{2x + 5}{x^2 + 5x + 3} dx$

put $x^2 + 5x + 3 = t$

$\Rightarrow (2x + 5) dx = dt$

$\therefore I = \int \frac{dt}{t}$

$= \log t + c = \log (x^2 + 5x + 3) + c$

(e) Let $I = \int \frac{7x^6 + 7^x \log 7}{x^7 + 7^x} dx$

put $x^7 + 7^x = t$

$\Rightarrow 7x^6 + 7^x \log 7 = \frac{dt}{dx}$

$\therefore I = \int \frac{dt}{t}$

$= \log t + c$

$= \log (x^7 + 7^x) + c$

Example 9:

Prove that:

(a) $\int \tan x \, dx = \log(\sec x) + c$ **(b)** $\int \cot x \, dx = \log(\sin x) + c$

(c) $\int \sec x \, dx = \log(\sec x + \tan x) + c$ **(d)** $\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x) + c$

Solution :

(a) Let $I = \int \tan x \, dx \Rightarrow I = \int \frac{\sin x}{\cos x} dx$

put $\cos x = t$

$\Rightarrow -\sin x \, dx = dt$

$\sin x \, dx = -dt$

$\therefore I = -\int \frac{dt}{t}$

$= -\log t + c = \log \frac{1}{t} + c$

$= \log \frac{1}{\cos x} + c = \log (\sec x) + c$

(b) Let $I = \int \cot x \, dx$

$= \int \frac{\cos x}{\sin x} dx$

put $\sin x = t \quad \therefore \cos x \, dx = dt$

$\therefore I = \int \frac{dt}{t}$

$= \log t + c = \log (\sin x) + c$

(c) Let $I = \int \sec x \, dx$

multiplying and dividing by $\sec x + \tan x$, we get

$I = \int \frac{\sec x}{(\sec x + \tan x)} (\sec x + \tan x) \, dx$

$= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x} dx$

put $\sec x + \tan x = t$

$\Rightarrow (\sec x \tan x + \sec^2 x)dx = dt \quad \therefore I = \int \frac{dt}{t}$

$= \log t + c$

$= \log (\sec x + \tan x) + c$

(d) Let $I = \int \text{cosec}\, x\, dx$

Multiplying and dividing by $(\text{cosec}\, x - \cot x)$, we get

$$I = \int \frac{\text{cosec}\, x}{(\text{cosec}\, x - \cot x)} (\text{cosec}\, x - \cot x)dx$$

$$= \int \frac{(\text{cosec}^2 x - \text{cosec}\, x \cot x)}{\text{cosec}\, x - \cot x} dx$$

put $\text{cosec}\, x - \cot x = t$

$(-\text{cosec}\, x \cot x + \text{cosec}^2 x)dx = dt$

$\therefore I = \int \frac{dt}{t}$

$= \log t + c \quad = \log (\text{cosec} x - \cot x) + c$

Example 10 :

Evaluate the following:

(a) $\int \frac{(3x+2)}{3x^2+4x-5} dx$ **(b)** $\int \frac{e^{2x}-1}{e^{2x}+1} dx$ **(c)** $\int \frac{1}{x-\sqrt{x}} dx$ **(d)** $\int \frac{3^x \log 3}{3^x+1} dx$

Solution :

(a) Let $I = \int \frac{(3x+2)}{3x^2+4x-5} dx$

put $3x^2 + 4x - 5 = t$

$(6x + 4)\, dx = dt$

$(3x + 2)\, dx = \frac{dt}{2} \quad \therefore I = \frac{1}{2} \int \frac{dt}{t}$

$= \frac{1}{2} \log t + c \quad = \frac{1}{2} \log (3x^2 + 4x - 5) + c$

(b) Let $I = \int \frac{e^{2x}-1}{e^{2x}+1}\,dx$

$$= \int \frac{e^x\left(e^x - \frac{1}{e^x}\right)}{e^x\left(e^x + \frac{1}{e^x}\right)}\,dx \quad = \quad \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

put $e^x + e^{-x} = t \quad \Rightarrow \quad (e^x - e^{-x})dx = dt$

$\therefore\ I = \int \frac{dt}{t}$

$= \log t + c \quad = \log\left((e^x + e^{-x}) + c\right.$

(c) Let $I = \int \frac{1}{x - \sqrt{x}}dx$

$$= \int \frac{1}{\sqrt{x}(\sqrt{x}-1)}dx$$

put $\sqrt{x} - 1 = t \quad \Rightarrow \quad \frac{1}{2\sqrt{x}}dx = dt$

$$\frac{1}{\sqrt{x}}dx = 2dt$$

$I = 2\int \frac{dt}{t}$

$= 2 \log t + c \quad = \quad 2 \log\left(\sqrt{x} - 1\right) + c$

(d) Let $I = \int \frac{3^x \log 3}{3^x + 1}dx$

Put $3^x + 1 = t$

$(3^x \log 3)\, dx = dt$

$\therefore\ I = \int \frac{dt}{t}$

$= \log t + c$

$= \log (3^x + 1) + c$

Example 11 :

Evaluate the following:

(a) $\int \frac{\sin x}{1+\cos x} dx$

(b) $\int \frac{\sin 2x}{1+\sin^2 x} dx$

(c) $\int \frac{\tan x}{2+\log(\sec x)} dx$

(d) $\int \frac{1+\tan x}{1-\tan x} dx$

(e) $\frac{\sec^2 x \tan x}{3+\sec^2 x} dx$

(f) $\int \frac{\sec^2 x + \text{cosec}^2 x}{\tan x - \cot x} dx$

Solution :

(a) Let $I = \int \frac{\sin x}{1+\cos x} dx$

put $1 + \cos x = t \quad \Rightarrow \quad -\sin x \, dx = dt \quad \Rightarrow \quad \sin x \, dx = -dt$

$\therefore I = -\int \frac{dt}{t}$

$= -\log t + c = -\log (1 + \cos x) + c$

(b) Let $I = \int \frac{\sin 2x}{1+\sin^2 x} dx$

$= \int \frac{2 \sin x \cos x}{1+\sin^2 x} dx$

Put $1 + \sin^2 x = t \quad \Rightarrow \quad 2 \sin x \cos x \, dx = dt$

$I = \int \frac{dt}{t}$

$= \log t + c = \log (1 + \sin^2 x) + c$

(c) Let $I = \int \frac{\tan x}{2+\log(\sec x)} dx$

Put $2 + \log (\sec x) = t \quad \Rightarrow \quad \frac{1}{\sec x} \sec x \tan x \, dx = dt$

$\tan x \, dx = dt$

$\therefore \quad I = \int \frac{dt}{t}$

$= \log t + c = \log [2 + \log (\sec x)] + c$

(d) Let $I = \int \frac{1+\tan x}{1-\tan x} dx$

$$= \int \frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}} dx = \int \frac{\cos x + \sin x}{\cos x - \sin x} dx$$

Put $\cos x - \sin x = t$

$(-\sin x - \cos x)\, dx = dt$

$(\sin x + \cos x)\, dx = -dt$

$$= -\int \frac{dt}{t} = -\log t + c$$

$$= -\log (\cos x - \sin x) + c$$

(e) Let $I = \int \frac{\sec^2 x \tan x}{3+\sec^2 x} dx$

Put $3 + \sec^2 x = t$

$\Rightarrow 2\sec x \sec x \tan x\, dx = dt$

$$\sec^2 x \tan x\, dx = \frac{dt}{2}$$

$$I = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log t + c$$

$$= \frac{1}{2}\log (3 + \sec^2 x) + c$$

(f) Let $I = \int \frac{\sec^2 x + \cos ce^2 x}{\tan x - \cot x} dx$

Put $\tan x - \cot x = t$

$\Rightarrow \sec^2 x + \text{cosec}^2 x\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{t}$$

$$= \log t + c = \log (\tan x - \cos x) + c$$

EXERCISE 20.3

Evaluate the following:

I. Two marks questions:

(a) $\int \frac{3x^2}{1+x^3}\,dx$ (b) $\int \frac{4x+3}{2x^2+3x+5}\,dx$ (c) $\int \frac{e^x-1}{e^x-x}\,dx$

(d) $\int \frac{9x^8+9^x\log 9}{x^9+9^x}\,dx$ (e) $\int \frac{\cos x}{2+\sin x}\,dx$ (f) $\int \frac{1}{x(2\log x+5)}\,dx$

(g) $\int \frac{3\sin x}{3+4\cos x}\,dx$

II. Five marks questions:

(a) $\int \frac{1}{\sqrt{x}+x}\,dx$ (b) $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$ (c) $\int \frac{e^{2x}+1}{e^{2x}-1}\,dx$

(d) $\int \frac{1+\cot x}{1-\cot x}\,dx$ (e) $\int \frac{\cot x}{3+\log(\sin x)}\,dx$ (f) $\int \frac{\operatorname{cosec}^2 x\cot x}{4+5\operatorname{cosec}^2 x}\,dx$

ANSWERS 20.3

I. (a) $\log(1+x^3)+c$ (b) $\log(2x^2+3x+5)+c$

(c) $\log(e^x-x)+c$ (d) $\log(x^9+9^x+c$

(e) $\log(2+\sin x)+c$ (f) $\frac{1}{2}\log(2\log x+5)+c$

(g) $\frac{-3}{4}\log(3+4\cos x)+c$

II. (a) $2\log\left(1+\sqrt{x}\right)+c$ (b) $-\log(1+\cos^2 x)+c$

(c) $\log(e^x-e^{-x})+c$ (d) $\log(\sin x-\cos x)+c$

(e) $\log(3+\log\sin x)+c$ (f) $\frac{-1}{10}\log(4+5\operatorname{cosec}^2 x)+c$

TYPE III : $\int [f(x)]^n f^1(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}+c$

WORKED EXAMPLES

Example 12 :

Evaluate the following:

(a) $\int (2x+3)\left(x^2+3x+5\right)^{\frac{3}{2}} dx$

(b) $\int \frac{1}{x}(\log x)^3\, dx$

(c) $\int \frac{\left(\sqrt{x}+1\right)^3}{\sqrt{x}} dx$

(d) $\int e^x \left(e^x+2\right)^{\frac{5}{2}} dx$

(e) $\int \frac{5}{x(3+2\log x)^5} dx$

(f) $\int x\sqrt{3x^2+5}\, dx$

Solution :

(a) Let $I = \int (2x+3)(x^2+3x+5)^{\frac{3}{2}} dx$

put $x^2+3x+5=t \Rightarrow (2x+3)dx = dt$

$$I = \int t^{\frac{3}{2}}\, dt = \frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$$

$$= \frac{2}{5}\, t^{\frac{5}{2}} + c$$

$$= \frac{2}{5}\, (x^2+3x+5)^{\frac{5}{2}} + c$$

(b) Let $I = \int \frac{1}{x}(\log x)^3 dx$

put $\log x = t$

$$\Rightarrow \frac{1}{x} dx = dt$$

$$\therefore\ I = \int t^3 dt$$

$$= \frac{t^4}{4} + c = \frac{(\log x)^4}{4} + c$$

(c) Let $I = \int \frac{(\sqrt{x}+1)^3}{\sqrt{x}}\, dx$

put $\sqrt{x}+1 = t \quad \Rightarrow \quad \frac{1}{2\sqrt{x}}dx = dt \quad \Rightarrow \quad \frac{1}{\sqrt{x}}dx = 2dt$

$\therefore \quad I = 2\int t^3 dt$

$= \frac{2t^4}{4} + c \quad = \frac{(\sqrt{x}+1)^4}{2} + c$

(d) Let $I = \int e^x (e^x + 2)^{\frac{5}{2}} dx$

put $e^x + 2 = t \quad \Rightarrow \quad e^x dx = dt$

$\therefore \quad I = \int t^{\frac{5}{2}} dt$

$= \quad \frac{t^{\frac{5}{2}+1}}{\frac{5}{2}+1} + c \quad = \quad \frac{2}{7}t^{\frac{7}{2}} + c = \frac{2}{7}(e^x+2)^{\frac{7}{2}} + c$

(e) Let $I = \int \frac{5}{x(3+2\log x)^5}\, dx$

put $3 + 2\log x = t$

$\frac{2}{x}dx = dt \quad \Rightarrow \quad \frac{1}{x}dx = \frac{dt}{2}$

$\therefore \quad I = \frac{5}{2}\int \frac{dt}{t^5}$

$= \frac{5}{2}\int t^{-5} dt \quad = \quad \frac{5}{2} \times \frac{t^{-5+1}}{-5+1} + c$

$= \frac{5}{2} \times \frac{t^{-4}}{-4} + c \quad = \quad \frac{-5}{8}(3+2\log x)^{-4} + c$

$= \frac{-5}{8\ (3+2\log x)^4} + c$

(f) Let $I = \int x\sqrt{3x^2+5}\ dx$

put $3x^2+5=t \Rightarrow 6x\ dx = dt$

$$x\ dx = \frac{dt}{6}$$

$$\therefore\ I = \frac{1}{6}\int \sqrt{t}\ dt$$

$$= \frac{1}{6}\times\frac{2}{3}t^{\frac{3}{2}}+c = \frac{1}{9}t^{\frac{3}{2}}+c = \frac{1}{9}(3x^2+5)^{\frac{3}{2}}+c$$

Example 13 :

Integrate the following w.r.t. 'x'

(a) $\sin^3 x\cos x$ **(b)** $\sec^3 x\tan x$ **(c)** $\dfrac{\cos x}{(1+\sin x)^3}$

(d) $\sec^2 x\sqrt{1+\tan x}$ **(e)** $\sin 2x\sqrt{1+\sin^2 x}$

(f) $\dfrac{\sec^2 x+\sec x\cdot\tan x}{(\sec x+\tan x)^{\frac{3}{2}}}$ **(g)** $\dfrac{1-\sin 2x}{(x+\cos^2 x)^2}$

Solution :

(a) Let $I = \int \sin^3 x\ \cos x\ dx$

put $\sin x = t \Rightarrow \cos x\ dx = dt$

$$\therefore\ I = \int t^3 dt$$

$$= \frac{t^4}{4}+c = \frac{\sin^4 x}{4}+c$$

(b) Let $I = \int \sec^3 x\tan x\ dx$

put $\sec x = t$

$\sec x\tan x\ dx = dt$

$$\therefore\ I = \int t^2 dt$$

$$= \frac{t^3}{3}+c = \frac{\sec^3 x}{3}+c$$

(c) Let $I = \int \frac{\cos x}{(1+\sin x)^3}\, dx$

put $1 + \sin x = t$

$\cos x\, dx = dt$

$\therefore\ I = \int \frac{dt}{t^3}$

$= \frac{t^{-3+1}}{-3+1} + c = \frac{-1}{2t^2} + c$

$= \frac{-1}{2(1+\sin x)^2} + c$

(d) Let $I = \int \sec^2 x\sqrt{1+\tan x}\, dx$

put $1 + \tan x = t$

$\sec^2 x\, dx = dt$

$\therefore\ I = \int \sqrt{t}\, dt$

$= \frac{2}{3}t^{\frac{3}{2}} + c = \frac{2}{3}(1+\tan x)^{\frac{3}{2}} + c$

(e) Let $I = \int \sin 2x\sqrt{1+\sin^2 x}\, dx$

put $1 + \sin^2 x = t$

$2 \sin x \cos x\, dx = dt$

$\sin 2x\, dx = dt$

$\therefore\ I = \int \sqrt{t}\, dt$

$= \frac{2}{3}t^{\frac{3}{2}} + c = \frac{2}{3}(1+\sin^2 x)^{\frac{3}{2}} + c$

(f) Let $I = \int \frac{\sec^2 x + \sec x + \tan x}{(\sec x + \tan x)^{\frac{3}{2}}}\, dx$

put $\sec x + \tan x = t \Rightarrow (\sec x \tan x + \sec^2 x)\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{t^{\frac{3}{2}}}$$

$$= \frac{t^{\frac{-3}{2}+1}}{\frac{-3}{2}+1} + c \quad = \quad -2t^{\frac{-1}{2}} + c \quad = \quad \frac{-2}{\sqrt{t}} + c$$

$$= \frac{-2}{\sqrt{\sec x + \tan x}} + c$$

(g) Let $I = \int \frac{1 - \sin 2x}{(x + \cos^2 x)^2}\, dx$

put $x + \cos^2 x = t$

$1 - 2\cos x \sin x\, dx = dt$

$(1 - \sin 2x)\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{t^2}$$

$$= \frac{t^{-2+1}}{-2+1} + c \quad = \quad -\frac{1}{t} + c$$

$$= \frac{-1}{x + \cos^2 x} + c$$

EXERCISE 20.4

Integrate the following *w.r.t. x:*

I. Two marks questions:

(a) $\int 2x(x^2+2)^{\frac{2}{3}}\, dx$

(b) $\int \frac{2}{x}(\log x)^2\, dx$

(c) $\int e^x \sqrt{e^x + 1}\, dx$

(d) $\int \cos^2 x \sin x\, dx$

(e) $\int \text{cosec}^4 x \cot x\, dx$

II. Three marks questions:

(a) $\int 6x + 5(3x^2 + 5x - 4)^{\frac{5}{3}}\, dx$

(b) $\int \frac{3}{x(2 + 3\log x)^{\frac{2}{3}}}\, dx$

(c) $\int \frac{1+e^x}{(x+e^x)^5}\,dx$

(d) $\int \frac{1}{2\sqrt{x}\left(\sqrt{x}-1\right)^{\frac{2}{5}}}\,dx$

(e) $\int \frac{5^x \log 5}{(5^x+3)^7}\,dx$

(f) $\int \operatorname{cosec}^2 x\sqrt{1+\cot x}\,dx$

(g) $\int \frac{\sin 2x}{(1-\cos^2 x)^3}\,dx$

(h) $\int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{(\operatorname{cosec} x - \cot x)^{\frac{1}{4}}}\,dx$

ANSWERS 20.4

I. (a) $\frac{3}{5}(x^2+2)^{\frac{5}{3}}+c$ (b) $\frac{2}{3}(\log x)^3+c$ (c) $\frac{2}{3}(e^x+1)^{\frac{3}{2}}+c$

(d) $\frac{-\cos^3 x}{3}+c$ (e) $\frac{-\operatorname{cosec}^4 x}{4}+c$

II. (a) $\frac{3}{8}(3x^2+5x-4)^{\frac{8}{3}}+c$ (b) $3(2+3\log x)^{\frac{1}{3}}+c$

(c) $\frac{-1}{4(x+e^x)^4}+c$ (d) $\frac{5}{3}\left(\sqrt{x}-1\right)^{\frac{3}{5}}+c$

(e) $\frac{-1}{6(5^x+3)^6}+c$ (f) $\frac{-2}{3}(1+\cot x)^{\frac{3}{2}}+c$

(g) $\frac{-1}{2(1-\cos^2 x)^2}+$ c (h) $\frac{4}{3}(\operatorname{cosec} x - \cot x)^{\frac{3}{4}}+c$

20.7 Integration by Resolving into Partial Fractions

In this topic students will learn to evaluate the integrals of proper rational functions and improper rational functions by resolving into partial fractions.

WORKED EXAMPLES

Example 14 : Evaluate

(a) $\int \frac{3}{(x+1)(x+2)}\,dx$

(b) $\int \frac{x+2}{(2x-1)(x-3)}\,dx$

(c) $\int \frac{2x+3}{(x+1)^2(x-3)}dx$ **(d)** $\int \frac{2x}{(x^2+4x+4)(x-1)}dx$

(e) $\int \frac{x-1}{(x^2-9)x}dx$

Solution: (a) Let $I = \int \frac{3}{(x+1)(x+2)}dx$

Consider, $\frac{3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$ ..... (1)

$3 = A(x+2) + B(x+1)$ ..... (2)

put $x = -2$ in (2), we get

$3 = A(-2+2) + B(-2+1)$

$3 = -B$

$\therefore B = -3$

put $x = -1$ (2) we get

$3 = A(-1+2) + B(-1+1)$

$3 = A$

$\therefore A = 3$

Now, (1) $\Rightarrow \frac{3}{(x+1)(x+2)} = \frac{3}{x+1} - \frac{3}{x+2}$

$\therefore I = \int\left(\frac{3}{x+1} - \frac{3}{x+2}\right)dx \qquad \left(\therefore \int \frac{1}{ax+b}dx = \frac{\log(ax+b)}{a} + c\right)$

$= 3\int \frac{1}{x+1}dx \quad -3\int \frac{1}{x+1}dx$

$= 3\log(x+1) - 3\log(x+2) + c$

(b) Let $I = \int \frac{x+2}{(2x-1)(x-3)}dx$

Consider, $\frac{x+2}{(2x-1)(x-3)} = \frac{A}{2x-1} + \frac{B}{x-3}$ .....(1)

$x+2 = A(x-3) + B(2x-1)$ .....(2)

Substituting $x = 3$ in (2), we get

$$3 + 2 = A(3 - 3) + B(6-1)$$

$$5 = 5B$$

$$B = 1$$

Substituting $x = \frac{1}{2}$ in (2), we get

$$\frac{1}{2} + 2 = A\left(\frac{1}{2} - 3\right) + B\left(2 \times \frac{1}{2} - 1\right)$$

$$\frac{5}{2} = A\left(\frac{-5}{2}\right) \qquad \therefore \ A = -1$$

Now, (1) $\Rightarrow \dfrac{x+2}{(2x-1)(x-3)} = \dfrac{-1}{2x-1} + \dfrac{1}{x-3}$

$$\therefore \ I = \int \left(\frac{-1}{2x-1} + \frac{1}{x-3}\right) dx$$

$$= -\int \frac{1}{2x-1} dx + \int \frac{1}{x-3} dx$$

$$= -\frac{\log(2x-1)}{2} + \log(x-3) + c$$

(c) Let $I = \int \dfrac{2x+3}{(x+1)^2(x-3)} dx$

Consider

$$\frac{2x+3}{(x+1)^2(x-3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3} \qquad \text{..... (1)}$$

$$2x+3 = A(x+1)(x-3) + B(x-3) + (x+1)^2 \qquad \text{..... (2)}$$

put $x = -1$ in (2), we get

$2(-1) + 3 = A(-1 + 1)(-1 - 3) + B(-1 - 3) + C(-1 + 1)^2$

$1 = -4B$

$B = \dfrac{-1}{4}$

put $x = 3$ in (2), we get

$2(3) + 3 = A(3 + 1)(3 - 3) + B(3 + 3) + C(3 + 1)^2$

$6 + 3 = C(4)^2$

$9 = 16C$

$$C = \frac{9}{16}$$

put $x = 0$ in (2), we get

$2(0) + 3 = A(0 + 1)(0 - 3) + B(0 - 3) + C(0 + 1)^2$

$3 = -3A - 3B + C$

$$\Rightarrow \quad 3 = -3A - 3\times\frac{-1}{4} + \frac{9}{16}$$

$$3A = -3 + \frac{3}{4} + \frac{9}{16}$$

$$A = -1 + \frac{1}{4} + \frac{3}{16} = \frac{-16 + 4 + 3}{16}$$

$$\therefore \quad A = \frac{-9}{16}$$

$$\text{Now, (1)} \Rightarrow \frac{2x+3}{(x+1)^2(x-3)} = \frac{-9}{16(x+1)} - \frac{1}{4(x+1)^2} + \frac{9}{16(x-3)}$$

$$= \frac{-9}{16}\int\frac{1}{x+1}dx - \frac{1}{4}\int\frac{1}{(x+1)^2}dx + \frac{9}{16}\int\frac{1}{x-3}dx$$

$$= \frac{-9}{16}\log(x+1) - \frac{1}{4(x+1)} + \frac{9}{16}\log(x-3) + c$$

(d) Let $I = \int\frac{2x}{(x^2+4x+4)(x-1)}dx$

Consider

$$\frac{2x}{(x^2+4x+4)(x-1)} = \frac{2x}{(x+2)^2(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1} \quad \text{.....(1)}$$

$2x = A(x + 2)(x - 1) + B(x - 1) + C(x + 2)^2$ .....(2)

put $x = 1$ in (2), we get

$2(1) = A(1 + 2)(1 - 1) + B(1 - 1) + C(1 + 2)^2$

$2 = C(3)^2$

$\therefore \quad C = \frac{2}{9}$

put $x = -2$ in (2), we get

$2(-2) = A(-2 + 2)(-2 - 1) + B(-2 - 1) + C(-2 + 2)^2$

$-4 = -3B$

$\therefore \quad B = \frac{4}{3}$

put $x = 0$ in (2), we get

$2(0) = A(0 + 2)(0 - 1) + B(0 - 1) + C(0 + 2)^2$

$0 = -2A - B + 4C$

$$0 = -2A - \frac{4}{3} + 4\left(\frac{2}{9}\right)$$

$$2A = \frac{-4}{3} + \frac{8}{9}$$

$$2A = \frac{-12 + 8}{9}$$

$$2A = \frac{-4}{9}$$

$$A = \frac{-2}{9}$$

(1) $\Rightarrow$

$$\frac{2x}{(x+2)^2(x-1)} = \frac{-2}{9(x+2)} + \frac{4}{3(x+2)^2} + \frac{2}{9(x-1)}$$

$$\therefore \quad I = \int \left[\frac{-2}{9(x+2)} + \frac{4}{3(x+2)^2} + \frac{2}{9(x-1)} \right] dx$$

$$= \frac{-2}{9}\int \frac{1}{x+2}dx + \frac{4}{3}\int \frac{1}{(x+2)^2} + \frac{2}{9}\int \frac{1}{x-1}dx$$

$$= \frac{-2}{9}\log(x+2) - \frac{4}{3(x+2)} + \frac{2}{9}\log(x-1) + c$$

(e) Let $I = \int \frac{x-1}{(x^2-9)x}dx$

Consider,

$$\frac{x-1}{(x^2-9)(x)} = \frac{x-1}{(x+3)(x-3)(x)}$$

Now, consider

$$\frac{x-1}{(x+3)(x-3)x} = \frac{A}{x+3} + \frac{B}{x-3} + \frac{C}{x} \qquad \text{.....(1)}$$

$x - 1 = A(x - 3)\, x + B\,(x + 3)\, x + C\,(x + 3)\,(x - 3)$ .... (2)

put $x = 0$ in (2), we get

$-1 = C\,(3)\,(-3)$

$-1 = -9C$

$\therefore C = \frac{1}{9}$

put $x = 3$ in (2), we get

$3 - 1 = A\,(3 - 3)\,3 + B\,(3 + 3)\,3 + C\,(3 + 3)\,(3 - 3)$

$2 = B(6)\,3$

$2 = 18B$

$\therefore B = \frac{2}{18} = \frac{1}{9}$

put $x = -3$ in (2), we get

$-3 - 1 = A\,(-3 - 3)\,(-3) + B\,(-3 + 3)\,(-3) + C(-3 + 3)\,(-3 - 3)$

$-4 = A(-6)\,(-3)$

$-4 = +18A$

$\therefore A = \frac{-4}{18} = \frac{-2}{9}$

Now,

(1) $\Rightarrow$

$$\frac{x-1}{(x+3)(x-3)x} = \frac{-2}{9(x+3)} + \frac{1}{9(x-3)} + \frac{1}{9x}$$

$$\therefore\ I = \int \left(\frac{-2}{9(x+3)} + \frac{1}{9(x-3)} + \frac{1}{9x} \right) dx$$

$$= \frac{-2}{9}\int \frac{1}{x+3}dx + \frac{1}{9}\int \frac{1}{x-3}dx + \frac{1}{9}\int \frac{1}{x}dx$$

$$= \frac{-2}{9}\log(x+3) + \frac{1}{9}\log(x-3) + \frac{1}{9}\log x + c$$

Example 15 :

(a) $\int \frac{1}{x\left[(\log x)^2 - 3\log x + 2\right]}dx$ **(b)** $\int \frac{e^x}{4e^{2x} - 4e^x - 3}dx$

(c) $\int \frac{2x^2 + 5}{2x^2 + 5x + 3}dx$ **(d)** $\int \frac{6x^2 + 2x + 3}{(x+1)(3x+2)}dx$

Solution:

(a) Let $I = \int \frac{1}{x\left[(\log x)^2 - 3\log x + 1\right]}dx$

put $\log x = t \quad \Rightarrow \frac{1}{x}dx = dt$

$\therefore I = \int \frac{dt}{t^2 - 3t + 2}$

Now consider, $\frac{1}{t^2 - 3t + 2} = \frac{1}{(t-1)(t-2)} = \frac{A}{t-1} + \frac{B}{t-2}$

$1 = A(t-2) + B(t-1)$

put $t = 1$, (2) we get, $1 = A(1-2)$

$A = -1$

put $t = 2$ in (2), we get, $1 = B\ (2-1)$

$B = 1$

$$\therefore (1) \Rightarrow \frac{1}{(t-1)(t-2)} = \frac{-1}{t-1} + \frac{1}{t-2}$$

Now, $I = \int \left(-\frac{1}{t-1} + \frac{1}{t-2} \right) dt$

$$= -\log(t-1) + \log(t-2) + c$$

$$= -\log(\log x - 1) + \log(\log x - 2) + c$$

$$= \log\left[\frac{\log x - 2}{\log x - 1} \right] + c$$

(b) Let $I = \int \frac{e^x}{4e^{2x} - 4e^x - 3}\, dx$

Put $e^x = t \Rightarrow e^x dx = dt$

$$\therefore I = \int \frac{dt}{4t^2 - 4t - 3}$$

Consider

$$\frac{1}{4t^2 - 4t - 3} = \frac{1}{(2t+1)(2t+3)} = \frac{A}{2t+1} + \frac{B}{2t-3} \quad .....(1)$$

$1 = A(2t - 3) + B(2t + 1)$

Put $t = \frac{3}{2}$, we get $1 = A\left(2 \cdot \frac{3}{2} - 3 \right) + B\left(2 \cdot \frac{3}{2} + 1 \right)$

$$1 = 4B \qquad \therefore B = \frac{1}{4}$$

put $t = -\frac{1}{2}$, we get

$$1 = A\left[2 \cdot \left(\frac{-1}{2} \right) - 3 \right] + B\left[2 . \left(\frac{-1}{2} \right) + 1 \right]$$

$$1 = -4A \qquad \therefore A = \frac{-1}{4}$$

$\therefore (1) \Rightarrow \dfrac{1}{(2t+1)(2t-3)} = \dfrac{-1}{4(2t+1)} + \dfrac{1}{4(2t-3)}$

Now, $I = \int \left[\dfrac{-1}{4(2t+1)} + \dfrac{1}{4(2t-3)} \right] dt = -\dfrac{1}{4}\int \dfrac{dt}{2t+1} + \dfrac{1}{4}\int \dfrac{1}{2t-3} dt$

$= -\dfrac{1}{4}\dfrac{\log(2t+1)}{2} + \dfrac{1}{4}\cdot\dfrac{1}{2}\log(2t-3) + c$

$= -\dfrac{1}{8}\log\left[2e^x+1\right] + \dfrac{1}{8}\log\left(2e^x-3\right) + c$

(c) Let $I = \int \dfrac{2x^2+5}{2x^2+5x+3} dx$

Consider $\dfrac{2x^2+5}{2x^2+5x+3}$ This is an improper rational function.

$\therefore$ convert improper into proper by division

1 = quotient

$\therefore 2x^2+5x+3 \;\big|\; 2x^2+5$

$2x^2+5x+3$

(–) (–) (–)

$-5x+2$ = remainder

$\therefore \quad \dfrac{x^2+5}{2x^2+5x+3} = 1 + \dfrac{2-5x}{2x^2+5x+3}$ .....(1)

Now consider

$\dfrac{2-5x}{2x^2+5x+3} = \dfrac{2-5x}{(x+1)(2x+3)} = \dfrac{A}{x+1} + \dfrac{B}{2x+3}$ ..... (2)

$2 - 5x = A(2x + 3) + B(x + 1)$ ..... (3)

put $x = 1$ in (3), we get

$2 + 5 = A(-2 + 3) + B(-1 + 1)$

$7 = A$

$\therefore A = 7$

put $x = \frac{-3}{2}$ in (3), we get

$$2 - 5\left(\frac{-3}{2}\right) = A\left(2 \times \frac{-3}{2} + 3\right) + B\left(\frac{-3}{2} + 1\right)$$

$$2 + \frac{15}{2} = B\left(-\frac{3}{2} + 1\right)$$

$$\frac{19}{\not 2} = B\left(\frac{-1}{\not 2}\right)$$

$\therefore \quad B = -19$

$$\therefore \quad (1) \Rightarrow \frac{x^2 + 5}{2x^2 + 5x + 3} = 1 + \frac{7}{x+1} - \frac{19}{2x+3}$$

Now,

$$I = \int \left(1 + \frac{7}{x+1} - \frac{19}{2x+3}\right) dx$$

$$= \int dx + 7\int \frac{1}{x+1} dx - 19\int \frac{1}{2x+3} + c$$

$$= x + 7 \log (x+1) \ \frac{-19}{2} \ \log (2x+3) + c$$

(d) Let $I = \int \frac{6x^2 + 2x + 3}{(x+1)\ (3x+2)} dx$

Consider $\frac{x^2 + 2x + 3}{(x+1)(3x+2)} = \frac{x^2 + 2x + 3}{3x^2 + 5x + 2}$

This is an improper rational function

$$\begin{array}{r|l} & 2 \\ \hline 3x^2 + 5x + 2 & 6x^2 + 2x + 3 \\ & 6x^2 + 10x + 4 \\ & (-) \quad (-) \quad (-) \\ \hline & -8x - 1 \end{array}$$

$$\therefore \quad \frac{6x^2+2x+3}{3x^2+5x+2} = 2 - \frac{8x+1}{3x^2+5x+2}$$

$$\frac{6x^2+2x+3}{3x^2+5x+2} = 2 - \frac{(8x+1)}{(x+1)(3x+2)} \quad \text{.....(1)}$$

Now consider,

$$\frac{8x+1}{(x+1)(3x+2)} = \frac{A}{x+1} + \frac{B}{3x+2} \quad \text{.....(2)}$$

$$8x + 1 = A(3x + 2) + B\,(x + 1) \quad \text{.....(3)}$$

Substituting $x = -1$ in (3), we get

$$8(-1) + 1 = A[3(-1) + 2] + B(-1 + 1)$$

$$-7 = -A \quad \therefore \; A = 7$$

Substituting $x = \frac{-2}{3}$ in (3), we get

$$8\left(\frac{-2}{3}\right)+1 = A\left[3\left(\frac{-2}{3}\right)+2\right] + B\left[\frac{-2}{3}+1\right]$$

$$\frac{-16+3}{3} = B\left(\frac{-2+3}{3}\right)$$

$$-13 = B$$

$$B = -13$$

$$\therefore \; (I) \Rightarrow \frac{6x^2+2x+3}{3x^2+5x+2} = 2 - \frac{7}{x+1} + \frac{13}{3x+2}$$

$$I = \int\left(2 - \frac{7}{x+1} + \frac{13}{3x+2}\right)dx$$

$$= 2\int dx - 7\int \frac{1}{x+1}\,dx + 13\int \frac{1}{3x+2}\,dx$$

$$= 2x - 7\log(x+1) + \frac{13}{3}\log(3x+2) + c$$

Exercise 20.5

Evaluate the following:

I. Five marks questions:

(a) $\int \frac{4x+5}{(x-1)(x+2)}dx$

(b) $\int \frac{3x+2}{(2x+3)(3x-1)}dx$

(c) $\int \frac{1}{x(x+1)(x+2)}dx$

(d) $\int \frac{5x+7}{(x-2)^2(x+3)}dx$

(e) $\int \frac{5}{(x^2+6x+9)(x-3)}dx$

(f) $\int \frac{2x-1}{(x^2-4)(x+1)}dx$

(g) $\int \frac{3e^x}{e^{2x}+5e^x+6}dx$

(h) $\int \frac{6}{x(2(\log x)^2+7\log x+5)}dx$

(i) $\int \frac{3x^2+2x+3}{(x+1)(3x+2)}dx$

(j) $\int \frac{x^2+3x-2}{x^2-4x-12}dx$

Answers 20.5

I. (a) $3\log(x-1)+\log(x+2)+c$

(b) $\frac{5}{22}\log(2x+3)+\frac{9}{33}\log(3x-1)+c$

(c) $\frac{1}{2}\log x-\log(x+1)+\frac{1}{2}\log(x+2)+c$

(d) $\frac{8}{25}\log(x-2)-\frac{17}{5}(x-2)-\frac{8}{25}\log(x+3)+c$

(e) $\frac{-5}{36}\log(x+3)+\frac{5}{6(x+3)}+\frac{5}{36}\log(x-3)+c$

(f) $\frac{-5}{4}\log(x+2)+\frac{1}{4}\log(x-2)+\log(x+1)+c$

(g) $-6\log(e^x+2)+9\log(e^x+3)+c$

(h) $2\log(\log x+1)-2\log(2\log x+5)+c$

(i) $x-4\log(x+1)+3\log(3x+2)+c$

(j) $x+\frac{1}{2}\log(x+2)+\frac{13}{2}\log(x-6)+c$

20.8 Integration by parts

Theorem: If u and v are two functions of x, then

$$\int uvdx = u\int vdx - \int\left(\frac{dx}{dx}\int vdx\right)dx$$

Proof : If u and w are functions of x, then

$$\frac{d}{dx}(uw) = u\frac{dw}{dx} + w\frac{du}{dx} \quad \text{(product rule in diffn.)}$$

Integrating both sides w.r.t. 'x', we get

$$\text{uw} = \int\left(u\frac{dw}{dx}\right)dx + \int\left(w\frac{du}{dx}\right)dx$$

$$\therefore \int u\frac{dw}{dx}dx = uw - \int\left(\frac{du}{dx}w\right)dx \quad .....(1)$$

Let $\frac{dw}{dx} = v$, then $w = \int vdx$

$\therefore$ From (1)

$$\boxed{\int uvdx = u\int vdx - \int\left(\frac{du}{dx}\int vdx\right)dx}$$

Note : Choose the first and second function in the order 'LATE' [i.e., Logarthmic, Algebraic, Trigonometric and Exponential]

Worked Examples

Example 16 : Evaluate

(a) $\int xe^x dx$ **(b)** $\int \log x dx$ **(c)** $\int x\log x dx$

Solution: We know that

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \quad .....(1)$$

Now consider, $\int xe^x dx$

Here, u = x $\quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1 \quad v = \int e^x dx = e^x$

$\therefore \quad \int xe^x dx = xe^x - \int e^x (1) dx \qquad$ [From (1)]

$= xe^x - e^x + c$

(b) $\int \log x(1) dx$

$= \log x\,(x) - \int x.\frac{1}{x} dx = x \log x - \int 1\, dx$

$= x \log x - x + c$

(c) $\int \log x.\, x\, dx$

$= \log x \times \frac{x^2}{2} - \int \frac{x^2}{2} \,.\, \frac{1}{x} dx$

$= \frac{x^2}{2} \log x - \frac{1}{2} \int x dx$

$= \frac{x^2}{2} \log x - \frac{x^2}{4} + c$

Example 17 : Evaluate

(a) $\int x^2 \log x dx$ **(b)** $\int x^2 e^x dx$ **(c)** $\int (1+x) \log x dx$

Solution :

(a) $\int x^2 \log x dx = \log x.\frac{x^3}{3} - \int \frac{x^3}{3}.\frac{1}{x} dx$

$= \frac{x^3}{3} \log x - \frac{1}{3} \int x^2 dx$

$= \frac{x^3}{3} \log x - \frac{x^3}{9} + c$

(b) $\int x^2 e^x dx = x^2 e^x - \int e^x .(2x)dx$

$= x^2 e^x - 2\int xe^x dx$

$= x^2 e^x - 2\left[xe^x - \int e^x (1)dx\right]$

$= x^2 e^x - 2xe^x + 2e^x + c$

(c) $\int (1+x)\log x dx = \int \log x(1+x)dx$

$= \log x\left(x + \frac{x^2}{2}\right) - \int\left(x + \frac{x^2}{2}\right)\frac{1}{x}dx$

$= \log x\left(x + \frac{x^2}{2}\right) - \int\left(1 + \frac{x}{2}\right)dx$

$= \left(x + \frac{x^2}{2}\right)\log x - x - \frac{x^2}{4} + c$

Example 18 : Evaluate

(a) $\int x\cos x\ dx$ **(b)** $\int x\sec^2 x\ dx$ **(c)** $\int x^2 \sin x\ dx$

Solution:

(a) $\int x\cos x\ dx$

$= x(\sin x) - \int(\sin x)(1)dx$

$= x\sin x - \cos x + c$

(b) $\int x\sec^2 x\ dx$

$= x\tan x - \int \tan x(1)dx$

$= x\tan x - \log(\sec x) + c$

(c) $\int x^2 \sin x\ dx$

$= x^2(-\cos x) - \int -\cos x(2x)dx$

$$= -x^2\cos x + 2\int x\cos x\ dx$$

$$= -x^2\cos x + 2\int x(+\sin x - \int(+\sin x)(1)dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

Example 19: Evaluate

(a) $\int xe^{2x}dx$ **(b)** $\int x\log 3x dx$ **(c)** $\int x\cos(2x+3)dx$

Solution : (a) $\int xe^{2x}dx$

$$= x\frac{e^{2x}}{2} - \int\frac{e^{2x}}{2}(1)dx \quad = \quad x\frac{e^{2x}}{2} - \frac{e^{2x}}{4} + c$$

(b) $\int x\log 3x\ dx$

$$= \int \log 3x.x\ dx$$

$$= \log 3x.\frac{x^2}{2} - \int\frac{x^2}{2}.\frac{1}{3x}.3dx$$

$$= \frac{x^2}{2}\log 3x - \frac{1}{2}\int x\ dx$$

$$= \frac{x^2}{2}\log 3x - \frac{x^2}{4} + c$$

(c) $\int x\cos(2x+3)dx$

$$= x\frac{\sin(2x+3)}{2} - \int\frac{\sin(2x+3)}{2} \quad .....(1)\ dx$$

$$= \frac{x\sin(2x+3)}{2} + \frac{\cos(2x+3)}{4} + c$$

EXERCISE 20.6

I. Three marks questions:

Evaluate:

(a) $\int 2\log x\ dx$ (b) $\int x^3\log x$ (c) $\int\frac{1}{x^2}\log x\ dx$

(d) $\int xe^{3x+5}dx$ (e) $\int x\sin x\,dx$ (f) $\int x\operatorname{cosec}^2 x\,dx$

(g) $\int x^2\cos x\,dx$ (h) $\int x\sin(5x+7)dx$

ANSWERS 20.6

(a) $2x\log x-2x+c$ (b) $\frac{x^4}{4}\log x-\frac{x^4}{16}+c$

(c) $\frac{-1}{x}\log x-\frac{1}{x}+c$ (d) $\frac{xe^{3x+5}}{3}-\frac{e^{3x+5}}{9}+c$

(e) $-x\cos x+\sin x+c$ (f) $-x\cot x+\log(\sin x)+c$

(g) $x^2\sin x+2x\cos x-2\sin x+c$ (h) $\frac{-x\cos(5x+7)}{5}+\frac{\sin(5x+7)}{25}+c$

****

Chapter 21

DEFINITE INTEGRAL AND ITS APPLICATIONS TO AREAS

21.1 Fundamental theorem of integral calculus.

Let *f(x)* be a continuous function on the closed interval [*a, b*] and F(x) be an anti-derivative of *f(x)* than

$$\int_a^b f(x)dx = \mathrm{F}(b - \mathrm{F}(a)$$

$\mathrm{F}(b) - \mathrm{F}(a)$ is also denoted by $\left[\mathrm{F}(x)\right]_a^b$

$$\therefore \int_a^b f(x)dx = \left[\mathrm{F}(x)\right]_a^b = \mathrm{F(b)} - \mathrm{F(a)}$$

Here **'a'** is called as lower limit or inferior limit and **'b'** is called as **upper limit** or superior limit.

Note: The value of a definite integral of a function is unique and it does not depend on different forms of indefinite integral.

If $\int f(x)\,dx = \mathrm{F}(x) + c$ then,

$$\int_a^b f(x)dx = \left[\mathrm{F}(x) + c\right]_a^b = \left[\mathrm{F}(b) + c\right] - \left[\mathrm{F}(a) + c\right]$$

$$= \mathrm{F}(b) - \mathrm{F}(a)$$

Thus the value of $\int_a^b f(x)dx$ is same even if we take $\int f(x)dx = \mathrm{F}(x) + \mathrm{C}$

Worked Examples

Example 1 :

Evaluate

(a) $\int_1^2 x\,dx$ **(b)** $\int_0^1 e^{2x}$ **(c)** $\int_0^{\frac{\pi}{2}} Sinx\,dx$ **(d)** $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} Sin^2 x\,dx$

Solution: (a) $\int_1^2 x\,dx = \left.\frac{x^2}{2}\right]_1^2$

$$= \frac{2^2}{2} - \frac{1}{2} = 2 - \frac{1}{2} = \frac{3}{2}$$

(b) $\int_0^1 e^{2x}\, dx = \left.\frac{e^{2x}}{2}\right]_0^1$

$= \frac{e^2}{2} - \frac{e^0}{2} = \frac{e^2}{2} - \frac{1}{2}$

$= \frac{e^2 - 1}{2}$

(c) $\int_0^{\frac{\pi}{2}} Sinx\, dx = -\cos x \Big]_0^{\frac{\pi}{2}}$

$= -\cos\frac{\pi}{2} - (-\cos 0)$

$= -0 + 1 = 1$

(d) $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} Sec^2 x\, dx = \tan x \Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right) = \tan\frac{\pi}{4} + \tan\frac{\pi}{4}$

$= 1 + 1 = 2$

Example 2 : Evaluate

(a) $\int_0^1 \left(2x^2 + \frac{1}{x}\right) dx$ **(b)** $\int_0^3 \frac{x+3}{x+2}\, dx$

(c) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x}\, dx$ **(d)** $\int_0^1 \frac{e^x + 1}{e^x}\, dx$

Solution :

(a) $\int_0^1 \left(2x^2 + \frac{1}{x}\right) dx$

$= \frac{2x^3}{3} + \log x \Big]_0^1 = \left(\frac{2}{3} + \log 1\right) - \left(\frac{-2}{3} + \log 0\right)$

$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

(b) $\int_0^3 \frac{x+3}{x+2}\, dx$

$= \int_0^3 \frac{x+2+1}{x+2}\, dx \quad = \int_0^3 \frac{x+2}{x+2}\, dx + \int_0^3 \frac{1}{x+2}\, dx$

$= x\Big]_0^3 + \log(x+2)\Big]_0^3 = 3 - 0 + \log 5 - \log 2$

$= 3 + \log\left(\frac{5}{2}\right)$

(c) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)}\, dx$

from half angle

$= \int_0^{\frac{\pi}{2}} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\, dx = \int_0^{\frac{\pi}{2}} \tan\frac{x}{2}\, dx$ $\left[\begin{array}{l}\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \\ \cos x = 2\cos^2\frac{x}{2} - 1\end{array}\right]$

$= 2\log\left(\sec\frac{x}{2}\right)\Big]_0^{\frac{\pi}{2}} = 2\log\left(\sec\frac{\pi}{4}\right) - 2\log(\sec 0)$

$= 2\log\sqrt{2} - 2\log(1)$

$= 2\left(\log\sqrt{2}\right) \qquad [\because \log 1 = 0]$

(d) $\int_0^1 \frac{e^x + 1}{e^x}\, dx$

$= \int_0^1 1 + \frac{1}{e^x}\, dx = x - e^{-x}\Big]_0^1 = 1 - e^{-1} - (0 - e^{-0})$

$= 1 - \frac{1}{e} + 1 = 2 - \frac{1}{e}$

$= \frac{2e-1}{e}$

Example 3 : Evaluate

(a) $\int_0^3 \frac{2x}{(x^2-4)^2}\, dx$ **(b)** $\int_0^1 (6x+1)\sqrt{3x^2+x+5}\, dx$

(c) $\int_1^2 xe^x\,dx$ **(d)** $\int_0^{\frac{\pi}{2}} x\cos x\,dx$

Solution :

(a) Let $I = \int_0^3 \frac{2x}{\left(x^2-4\right)^2}\,dx$

Put $x^2 - 4 = t \Rightarrow 2x\,dx = dt$

when $x = 0$, $t = -4$

$x = 3$, $t = 5$

$$I = \int_{-4}^{5} \frac{dt}{t^2} = \frac{-1}{t}\Big]_{-4}^{5}$$

$$= \frac{-1}{5} - \frac{1}{4} = \frac{-4-5}{20}$$

$$= \frac{-9}{20}$$

(b) $\int_0^1 (6x+1)\sqrt{3x^2+x+5}\,dx$

Put $3x^2 + x + 5 = t \Rightarrow (6x + 1)\,dx = dt$

when $x = 0$ $t = 5$

when $x = 1$ $t = 9$

$$I = \int_5^9 \sqrt{t}\,dt = \frac{2}{3}t^{\frac{3}{2}}\Big]_5^9$$

$$= \frac{2}{3}\left[9^{\frac{3}{2}} - 5^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[27 - 5\sqrt{5}\right]$$

(c) $\int_1^2 xe^x\,dx$

$$= \int_1^2 xe^x\,dx$$

$$= xe^x - e^x\Big]_1^2 = (2e^2 - e^2) - (e - e)$$

$$= e^2$$

(d) $\int_0^{\frac{\pi}{2}} x\cos x\ dx$ (by parts)

$= x\sin x - \int \sin x\,dx = x\sin x + \cos x$

$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x\cos\ dx = x\sin x + \cos x\Big]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$

$= \left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - \left(\frac{-\pi}{2}\sin\left(\frac{-\pi}{2}\right) + \cos\frac{-\pi}{2}\right)$

$= \left(\frac{\pi}{2}+0\right) - \left(\frac{\pi}{2}+0\right) = 0$

Exercise 21.1

Evaluate:

I. One mark questions:

(a) $\int_0^1 x^2\,dx$ (b) $\int_1^2 \left(x+e^x\right)dx$ (c) $\int_0^{\frac{\pi}{2}} \sin 2x\ dx$

(d) $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \operatorname{cosec}^2 x\ dx$ (e) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)dx$

II. 3 and 5 mark questions:

(a) $\int_1^2 \frac{2x+5}{\left(x^2+5x+3\right)^2}dx$ (b) $\int_{-1}^1 (10x+3)\sqrt{5x^2+3x+7}dx$ (c) $\int_0^1 \frac{2x+3}{3x+5}\,dx$

(d) $\int_0^{\frac{\pi}{2}} x\sin x\ dx$ (e) $\int_2^3 \frac{1}{(x+1)(x+2)}dx$

Answers 21.1

I. (a) $\frac{1}{3}$ (b) $\frac{3}{2}+e^2-e$ (c) 1 (d) –2 (e) 2

II. (a) $\frac{8}{153}$ (b) $\frac{2}{3}\left[15\sqrt{15}-27\right]$ (c) $\frac{2}{3}\left[1-\frac{1}{2}\log\left(\frac{8}{5}\right)\right]$ (d) 1 (e) $\log\frac{16}{15}$

21.2 Application of definite integrals to area bonded by curves and lines with the axes.

Introduction:

In this topic, we shall use definite integrals to find the areas enclosed by curves. Here we shall study areas enclosed by parabolas, straight lines, etc.,

Area enclosed by simple curves:

Theorem: Let $f(x)$ be a continuous non-negative function in interval $[a, b]$. Then the area of the region bounded by the curve $y = f(x)$, the x – axis and the ordinates $x = a$ and $x = b$, is given by $\int_a^b f(x)dx$ or $\int_a^b y\,dx$

Proof :

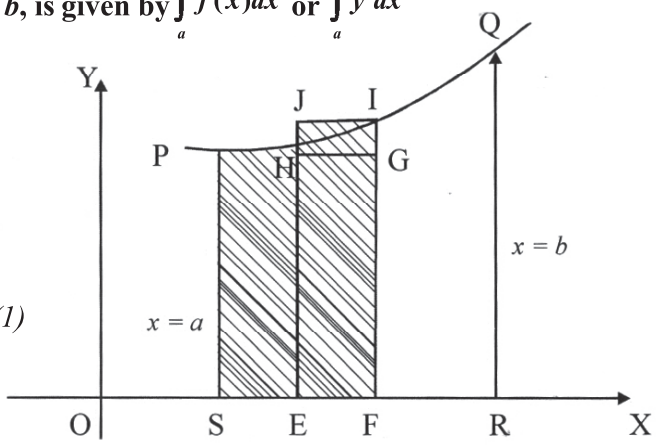


fig (1)

Let H(x, y) be any point on the curve and S $(x + \delta x, y + \delta y)$ be a neighboring point on the curve HE and GF be perpendicular to x-axis. Let A be the area of SEHPS and A+ δA be the area of SFIPS. Thus the area EFIHE = (A + δA) – A = SA

From fig 1, we have

Area EFGHE < Area EFIHE < Area EFITE

$y.\delta x < \delta A < (y + \delta y)\,\delta x$

Dividing throughout by δx

$$y < \frac{\delta A}{\delta x} < y + \delta y$$

Applying the limit as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} y < \lim_{\delta x \to 0} \frac{\delta A}{\delta x} < \lim_{\delta x \to 0} (y + \delta y)$$

$$\Rightarrow\ y \le \frac{dA}{dx} \le y \quad (\therefore\ \delta y \to 0 \text{ as } \delta x \to 0)$$

$\Rightarrow \frac{dA}{dx} = y \Rightarrow dA = y\,dx \Rightarrow A = \int y\,dx$

$A = F(x) + C$

When $x = a$ then HE coincides with SP and the area A becomes zero

$0 = F(a) + C$ ——— (1)

Again, when $x = b$, than HE coincides with QR, then the area becomes, the required area A thus, we have

$A = F(b) + C$ ——— (2)

From (1) and (2) we have

$F(b) - F(a) = A$

$F(b) - F(a) = \int_a^b f(x)dx$

Thus the area a bounded by the curve, $y = f(x)$, x –axis and the ordinates $x = a$ and $x = b$ is given by

$$\boxed{A = \int_a^b y dx = \int_a^b f(x)dx}$$

Note: Note: If the function $f(x)$ is continuous non- positive in the closed interval $[a, b]$ then the curve $y = f(x)$ lies below the x-axis and the definite integral $\int_a^b f(x)dx$ is negative since the area of a region is always non- negative the area of the region bounded by the curve $y = f(x)$ the x-axis the ordinates $x = a, x = b$ is given by $\left|\int_a^b f(x)\,dx\right|$ or $\left|\int_a^b y\,dx\right|$

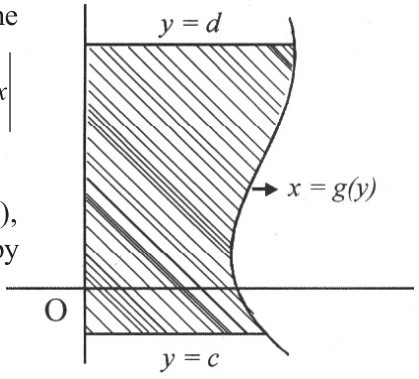


fig (2)

Similarly,

The area of the region bounded by the curve $x = g(y)$, the y - axis and the abscissae $y = c$, $y = d$ is given by

$\left|\int_c^d g(y)\,dy\right|$ or $\left|\int_c^d x\,dy\right|$

21.3 Areas between curves

If $f(x)$, $g(x)$ are both continuous in $[a,b]$ and $o \le g(x) \le f(x)$ for all $x \in [a,b]$, then the area of the region between the graphs of $y = f(x)$, $y = g(x)$ and the ordinates $x = a, x = b$ is given by $\int_a^b [f(x) - g(x)]\, dx$

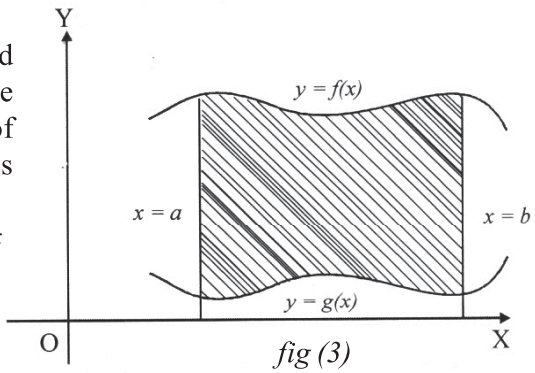


fig (3)

Similarly the area of the region between the graphs of $x = f(y)$, $x = g(y)$ and the abscissae $y = c, y = d$ is given $\int_c^d [f(y) - g(y)]\, dy$

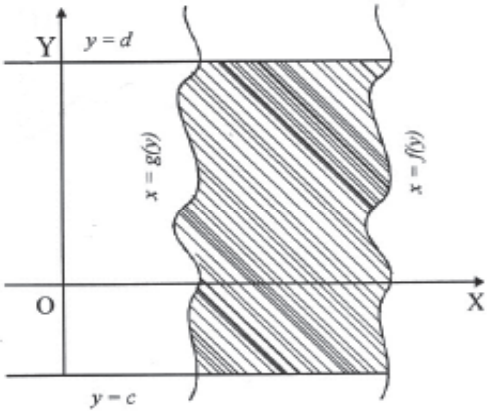


fig (4)

Worked Examples

Example 1 :

Find the area bounded by the curve $y = x^2$, x - axis and the lines $x = 1$ and $x = 3$.

Solution:

Required area $[A] = \int_1^3 y\, dy$

$$= \int_1^3 x^2\, dy = \frac{x^3}{3}\Big]_1^3$$

$$= \frac{27}{3} - \frac{1}{3}$$

$$= \frac{26}{3} \text{ sq. units}$$

Example 2 :

Find the area bounded by the curve $x = 2y^2$, y - axis and the abscissae $y = 2$ and $y = 4$.

Solution :

$$\text{Required area} = \int_2^4 x\,dy$$

$$= \int_2^4 2y^2\,dy = \frac{2y^3}{3}\Big]_2^4$$

$$= \frac{2(64)}{3} - \frac{2(8)}{3}$$

$$= \frac{128}{3} - \frac{16}{3} = \frac{112}{3} \text{ sq. units}$$

Example 3 :

Find the area bounded by the parabola $y^2 = 16x$ and its latus rectum

Solution :

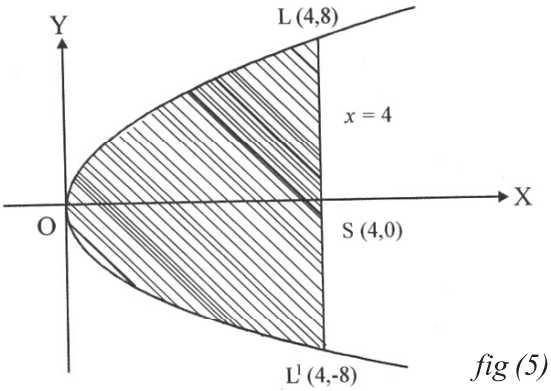


fig (5)

Since the parabola is symmetrical about x - axis

Required area = 2 area OSLO

$$= 2\int_0^4 y\,dx = 2\int_0^4 4\sqrt{x}\,dx = 8\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$$

$$= \frac{16}{3}\left[4^{\frac{3}{2}} - 0\right] = \frac{16}{3}\left(2^2\right)^{\frac{3}{2}}$$

$$= \frac{16}{3} \times 8$$

$$= \frac{128}{3} \text{ sq. units}$$

Example 4 :

Find the area bounded by the parabola $y^2 = 4x$ and the line $x - y = 0$

Solution :

The given curve is $y^2 = 4x$ ——— (1)

and the line is $x - y = 0$ ——— (2)

Solving (1) and (2) we get the points of intersection

Put $y = x$ is (1)

$x^2 = 4x$

$x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0,\ x = 4$

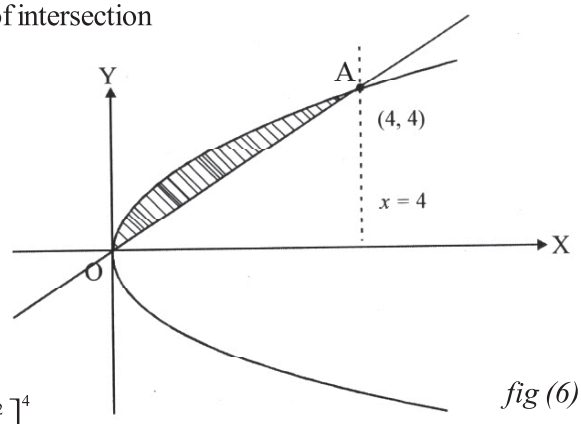


fig (6)

$\therefore$ The required area is

$$A = \int_0^4 \left[f(x) - g(x) \right] dx$$

$$= \int_0^4 \left[2\sqrt{x} - x \right] dx = 2.\frac{2}{3}.x^{\frac{3}{2}} - \frac{x^2}{2} \Bigg]_0^4$$

$$= \frac{4}{3}.4^{\frac{3}{2}} - \frac{16}{2} = \frac{4}{3}\left(2^2\right)^{\frac{3}{2}} - \frac{16}{2}$$

$$= \frac{32}{3} - \frac{16}{2} = \frac{32}{3} - 8 = \frac{32 - 24}{3}$$

$$= \frac{8}{3} \text{ sq. units.}$$

Example 5 :

Find the area of the region between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$

Solution :

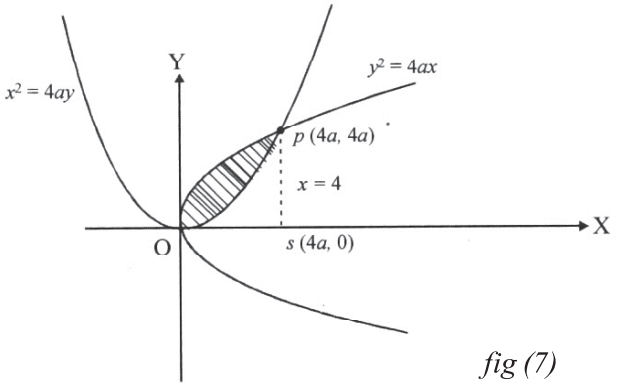


fig (7)

Given curve are $y^2 = 4ax$ ——— (1)

$x^2 = 4ay$ ——— (2)

To find the points of intersection, solve (1) and (2)

Put $y = \frac{x^2}{4a}$ in (1)

$\Rightarrow \left(\frac{x^2}{4a}\right)^2 = 4ax$

$x^4 = 64a^3 x \Rightarrow x^4 - 64a^3 x = 0$

$x(x^3 - 64\ a^3) = 0$

$x = 0,\ x = 4a$

$\therefore$ Required area is

$$A = \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= 2\sqrt{a}\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{12a}\Big]_0^{4a} = \frac{4}{3}\sqrt{a}.(4a)^{\frac{3}{2}} - \frac{(4a)^3}{12a}$$

$$= \frac{4}{3}\sqrt{a}.\ 8.\ a^{\frac{3}{2}} - \frac{64a^2}{12} = \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3} \text{ sq. units}$$

Example 6 :

Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$

Solution :

The given parabola is $y^2 = 4x$

$$\Rightarrow x = \frac{y^2}{4} \text{ ——— (1)}$$

and the given line is

$$y = 2x - 4$$

$$\Rightarrow x = \frac{y+4}{2} \text{ ——— (2)}$$

solving (1) and (2) we get

$$\frac{y^2}{4} = \frac{y+4}{2}$$

$\Rightarrow y^2 = 2y + 8 \Rightarrow y^2 - 2y - 8 = 0$

$\Rightarrow (y + 2)(y - 4) = 0 \Rightarrow y = -2, 4$

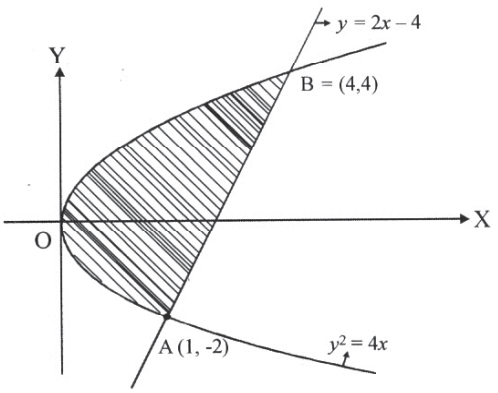


fig (8)

$\therefore$ The points of intersection are A(1, –2) and B (4, 4)

$\therefore$ The required area

$$A = \int_{-2}^{4}\left(\frac{y+4}{2} - \frac{y^2}{4}\right)dy$$

$$= \frac{1}{2}\frac{y^2}{2} + 2y - \frac{1}{4}\cdot\frac{y^3}{3}\Big]_{-2}^{4}$$

$$= \left(\frac{1}{2}\cdot\frac{4^2}{2} + 2(4) - \frac{1}{4}\cdot\frac{(4)^3}{3}\right) - \left(\frac{1}{2}\cdot\frac{(-2)^2}{2} + 2(-2) - \frac{1}{4}\cdot\frac{(-2)^3}{3}\right)$$

$$= \left(4+8-\frac{16}{3}\right) - \left(1-4+\frac{2}{3}\right) = 15 - \frac{16}{3} - \frac{2}{3}$$

$$= \frac{27}{3} = 9 \text{ sq. units}$$

Example 7 :

Find the area of the region included between the curve $4y = 3x^2$ and the line $3x - 2y + 12 = 0$

Solution :

The given curve is $4y = 3x^2$

i.e $y = \frac{3}{4}x^2$ ——— (1)

The given line is

$3x - 2y + 12 = 0$

$\Rightarrow y = \frac{3x+12}{2}$ ——— (2)

solving (1) and (2) we get

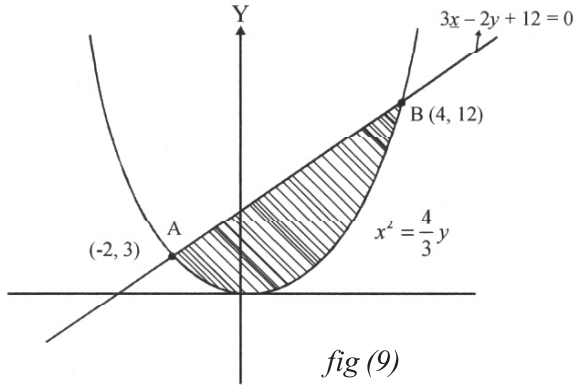


fig (9)

$$\frac{3x+12}{2} = \frac{3}{4}x^2$$

$\Rightarrow 3x^2 - 6x - 24 = 0 \Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow (x + 2)(x - 4) = 0 \Rightarrow x = -2, x = 4$

$\therefore$ The points of intersection are A(–2, 3) and B (4, 12)

$\therefore$ Required area

$$A = \int_{-2}^{4}\left(\frac{3x+12}{2} - \frac{3x^2}{4}\right)dx$$

$$= \left(\frac{3}{2}\cdot\frac{x^2}{2} + 6(x) - \frac{3}{4}\cdot\frac{x^3}{3}\right)_{-2}^{4}$$

$$= \frac{3x^2}{4} + 6x - \frac{x^3}{4}\Big]_{-2}^{4}$$

$$= \left(\frac{3}{2}\cdot\frac{16}{4} + 6(4) - \frac{64}{4}\right) - \left(3.\frac{4}{4} + 6(-2) + \frac{8}{4}\right)$$

$$= [12 + 24 - 16] - [3 - 12 + 2]$$

$$= 20 - (-7) = 20 + 7$$

$$= 27 \text{ sq. units}$$

EXERCISE 21.2

I. Two marks questions:

1. Find the area bounded by the curve $y = x^2$ x - axis and the ordinates $x = 0, x = 1$.
2. Find the area bounded by the curve $y^2 = 8x$, x - axis and the lines $x = 0, x = 2$.
3. Find the area bounded by the curve $x^2 = 8y$, y - axis and abscissas $y = 3, y = 6$.
4. Find the area bounded by the curve $3x^2 = 4y$, y - axis and the lines $y = 1, y = 2$.

II. Five marks questions:

1. Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.
2. Find the area bounded by the parabola $y^2 = 4x$ and $x^2 = 4y$.
3. Find the area bounded by the curve $y^2 = 5x$ and the line $y = x$.
4. Find the area enclosed between the parabola $x^2 = 4y$ and the line $x = 4y - 2$.
5. Find the area enclosed between the parabola $y^2 = x$ and the line $x + y = 2$.
6. Find the area enclosed between the parabola $y^2 = 4ax$ and the line $y = mx$.

ANSWERS 21.2

I. (1) $\frac{1}{3}$ sq. units (2) $\frac{16}{3}$ sq. units

(3) $\sqrt{3}\left(16 - 4\sqrt{2}\right)$ sq. units (4) $\frac{4\sqrt{3}}{9}\left(2\sqrt{2} - 1\right)$ sq. units

II. (1) $\frac{8a^2}{3}$ sq. units (2) $\frac{16}{3}$ sq. units (3) $\frac{25}{6}$ sq. units

(4) $\frac{9}{8}$ sq. units (5) $\frac{3}{2}$ sq. units (6) $\frac{8a^2}{3m^3}$ sq. units

21.4 Application definite integrals to cost and revenue function.

Students have already learnt to calculate marginal cost and marginal revenue. In this topic we will learn to calculate total cost and marginal revenue

$\boxed{MC = \frac{dc}{dQ}}$ where 'C' is the total cost, Q is the quantity produced.

$\boxed{\text{Total cost (C)} = \int_a^d MC\, dx}$ [MC = marginal cost]

Where 'a' & 'b' are the lower and upper limits respectively of the total cost function with respect to marginal cost.

In case of an indefinite integral is used in finding the total cost function, then the constant 'c' is considered as fixed cost.

$\boxed{MC = \frac{dR}{dQ}}$ where R is the total revenue

$\boxed{\text{Total revenue (R)} = \int_a^b MR\, dx}$

Where 'a' & 'b' are the lower and upper limits respectively of the total revenue with respect to marginal revenue. In case of indefinite integrals, the constant will always be taken as zero.

APPLICATION TO COMMERCE

Example 1 :

Compute the total cost for the marginal cost function $f^1(x) = 6x^2 - 6x + 12$ as summing that the fixed cost is ₹500.

Solution :

$f^1(x) = 6x^2 - 6x + 12 = M.C$

Fixed cost = ₹500

$$\text{Total cost} = \int f^1(x)\,.\,dx$$

$$= \int 6x^2 - 6x + 12.\, dx$$

$$= \frac{6x^3}{3} - \frac{6x^2}{2} + 12x + C$$

$\therefore$ T.C $= 2x^3 - 3x^2 + 12x + 500$

Example 2 :

The marginal cost function is given by $f^1(x) = 15x^2 + 2x + 1$ where x is the level of output. Find the total cost, Average cost, total variable cost, given that the fixed cost ₹100.

Sol. Given

$f^1(x) = 15x^2 + 2x + 1 = \text{M.C}$

$$\text{Total cost} = \int f^1(x)\,.\,dx = \int 15x^2 + 2x + 1.\,dx$$

$$\text{T. C} = \frac{15x^3}{3} + \frac{2x^2}{2} + x + \text{C}$$

$\text{C} = ₹\ 100$

$\text{T. C} = 5x^3 + x^2 + x + 100$

$\text{Total variable cost} = 5x^3 + x^2 + x.$

$$\text{Average cost} = \frac{\text{Total cost}}{x} = 5x^2 + x + 1 + \frac{100}{x}$$

$\text{Average variable cost} = 5x^2 + x + 1.$

Example 3 :

The marginal cost function of a first is $3x^2 - 20x + 100$ where 'x' is the level of output. Final the total cost function if the total fixed cost is ₹ 500 what is the average cost.

Solution :

$$\text{Total cost} = \int 3x^2 - 20x + 100\ dx$$

$$= \frac{3x^3}{3} - \frac{20x^2}{2} + 100x + \text{C}$$

$\text{T. C} = x^3 - 10x^2 + 100x + 500$

$$\text{A.C} = x^2 - 10x + 100 + \frac{500}{x}$$

Example 4 :

If the marginal revenue = $76 - x^2$ find the maximum total revenue. Also find the total and the average revenues. Write the demand function.

Solution :

$f^1(x) = \text{M.R} = 76 - x^2$

$$\text{Total Revenue} \quad \int 76 - x^2\,.\,dx$$

$$76x - \frac{x^3}{3} + \text{C}$$

when Output = 0 Total revenue = 0 $\therefore$ C = 0

Total revenue R = $76x - \frac{x^3}{3}$

Average revenue = $\frac{\text{Total Revenue}}{x}$

$= \frac{76x - \frac{x^3}{3}}{x} = 76 - \frac{x^2}{3}$

Average revenue = price per unit which is nothing but the demand function

$\therefore$ the demand function = $76 - \frac{x^2}{3}$

Example 5 : The marginal cost = 8 + 0.08*x* and the marginal revenue = 16. Find the total revenue, total cost and total profit. Assume that the fixed cost is nil .

Solution :

Under conditions of perfect competition, MR is constant when MR = MC; profit is maximized

$16 = 8 + 0.08x$

$8 = 0.08x$

$x = \frac{8}{0.08}$

$x = \frac{800}{8} = 100$ units

Total revenue $= \int \text{M.R}\ dx = \int 16\, dx$

$= 16x$

Total cost $= \int \text{M.C.}\ dx = \int (8 + 0.08x)dx$

$= 8x + 0.08\frac{x^2}{2} + \text{C}$

It is given fixed cost 0 $\therefore$ C = 0

T.C = $8x + 0.04x^2$

Total profit = $16x - (8x + 0.04x^2)$

= $8x - 0.04x^2$

Total profit = $16x - (8x + 0.04x^2)$

Putting $x = 100$ in the total profit function, we have

Total proof = $-0.04\,(100)^2 + 8(100)$

= $-\ 0.04\ (10000) + 800$

= 400 + 800 = ₹400

Example 6 :

The marginal cost curve for a super Bazaar is 1.16 – 04*x*. Find the total cost for C(0) = 14.4.

Solution:

C(0) = 14.4 refers to fixed cost or means the total cost. When output is zero. In the short rum analysis of cost, the only cost that anise even when there is no production is

$$\text{Total cost} = \int (1.16 - 0.04x)\,dx$$

$$= 1.16x - \frac{0.04x^2}{2} + C$$

$$\text{T.C.} = 1.16x - 0.02x^2 + 14.4$$

Example 7 :

A manufacturer produces x units of a product per week is marginal cost function $f^1(x) = 6x^2 + 2x$. Find the total cost when fixed cost = ₹50.

Solution:

$$\text{Marginal cost} = f^1(x) = 6x^2 + 2x$$

$$\text{Total cost} = \int f^1(x)\,dx$$

$$= \int 6x^2 + 2x.dx$$

$$= \frac{6x^3}{3} + \frac{2x^2}{2} + C$$

$$\text{T. C} = 2x^3 + x^2 + C$$

$$\because\ C = ₹50$$

$$\text{T. C} = 2x^3 + x^2 + 50$$

Example 8 :

The marginal cost function is $f^1(x) = 6x^2 + 2x + 1$ where x is the level of output. Find the total cost, average cost, total variable cost, average variable cost. Given that fixed cost = ₹70.

Solution:

$$\text{Marginal cost} = f^1(x) = 6x^2 + 2x + 1$$

$$\text{Total cost} = \int f^1(x)\,dx$$

$$= \int 6x^2 + 2x + 1.\,dx = \frac{6x^3}{3} + \frac{2x^2}{2} + x + C$$

$$\text{T. C} = 2x^3 + x^2 + x + 70 \text{ (As C = ₹70)}$$

Total variable cost $= 2x^3 + x^2 + x$

$$\text{Average cost} = 2x^3 + x + 1 + \frac{70}{x}$$

Average Variable cost $= 2x^3 + x + 1$

Example 9 :

Find the marginal revenue when the quantity sold is 5 units for the demand function $P = 300 - Q^2$. If fixed cost is 50 and variable cost is 3 at what quantity will profit be maximum

Solution :

M.R = price × quantiy

$= (300 - Q^2) \times Q = 300Q - Q^3$

at $Q = 5$

M.R $= 300\,(5) - 125$

$= 1500 - 125 = ₹1375$

T.C = V.C + F.C

$= 3Q + 50$

$$\text{M.C} = \frac{d(\text{T.C})}{dQ} = 3$$

Profit is maximum when M.R = M.C

$300Q - Q^3 = 3$

Example 10 :

Find the marginal cost is $f^1(x) = 1 - 2x + 12x^3$ where x is the output. Find the total cost, average cost, total variable cost and average variable cost given that fixed cost is ₹ 50.

Solution :

Given $f^1(x) = 1 - 2x + 12x^3$, Fixed cost = ₹ 50

$$\text{Total cost} = \int f^1(x)\,dx$$

$$= \int (1 - 2x + 12x^3)\,dx$$

$= x - x^2 + 3x^4 + C$

$= x - x^2 + 3x^4 + 50$ (Fixed cost = 50)

$$\text{Average cost} = \frac{\text{Total cost}}{x}$$

$$= \frac{x - x^2 + 3x^4 + 50}{x} = 1 - x + 3x^4 + \frac{50}{x}$$

Total variable cost = $x - x^2 + 3x^4$

Average variable cost = $1 - x + 3x^3$

Example 11 :

If the marginal revenue is $f^1(x) = 30 - \frac{x}{30}$. Find the total revenue and average revenue obtained from on output of 60 units.

Solution : Total revenue = $\int f^1(x)dx = \int\left(30 - \frac{x}{30}\right)dx$

$$= \int 30x - \frac{x^2}{60}$$

Total revenue for an output of 60 units

$$= 30x - \frac{x^2}{60}\Big]_0^{60}$$

$$= 30\,(60) - \frac{(60)^2}{60} = 1800 - 60$$

Total revenue = ₹ 1740

Average revenue = $\frac{\text{Total revenue}}{x}$

$$= \frac{30x - \frac{x^2}{60}}{x}$$

$$= 30 - \frac{x}{60}$$

Average revenue for an output of 60 units = $30 - \frac{60}{60} =$ ₹ 29

Example 12 :

Find the total variable cost of producing 10 units given that the marginal cost function is $f^1(x) = 275 - x - 0.3x^2$, where x is the number of units produced.

Solution :

Total cost = $\int f^1(x)dx$

$$= \int\left(275 - x - 0.3x^2\right)dx$$

Total cost $= 275x\frac{-x^2}{2} - 0.1x^3 + C$

Total variable cost $= 275x - \frac{x^2}{2} - 0.1x^3$

Total variable cost for producing 10 units

$$= 275(10) - \frac{(10)^2}{2} - (0.1)(10)^3$$

$$= 2750 - 50 - 100 = 2600$$

Total variable cost for producing 10 units = ₹ 2,600.

Example 13 :

If the marginal cost of a product is $2x^2 - 4x$, where x is output, find the total cost of producing 16 units.

Solution :

Given M.C $= 2x^2 - 4x$

$$\text{T.C} = \int \text{M.C}\, dx = \int_0^{16} (2x^2 - 4x)\, dx$$

$$= \frac{2x^3}{3} - \frac{4x^2}{2}$$

$$= \frac{2(16)^3}{3} - \frac{4(16)^2}{2}$$

Total cost $= 2730.66 - 512$

$= ₹\ 2218.66$

Example 14 :

Find the total revenue in rupees by raising the output from 20 units to 40 units when the marginal revenue function is $\frac{5q^2}{12} + 16q - 100$ where q is the output.

Solution :

Total revenue $= \text{TR} = \int \text{MR}\, dq$

$$= \int_{20}^{40} \left(\frac{5q^2}{12} + 16q - 100 \right) dq$$

$$= \left[\frac{5q^3}{36} + 8q^2 - 100q \right]_{20}^{40}$$

$$= \left[\frac{5(40)^3}{36} + 8(40)^2 - 100(40)\right] - \left[\frac{5(20)^3}{36} + 8(20)^2 - 100(20)\right]$$

$= [8888.89 + 12800 - 4000] - [1111.11 + 3200 - 2000]$

$= ₹\ 15,377.78$

Example 15 :

If the marginal revenue = $10x - x^2$, find the maximum revenue. Write the demand function.

Solution :

Given MR = $10x - x^2$

Total revenue (R) = $\int$ MR dx

$$= \int (10x - x^2)dx = 10x^2 - \frac{x^3}{3} + C$$

when output = nil, total revenue = nil

$\therefore$ C = 0 Hence R = $10x^2 - \frac{x^3}{3}$

Demand function (P) = Average revenue = $\frac{\text{Total revenue}}{x}$

$$P = \frac{10x^2}{x} - \frac{x^3}{3}$$

$$P = 10x - \frac{x^2}{3}$$

EXERCISE 21.3

I. Two marks questions:

1. If the marginal cost of a firm is $f^1(x) = 10 + 6x - 6x^2$ where x is the output find the total cost amusing that that the fixed cost is ₹125.
2. If the marginal revenue is given by $f^1(x) = \frac{30 - x^2}{30}$. Find the revenue obtained from an output of 50 units.
3. Find the total revenue in rupees by raising the output from 10 units to 20 units when the marginal revenue function is $2q^2 - q$ where q is the output

II. Three and Five marks questions:

1. If the marginal cost function is $3x^2 - x + 5$ where x is the output, then find the total cost, average cost, total variable cost and average variable cost given that the fixed cost is ₹25.

2. If the marginal revenue is $f^1(x) = 20 - \frac{x}{20}$. Find the total revenue and average revenue obtained from an output of 30 units.

ANSWERS 21.3

I. (1) $10x + 3x^2 - 2x^3 + 125$ (2) 111.1 (3) 4516.6

II. (1) Total cost = $x^3 - \frac{x^2}{2} + 5x + 25$, Average cost = $x^2 - \frac{x}{2} + 5 + \frac{25}{x}$

Total variable cost = $x^3 - \frac{x^2}{2} + 5x$, Average variable cost = $x^2 - \frac{x}{2} + 5$

(2) Total revenue = ₹1110

Average revenue = ₹185

* * * * *

II P.U.C BASIC MATHEMATICS (CODE - 75)
Blue print of the Question paper (New syllabus 2014-15 onwards)
MODEL - 1

Unit / Chapter	Title of the Chapter	No. of Teaching hrs	Part A	Part B	Part C	Part D	Part E		Total Marks
			1 Mark	2 Mark	3 Mark	5 Mark	4 Mark	6 Mark	
UNIT-I	**ALGEBRA (42 hrs)**								
1	Matrices & Determinants	13	1	1	2	-	-	1	15
2	Permutation & Combination	08	1	1	1	-	-	-	06
3	Probability	05	-	1	1	-	-	-	05
4	Binomial theorem	06	-	-	-	1	1	-	09
5	Partial fraction	04	-	-	-	1	-	-	05
6	Mathematical logic	06	1	1	-	1	-	-	08
UNIT-II	**COMMERCIAL ARITHMETIC (34 hrs)**								
7	Ratios & proportions	10	1	1	1	1	-	-	11
8	Bill Discounting	06	-	1	1	-	-	-	05
9	Stocks & shares	04	1	-	1	-	-	-	04
10	Learning curve	04	-	-	-	1	-	-	05
11	Linear programming problem	06	-	-	-	1	-	-	05
12	Sales tax & value added tax	04	-	-	1	-	-	-	03
UNIT-III	**TRIGONOMETRY (12 hrs)**								
13	Heights & Distances	04	-	-	-	-	1	-	04
14	Compound angle, multiple angle, sub multiple angle & transformation formule	08	1	2	-	1	-	-	10
UNIT-IV	**ANALYTICAL GEOMETRY (10 hrs)**								
15	Circles	06	1	-	-	1	-	-	06
16	Parabola	04	-	1	1	-	-	-	05
UNIT-V	**CALCULUS (42 hrs)**								
17	Limit & Continuity of a function	08	1	1	-	-	-	1	09
18	Differential Calculus	10	1	1	1	1	-	-	11
19	Application of Derivative	08	-	1	2	-	-	-	08
20	Indefinite Integrals	08	1	1	2	-	-	-	09
21	Definite Integrals & its Application to areas	08	-	1	-	1	-	-	07
	Total No. of Teaching hrs / Marks	140 hrs	10	14	14	10	02	02	150

II P.U.C BASIC MATHEMATICS (CODE - 75)
Blue print of the Question paper (New syllabus 2014-15 onwards)
MODEL - 2

Unit / Chapter	Title of the Chapter	No. of Teaching hrs	Part A	Part B	Part C	Part D	Part E		Total Marks
			1 Mark	2 Mark	3 Mark	5 Mark	4 Mark	6 Mark	
UNIT-I	**ALGEBRA (42 hrs)**								
1	Matrices & Determinants	13	1	1	2	1	-	-	14
2	Permutation & Combination	08	1	1	1	-	-	-	06
3	Probability	05	-	1	1	-	-	-	05
4	Binomial theorem	06	-	-	-	1	1	-	09
5	Partial fraction	04	-	-	-	1	-	-	05
6	Mathematical logic	06	1	1	-	1	-	-	08
UNIT-II	**COMMERCIAL ARITHMETIC (34 hrs)**								
7	Ratios & proportions	10	1	1	1	1	-	-	11
8	Bill Discounting	06	-	1	1	-	-	-	05
9	Stocks & shares	04	-	-	1	-	-	-	03
10	Learning curve	04	-1	-	-	1	-	-	06
11	Linear programming problem	06	-	-	-	1	-	-	05
12	Sales tax & value added tax	04	-	-	1	-	-	-	03
UNIT-III	**TRIGONOMETRY (12 hrs)**								
13	Heights & Distances	04	-	-	-	-	1	-	04
14	Compound angle, multiple angle, sub multiple angle & transformation formule	08	1	2	-	1	-	-	10
UNIT-IV	**ANALYTICAL GEOMETRY (10 hrs)**								
15	Circles	06	1	-	-	-	-	1	07
16	Parabola	04	-	1	-1-	-	-	-	05
UNIT-V	**CALCULUS (42 hrs)**								
17	Limit & Continuity of a function	08	1	1	-	-	-	1	09
18	Differential Calculus	10	1	1	1	1	-	-	11
19	Application of Derivative	08	-	1	2	-	-	-	08
20	Indefinite Integrals	08	1	1	2	-	-	-	09
21	Definite Integrals & its Application to areas	08	-	1	-	1	-	-	07
	Total No. of Teaching hrs / Marks	140 hrs	10	14	14	10	02	02	150

II PUC BASIC MATHEMATICS
WEIGHTAGE GIVEN TO THE CURRICULUM

1.	Knowledge	:	35%	=	52.5Mark
2.	Understanding	:	30%	=	45 Mark
3.	Applications	:	25%	=	37.5 Mark
4.	Skill	:	10%	=	15 Mark
			Total		**150 Marks**

UNIT WISE WEIGHTAGE

	Unit		
I	Algebra	=	48 Marks
II	Commercial arithmetic	=	33 Marks
III	Trigonometry	=	14 Marks
IV	Analytical geometry	=	11 Marks
V	Calculus	=	44 Marks
	Total		**150 Marks**

INSTRUCTION TO QUESTION PAPER SETTERS

(New syllabus 2014-15 onwards)

Note: In the chapter (4) Binomial Theorem the proof of the Binomial Theorem for Positive Integral power is Excluded

PART - E

6 MARKS QUESTIONS MUST BE SELECTED FROM THE FOLLOWING TOPICS ONLY:

1. **Application of matrix:** (3 × 3) order about statement problem, formation of linear equation and solve them by matrix method
2. **Circle:** Problems on concylic (circle passes through 4 points)
3. **Limits:** standard theorem

   (i) prove that $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n.a^{n-1}$ for all n $\in$ Q (Q = set of rational n is + ve, – ve and fraction)

   (ii) Prove that $\lim_{x \to 0}\left[\frac{\sin x}{x}\right] = 1$, ($x$ = angle mesuered in radian) and deduce that

   $\lim_{x \to 0}\left[\frac{\tan x}{x}\right] = 1,$

4 MARKS QUESTION MUST BE SELECTED FROM THE FOLLOWING TOPIC ONLY.

1. **Binomial theorem**: Application problems like Evaluate : $(0.\ 98)^5$, $(1.01)^5$, $(102)^4$, $(0.97)^4$... ect upto 4 decimal.
2. **Heights and distances** : Application problems
3. **Linear programming problem (L.P.P)** : Statement problems on L.P.P for mation-tion of linear equations.
4. **Cost and revenue function** : Problems on total cost, total revenue, Marginal cost, Marginal revenue, Profit Maximization. ..... etc.

MODEL QUESTION PAPER - 1
II PUC : BASIC MATHEMATICS [CODE : 75]
(NEW SYLLABUS 2014-15)

Time: 3.15 hours **Max. Marks: 100**

Instructions:

(i) The question paper has 5 parts A, B, C, D, and E. Answer all the parts.

(ii) Part - A carries 10 marks, part - B carries 20 marks, part - C carries, part - D carries 30 marks and part - E carries 10 marks.

(iii) Write the question number properly as indicated in the question paper.

PART – A

I. Answer ALL the questions: **(1 × 10 = 10)**

1. If $A = \begin{bmatrix} 5 & -6 & 3 \end{bmatrix}, B = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ find AB
2. If ${}^{n}P_{4} = 360$ find n.
3. Write the truth value of the proposition "If 2 is not a prime numbers then $\sqrt{2}$ is an irrational number".
4. Find the mean proportion of 9 and 16
5. Define the 'Index of learning'.
6. If $\text{Sin } A = \frac{3}{5}$, find Sin 2A
7. Find the radius of the circle $x^2 + y^2 - 4x - 5 = 0$.
8. Evaluate : $\lim_{x \to 3} \left[\frac{x^3 - 27}{x - 3} \right]$
9. If $y = 2\sqrt{x} - \cos 2x + 2$, find $\frac{dy}{dx}$.
10. Evaluate : $\int \left(\frac{1}{x} - \sin x + 3 \right) dx$.

PART – B

II. Answer any 10 questions: **2 × 10 = 20**

11. If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Show that $A^2 - 4A + 3I = 0$ (I = Identify matrix of 2nd order)
12. There are 15 points in a plane of which 5 are collinear. Find the number of straight lines can be formed.
13. A boy drawn at random 3 balls from a bag containing 9 red and 5 while balls. What is the probability of getting (i) all red balls (ii) 2 red and 1 white.
14. Write the converse and contrapositive of the compound proposition "If prashanth got first class in mathematics then prof. John will gift him a watch".
15. A certain number is subtracted from each of the two term of the ratio 21 : 35 to give a new ratio 3 : 10. Find the number which is subtracted?
16. Find the Banker's discount on ₹ 1000 due 6 months hence at 10% per annum.
17. Derive the value for cos 105°.
18. Transform 2sin 40° × cos 20° into sum.
19. Find the focus and the equation of Directrix of the parabola $x^2 - 16y = 0$.
20. Evaluate : $\lim_{x \to 0} \left(\frac{\sin 3x . \tan 4x}{x^2} \right)$.
21. If $y = x^x$, find $\frac{dy}{dx}$.
22. If $S = 4t^3 - 6t^2 + t - 7$ (S = distance in mt, t = time sec)
    Find the velocity and acceleration at t = 2 sec.
23. Evaluate : $\int \frac{x}{x^2 + 4} dx$.
24. Evaluate : $\int_0^{\pi/4} \sec^2 3x \, dx$.

PART – C

III. Answer any 10 questions: **(3 × 10 = 30)**

25. If $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ verify A. (adj A) = (adj A). A = |A|.I and (I=Identity Matrix of 2nd order).
26. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)$.

27. Find the number of permutation of the letters of the word 'COMMISSION'. If the word
    (i) Start with M and end with M
    (ii) 2S's are together
    (iii) 2O's are not together
28. In a class of 80 student 40 are taking Mathematics 25 are taking Statistics. If each student has taken at least one of these subject. Find the probability that the student is taking
    (i) both Mathematics and Statistics
    (ii) only Mathematics
    (iii) Only Statistics
29. A sum of ₹. 2415 has to be divided among three person A, B and C in such proportion that A's share to B's is 4 : 5 and B's share to C's share is 9 : 16. How much does each get?
30. A bill of ₹ 5000 was drawn an 10/4/2013 at 3 months and was discounted on 1/5/ 2013 at 12% p.a. For what sum was the bill Discounted and also find the Banker's gain.
31. Find the Interest earned on ₹ 4897.50 cash invested in 15% stock at 81.50, given that brokerage is 0.125%.
32. The price of a washing machine, inclusive of sales tax is ₹ 13,530. If the sales tax is 10%, find its basic price?
33. Find the equation of the parabola given that the ends of the latus rectum are L (3, 6) is Focus is (0,–3) and Directrix is y = 3
34. If $x = a\cos^4 t$, $y = b\sin^4 t$. find $\frac{dy}{dx}$ at $t = \pi/4$.
35. The height of a cone is 30 cm and it is constant the radius of the base is increasing at the rate 0.25cm / sec. Find the rate of increase of volume of the cone when the radius is 10 cm?
36. Find the maximum and minimum value of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$.
37. Evaluate: $\int \frac{5x}{(x-3)(x+4)} dx.$
38. Evaluate: $\int x^2 . \cos 3x \, dx.$

PART – D

IV. Answer any 6 questions: **(6 × 5 = 30)**

39. Find the term Independent of 'x' in the expansion of $\left(\sqrt{x}+\frac{1}{3x^2}\right)^{10}$.

40. Resolve $\frac{2x+1}{(x-1)(x+2)(3x-1)}$ into partial fractions.

41. Verify: $\sim(p\to q)\vee[(\sim p\wedge q)\leftrightarrow\sim q]$ is a Tautology or contradiction or neither.

42. A jar contains two liquids X and Y in the ratio 7 : 5. When 6 litres of the mixture is drawn and the jar in filled with the same quantity of Y, the ratio of X and Y becomes 7 : 9. Find the quantity X in the jar initially.

43. Samsung company which manufacture LCD TV. The 1st lot of 10 unit was completed in 1400 labour hrs. Find each subsequent lot, the commutative production was doubled. And it has observed that 90% learing effect applies to all labour related cost. The anticipated production is 320 unit of LCD TV find total labour cost required to manufacture 320 unit and also find the total labour cost at ₹ 20/per hrs.

44. Using the graphical method, solve the following. LPP:-

    Objective function : minimize $z = 1.5x + 2.5y$

    Subject to constraints : $x + 3y \geq 3$

    $x + y \geq 2$

    and $x \geq 0, y \geq 0.$

45. Show that $\frac{\sin^3\theta+\sin 3\theta}{\sin\theta}+\frac{\cos^3\theta-\cos 3\theta}{\cos\theta}=3.$

46. A circle has its centre on x-axis and passess through (5, 1) and (3, 4). Find its equation.

47. If $y=(x^2+a^2)^6$, show that $\left(x^2+a^2\right)\frac{d^2y}{dx^2}-10x\frac{dy}{dx}-12y=0.$

48. Find the area enclosed between the curve $y = 11x - 24 - x^2$ and the line $y = x$.

PART – E

V. Answer any 1 questions: **(1 × 10 = 10)**

49.(a) Transport corporation operate bus service between two villages. Data regarding the passenger traffic during the 1st three days of the week is given below along with the total revenue. (6)

Day	No. Passenger of travelled			Total Revenue (₹)
	Children	Senior citizen	Adult	
1	10	10	20	90
2	30	20	10	100
3	10	20	30	140

Find the bus fare charged per children, senior citizen and per adult. By using matrix method.

(b) The angle of elevations of the top of an unfinished tower at a point distance 120 mt from its base in 45°. How much higher must the tower be raised so that the angle of elevation at the same point many be 60°? (4)

50.(a) Prove that $\lim_{x \to a}\left(\frac{x^n - a^n}{x - a}\right) = n.a^{n-1}$,

for all 'n' belongs to rational (n is + ve, – ve or a fration). (6)

(b) Find the total Revenue obtained by raising the output from 10to 20 units. Where the marginal revenue function in given by $MR = 3\left(\frac{x^2}{20}\right) - 10x + 100$ (x = output).

(4)

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MODEL QUESTION PAPER - 2
II PUC : BASIC MATHEMATICS [CODE : 75]
(NEW SYLLABUS 2014-15)

Time: 3.15 hours **Max. Marks: 100**

Instructions:

(i) The question paper has 5 parts A, B, C, D, and E. Answer all the parts.

(ii) Part - A carries 10 marks, part - B carries 20 marks, part - C carries, part - D carries 30 marks and part - E carries 10 marks.

(iii) Write the question number properly as indicated in the question paper.

PART – A

I. Answer all the questions: **10 × 1 = 10**

1. Evaluate : $\begin{vmatrix} 4000 & 4001 \\ 4002 & 4003 \end{vmatrix}$.
2. Find n if ${}^{n}C_{8} = {}^{n}C_{9}$
3. Write symbolically "If gold is an elemont then water is a com pound.
4. Find the mean proportion of 36 and 4.
5. Write the formula for yield Percentage.
6. Write the value for sin 75°.
7. Find the centre of the of the circle $x^2 + y^2 = 25$
8. Evaluate : $\lim_{x \to 2} \left[\frac{x^2 - 2^2}{x - 2} \right]$.
9. Find $\frac{dy}{dx}$ if $y = \sqrt[3]{x}$.
10. Evaluate : $\int e^{3x}\, dx$.

PART B

II. Answer any Ten question. **(2 × 10 = 20)**

11. If $\begin{bmatrix} x+y & 3 \\ y & x-y \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$ find x and y.

12. Find the number of words formed by the letters of the word "DELHI" How many of them. start with D and end with I
13. Find the probability of getting a black Jack from a pack of 52 cards.?
14. If the truth value of p is true, q is false. Find the truth value of $\sim(p \rightarrow \sim q) \vee \sim p$.
15. If a : b = 3 : 5, b : c = 15 : 23 find a : c.
16. Find the legally due date for a bill date 22/04/2014 due 6 months hence.
17. Tanya bought a coat for ₹ 220 inclusive of sales tax 10%. How much was the sales tax?
18. If $\text{Tan A} = \frac{1}{2}$ & $\text{Tan B} = \frac{1}{3}$ prove that $\text{A} + \text{B} = \frac{\pi}{4}$.
19. If $y^2 = 16x$ find the (i) Focus (ii) Latus Rectum of the parabola.
20. Evaluate : $\lim_{\theta \to 0} \frac{\tan 5\theta . \sin^2 \theta}{\theta^3}$.
21. If $S = \sqrt{t-1}$ find velocity (S = distance, t = time)
22. If Total cost $C(x) = x^2 + 2x + 1$ find (i) marginal cost & (ii) Average cost.
23. Evaluate: $\int_1^2 \frac{1}{x}\, dx$.
24. Find the area enclosed by the curve $y = x^2 + 2x$ between the ordinates $x = 0$ and $x = 2$.

PART C

III. Answer any 10 questions: **3 × 10 = 30**

25. Show that, if in a determinant scalar multiple of the elements of any row or column is added to any other row or column. The value of the determinant remains unchanged.
26. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
27. Find the no. of parallelogram that can be formed from a set of 6 parallel lines intersecting another on the set of 11 parallel line.
28. A bag contains 7 white, 3 red and 4 black balls, one ball is picked up at random. What is the probability that (i) none is black (ii) ball is red (iii) ball is white.
29. Two quantities are in the ratio 3 : 4 If 10 is subtracted from each of them, the remainder are in the ratio 1 : 3. find the quantities.
30. Calculate the banker's discount on face value ₹ 1000 of the period is 73 days at 5% p.a banker's commissions.

31. What is the market value of 9.5 stock when an investment of ₹12,400 produce income ₹1472.5.

32. Ananya went to the grocery shop to purchase biscuits for ₹ 40, Rice for ₹ 50, and wheat for ₹ 50 sales tax on each item is 10%. How much should she pay to the shop keepers?

33. If $y = -4$ is the equation of the drectrix, axis $x = 3$ and the length of lotus rectum in 8 find the equation of the parabola.

34. Differentiate 'e^x' w.r.t. x by Ist priciples.

35. Divide 20 into two parts so that the product is maximum.

36. The side of a square is increasing at the rate of 10 cm/sec. If the side 20 cm. Find the rate of increase of its area.

37. Evaluate : $\int \frac{1}{1+\cos x}\,dx$.

38. Evaluate : $\int e^{\sin x}.\cos x\ dx$.

PART D

IV. Answer any SIX questions **(5 ×6 = 30)**

39. Evaluate : $\left(1+\sqrt{5}\right)^4 - \left(1-\sqrt{5}\right)^4$.

40. Resolve in to partial fractions $\frac{2x+1}{(x-1)(x-2)(x-3)}$.

41. Define Tautology and verify for tautology $\sim(p \wedge q) \rightarrow (\sim p \vee \sim q)$.

42. ₹. 5625 is Divided among A, B and C so that A receives one half as much as B and C together receive and B receives one fourth of what A and C together receive. Find the share of A, B and C.

43. XYZ company supplies water tankers to the government the 1st water tanker takes 20,000 labour hours. The government auditors suggest that there should be 90% learning effect rate the management expects an order of 8 water tankers in the next year what will be total labour hrs and the labour cost the company will incur at the rate of ₹. 20 per hour.

44. Solve the following L.P.P by graphically

Maximise : $z = 400x + 150y$

Subject to the constraints, $3x + y \leq 600$

$x \leq 150$

$y \leq 400$

$x \geq 0, y \geq 0$

45. Prove that $\sin 3A = 3 \sin A - 4 \sin^3 A$

46. Find the equation of the circle passing through the points (0, 0) (1, 1) and has its centre on x-axis.

47. If $e^y = \sin(x + y)$ prove that $\frac{dy}{dx} = \frac{\cos(x+y)}{e^y - \cos(x+y)}$.

48. Find the area between the curves $x^2 = y$ and $y^2 = x$.

PART E

V. Answer any one question **(10 × 1 = 10)**

49. (a) Evaluate : $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n.a^{n-1}$ for all rationals. (n is + ve, – ve & fraction)

(6)

(b) Expand $(0.99)^5$ using Binomial theorem upto 4 decimal. (4)

50.(a)A sales person has the following record of sales for the month of Jan, Feb and March 2014 for 3 products A, B and C. He has paid a commission at fixed rate per unit but at varying rates for products A, B and C.

Months	Sales (units)			Commission (₹)
	A	B	C	
January	9	12	2	800
February	15	5	4	900
March	6	10	3	950

Find the rate of commission payable on A, B & C per unit sold by using matrix method.

(6)

(b) A person standing on the bank of a river observes, that the angle subtended by a tree on the opposite bank is 60°. When he returns 40 meters from the bank he finds the angle to be 30°. Find the height of the tree and the breadth of the river.

(4).

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MODEL QUESTION PAPER - 3
II PUC : BASIC MATHEMATICS [CODE : 75]
(NEW SYLLABUS 2014-15)

Time: 3.15 hours **Max. Marks: 100**

Instructions:

(i) The question paper has 5 parts A, B, C, D, and E. Answer all the parts.

(ii) Part - A carries 10 marks, part - B carries 20 marks, part - C carries, part - D carries 30 marks and part - E carries 10 marks.

(iii) Write the question number properly as indicated in the question paper.

PART – A

I. Answer all the questions: **10 × 1 = 10**

1. If $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, find AB.
2. If $5P_r = 60$, find the value of r.
3. Negate : "4 is an even integer of 7 is a prime number".
4. Find the third proportional to 6 and 24.
5. Define yield.
6. If $\sin A = \frac{1}{2}$, find $\sin 2A$.
7. If the length of latus rectum of the parabola $x^2 = 4ky$ is 8, find k.
8. Evaluate : $\lim_{x \to -1} \left[\frac{x^3 + 1}{2x^2 + 5x + 3} \right]$.
9. Find $\frac{dy}{dx}$ if $x^2 - y^2 = a^2$.
10. Evaluate : $\int \frac{1}{7x + 8}\, dx$.

PART B

II. Answer any Ten question. **(10 × 2 = 20)**

11. If $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, show that $A^2 - 4A + 3I = 0$. (I = Identify matrix of 2nd order)
12. In how many ways can 6 gentlemen and 4 ladies be seated round a table so that no 2 ladies are together?

13. If $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}, P(A \cup B)=\frac{7}{12}$, find $P\left(\frac{B}{A}\right)$.

14. Write the converse and contrapositive of "If two straight lines are parallel then they do not intersect".

15. 2 numbers are in the ratio 3 : 5. If 5 is added to each term, then their ratio becomes 2:3. Find the numbers.

16. A banker pays ₹ 2380 on a bill of ₹ 2500, 73 days before the legal due date. Find the rate of discount charged by the banker.

17. Sanju purchases a bicycle costing ₹ 12,000. If the rate of sales tax is 9%, then calculate the total amount paid by him.

18. Prove that $\frac{\cos 2A - \cos 12A}{\sin 12A - \sin 2A} = \tan 7A$.

19. Find the equation of a circle whose end points of it's diameter are (3, 4) and (1, –2).

20. If $y=\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$, prove that $\frac{dy}{dx} = \sec^2 x$.

21. Evaluate : $\lim_{\theta \to 0} \frac{\sin 3\theta}{\tan 2\theta}$.

22. The demand function of a firm is $2x - 5y = 7$ (x is the output, y is price / unit) Find the marginal revenue.

23. Find : $\int \frac{3x^2}{1+x^3}\, dx$.

24. Evaluate $\int_1^2 x\, e^x\, dx$.

PART - C

III. Answer any TEN questions : **(10 × 3 = 30)**

25. Solve using cramer's rule : $3x + 4y = 7$ and $7x = y + 6$.

26. Using properties of determinants, prove : $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1 + a + b + c$.

27. A team of 8 players has to be selected from 14 players. In how many ways the selections can be made if
    (i) 2 particular players are always included
    (ii) 2 particular players are always excluded.

28. A box contains 4 defective and 6 non-defective bulbs. Find the probability that at least 3 bulbs are defective when 4 bulbs are selected at random.

29. 5 carpenters can earn ₹ 540 in 6 days, working 9 hours a day. How much will 8 carpenters earn in 12 days working 6 hours a day?

30. A bill for ₹ 2920 was drawn on september 11 for 3 months after date and was discounted at 16% p.a for ₹ 2875.20. On what date was the bill discounted?

31. What is the market value of 6% stock if it earns an interest of 4.5% after deducting the income tax of 4%.

32. The owner of a departmental store purchased an article of ₹ 1500 at 4% VAT and sells it at ₹ 1700 to the customer at 4% VAT. How much amount did the shopkeeper deposit to the Government as VAT?

33. Find the equation of a parabola whose focus is (–5, 0) and directrix $x = 5$.

34. If $x = a \sec\theta$, $y = b \tan\theta$, find $\frac{dy}{dx}$ at $\theta = \pi/4$.

35. The surface area of a spherical bubble is increasing at the rate of 0.8 cm^2/sec. Find the rate at which its volume is increasing when its radius is 2.5 cms.

36. Find the maximum and minimum values of the function $f(x) = 9x^2 + 12x + 2$.

37. Evaluate : $\int \frac{4x+3}{(x-1)(x+2)}\,dx$.

38. Evaluate : $\int \frac{1}{\sqrt{x}+x}\,dx$.

PART - D

IV. Answer any SIX questions: **(6 × 5 = 30)**

39. If $A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$, verify : A adj A = adj A. A = |A| . I (where I is the identify matrix of 3rd order).

40. Find the middle terms in the expansion of $\left(\sqrt{x} - \frac{4}{x^2}\right)^{11}$.

41. Resolve into partial fractions : $\frac{1+2x}{(x+2)^2(x-1)}$.

42. Examine whether $p \leftrightarrow q$ and $\left[(p \rightarrow q) \wedge (q \rightarrow p)\right]$ are logically equivalent.

43. Two taps fill a cistern separately in 20 mins and 40 mins respectively. Another pipe can drain off 30 litres per minute from the cistern. If all 3 pipes are opened together, the cistern fill in 72 mins what is the capacity of the cistern?

44. A company requires 1000 hours to produce the first 30 engines. If the learning effect is 90%, then find the total labour cost to produce a total of 120 engines, at the rate of ₹ 20 per hour.

45. Solve the following (LPP) graphically:

Maximize : $z = 60x + 15y$

Subject to Constraits : $x + y \leq 50$

$3x + y \leq 90$

$x, y \geq 0.$

46. Prove : $\sin 20° \times \sin 40° \times \sin 60° \times \sin 80° = \frac{3}{16}$.

47. If $y = x + \sqrt{x^2 - 1}$, prove that : $(x^2 - 1)y_2 + xy_1 - y = 0$.

48. Find the area enclosed between the parabola $y^2 = x$ and the line $x + y = 2$.

PART - E

V. Answer any ONE question : **(1 × 10 = 10)**

49. (a) Show that the points (2, – 4), (0, 0), (3, –1) and (3, –3) are conyclic. (6)

(b) From the top of a house 32 m high, the angle of elevation of the top of a tower is 45° and the angle of depression of the foot of the tower is 30°. Find the height of the tower. (4)

50. (a) Prove: $\underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1$, θ is in radians and hence show that $\underset{\theta \to 0}{\text{Lt}} \left[\frac{\tan\theta}{\theta}\right] = 1$, . (6)

(b) Find the value of $(1.05)^5$ upto 4 places of decimals, using Binomial theorem. (4)

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MODEL QUESTION PAPER - 4
II PUC : BASIC MATHEMATICS [CODE : 75]
(NEW SYLLABUS 2014-15)

Time: 3.15 hours **Max. Marks: 100**

Instructions:

(i) The question paper has 5 parts A, B, C, D, and E. Answer all the parts.

(ii) Part - A carries 10 marks, part - B carries 20 marks, part - C carries, part - D carries 30 marks and part - E carries 10 marks.

(iii) Write the question number properly as indicated in the question paper.

PART – A

I. Answer all the questions: **10 × 1 = 10**

1. Without expanding evaluate : $\begin{vmatrix} 500 & 503 \\ 506 & 509 \end{vmatrix}$.

2. Find r if ${}^{15}C_{r+3} = {}^{15}C_{2r-3}$.

3. Negate : If Ramya study hard then she will get the rank.

4. If a : b = 2 : 5 and b : c = 3 : 5 find a : c.

5. what rate of interest is realised by investing in 14.5% stock at 81?

6. find the value of sin 70° cos 20° + cos 70° sin 20°.

7. Find the equation of the point circle with centre at (4, – 3).

8. Evaluate : $\lim_{x \to -4} \frac{x^4 - 256}{x+4}$.

9. Find $\frac{dy}{dx}$ if $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

10. Evaluate : $\int \left(\frac{1+x^2}{x} \right) dx$.

PART - B

II. Answer any TEN questions: **(2 × 10 = 20)**

11. 11. Prove that if any two rows (or columns) of a determinant are interchanged then the value of the determinant changes only in sign.
12. If a convex polygon has 170 diagonals, find the number of sides of the polygon.
13. In a single throw of two dice, what is the probability of obtaining a total of 9?
14. If p, q, r are three propositions with truth values T, T, F respectively find the truth value of $p \to (\sim q \wedge r)$.
15. Two numbers are in the ratio 3 : 5. If 9 is subtracted from each the new numbers are in the ratio 12 : 23. Find the smaller number.
16. The Banker's gain on a certain Bill due six months hence is ₹ 10 the rate of interest being 10% p.a. Find the face value of the Bill.
17. If $\sin\theta = \frac{3}{5}$ and θ is acute, find the value of $\sin 3\theta$.
18. Show that $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$.
19. Find the equation of the parabola whose focus is (0, –6) and directrix $y = 6$.
20. Find the value of k if $f(x) = \begin{cases} \frac{e^{5x}-1}{2x}, \text{for } x \neq 0 \\ \frac{k}{2}, \text{ for } x = 0 \end{cases}$ is continuous at $x = 0$.
21. If $x = a\sec\theta$, $y = b\tan\theta$ find $\frac{dy}{dx}$.
22. Find a point on the parabola $y^2 = 18x$ at which ordinate increases at twice the rate of the abscissa.
23. Find two positive numbers whose sum is 14 and the sum of the squares of the numbers is minimum.
24. Integrate w.r.t x $\int (4x^2 - 2x + 7)^{3/2}(4x - 1)dx$.

PART - C

III. Answer any TEN questions **(3 × 10 = 30)**

25. If $A=\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}$ show that $A^2 - 7A - 2I = 0$. (I = Idtenity matrix of 2nd order)

26. Find the adjoint of the matrix $A=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$.

27. In how many ways 3 boys and 5 girls can be arranged in a row so that no two boys are together?

28. If A and B are events with $P(A)=\frac{5}{8}, P(B)=\frac{3}{8}$ and $P(A\cup B)=\frac{3}{4}$ find (i) P(B/A) (ii) P(A/B).

29. The driver of a car is travelling at the speed of 36kmph and spot a bus at 80mt ahead of him, after 1hr the bus 120mt behind the car. What is the speed of the bus.

30. A banker pays ₹. 2380 on a bill of ₹. 2500, 73 days before the legally due date. Find the rate of discount charged by the banker.

31. Find the interest earned on ₹ 4897.50 cash invested in 15% stock at 81.5 the brokerage given is 0.125.

32. If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+}}}\ldots\infty$ find $\frac{dy}{dx}$.

33. Find the co-ordinates of the vertex, focus and equation of directrix of the parabola $5x^2 + 24y = 0$.

34. Raju goes to purchase a motor cycle which is priced ₹ 35,640 including 10% on sales tax how ever the actual rate of sales tax at the time of purchase is 7%. Find the extra profit made by the shop keeper if he still charges the original listed price.

35. A circular plate of metal is heated so that its radius increases at the rate of 0.1mm per minute. At what rate is the plate's area increasing when the radius is 25cm.

36. The cost function $C = 500x - 20x^2 + \frac{x^3}{3}$ where x stands for output. Calculate the output when the marginal cost is equal to average cost.

37. Evaluate : $\int \frac{1}{e^x + e^{-x}}\,dx.$

38. Evaluate : $\int_0^{\pi/2} \sin 5x.\cos 3x\ dx.$

PART - D

IV. Answer any SIX questions: **(5 × 6 = 30)**

39. Find the coefficient of x^{-11} in the expansion of $\left(\sqrt{x}-\frac{2}{x}\right)^{17}$?

40. Resolve $\frac{2x^2+16x+29}{(x+3)^2(x+4)}$ into partial fractions.

41. Prove that, $\sim(p\leftrightarrow q)\equiv(p\wedge\sim q)\vee(q\wedge\sim p)$.

42. If 15 men working 12 hours per day perform job in 16 days how long will it take for 21 men working 10 hours daily to do the same job.

43. The time required to produce the first unit of a product is 1000 hours. If the manufacturer experience 80% learning effect. Calculate the cumulative average time per unit and the total time taken to produce altogether 8 units. Also find the labour charges for the production of 8 units at the rate of ₹ 10 per hour.

44. Solve L.P.P graphically, maximize $z = 4x + 3y$ subject to the constraints $x+2y\le 5$, $x+y\le 3$, $3x+y\le 7$, $x, y\ge 0$.

45. The angle of elevation of an object from a point 100m above a lake is 30° and angle of depression of its image in the lake is 45°. Find the height of the object above the lake.

46. Find the equation of the circle passing through the points (1, 1), (2, –1) and (3, 2).

47. If $y=\left(x+\sqrt{1+x^2}\right)^m$, prove that $\left(1+x^2\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-m^2y=0$.

48. Find the area enclosed between the curves $y^2 = 4x$ and $x^2 = 4y$.

PART - E

V. Answer any ONE question: **(1 × 10 = 10)**

49. (a) Salesman Venki has the following record of sales during 3 months of July, August and September for three products. A, B, C which have different rates of commission.

Month	Sales in Units			Total Commission (₹)
	A	B	C	
July	100	100	100	700
August	200	300	200	1700
September	400	900	100	3700

Using matrix method, find out the rates of commission on items A, B and C received by Venki. (6)

(b) Find the value of $(0.98)^3$ using binomial theorem upto 5 places of decimals. (4)

50. (a) Prove that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ and hence deduce that $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$. ($\theta$ in radian) (6)

(b) A company produces two types of leather belts A and B. A is of superior quality and B is of inferior quality. The respective profits are ₹ 10 and ₹ 5 per belt. The supply of raw material is sufficient for making 850 belts per day. For belt A, a special type of buckle is required and 500 are available per day. There are 700 buckles available for belt B per day. Belt A needs twice as much time as that required for belt B and the company can produce 500 belts if all of them were of the type A. Formulate a L.P.P model for the above problem. (4)

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KARNATAKA STATE WOMEN COMMISSION

Karnataka State Woman Commission is the Legislative Commission established on 06-08-1996 and is the important component of Karnataka State Women and Child Welfare Department. The different facilities provided to exploited women and the essential information to public about women are as follows:

FACILITIES:

1. Women are facilitated to get solution and help regarding the family problems such as 'dowry', 'divorce', 'property disputes', sexual harassment.
2. Provide counseling at 'Parivarik Lok Adalath' to solve those family litigations which are not solved in Family Court for a long time.
3. Women are provided help/suggestion to file a case in family court.
4. A relief fund of a minimum Rs. 20,000 to maximum Rs.2, 00,000 is provided to women who have attacked been by a person throwing acid on them causing grievous wounds.

INFORMATION TO PUBLIC ABOUT WOMEN:

1. Giving and taking 'Dowry' is an offence.
2. Minimum age for the girl's marriage is 18.
3. Better to keep the records of photos (with negatives) which are taken at the time of marriage, gifts, an amount should a woman get by right, etc. which would help at the time of any problems in future and they will serve the purpose.
4. When a woman is raped, she should be given first-aid by the government doctor and her clothes should be kept carefully for scrutiny later.
5. Do not forget to take medical attestation from government doctor when woman is exploited physically.
6. When a woman dies unnaturally, post mortem must be done by a government doctor.
7. Whenever, death, accident or dispute take place in a family, do not sign on any paper given by the relatives of the women as the woman was not sound mentally. It may lead to exploitation or cheating.
8. Since 1989, women have the right to get equal share in the parental property.
9. Immovable assets must be registered in both the names of husband and wife.
10. According to the judgment given by the Supreme Court, there should be a committee to listen to complaints in order to prevent sexual harassment on women at each working place.
11. At each taluk centre, there is a C.D.P office to provide suggestion/help/counseling for women subjected to injustice.

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